Dynamics of Classical and Quantum Fields: An Introduction
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Nonlocal Operators
Lecture - 48
Nonlocal particle hole operators - Fermions

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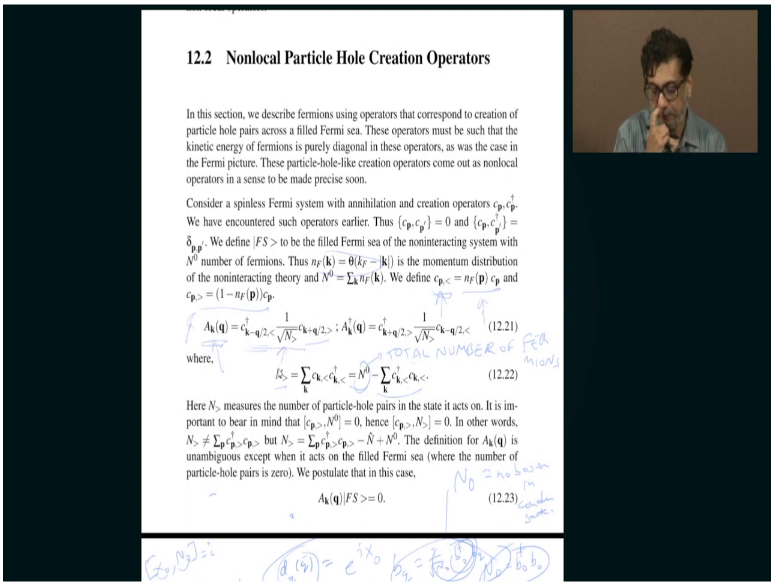

So today let us try and see if we can generalize the concepts of the previous lecture to fermions. So, today is going to be the last lecture of this MOOC's course and the topic I am going to discuss is basically of a research variety; that means, that it is you will not find this in anywhere else; because it is I mean, it is actually based on my own research. So, in fact, it is not even the fermion version is only there in my textbook in chapter 12.2. It is not there anywhere else.

So, remember that in the last class I was telling you that it is beneficial to introduce kind of non-local version of particle hole creation operator. That means you first create a hole then you create a particle both at the same time; that means, you remove a particle with a certain momentum and in the case of bosons you remove a particle with non-zero momentum and you put it back into the condensate; that means, you create a boson with momentum k equal to 0.

So, but then that corresponds to basically b dagger 0 b q, but that itself will not be useful. So, I have to see I have to introduce a creation operator which also behaves like a boson. So, if that is possible only if I multiply that by 1 by square root of the number of bosons in the condensate.

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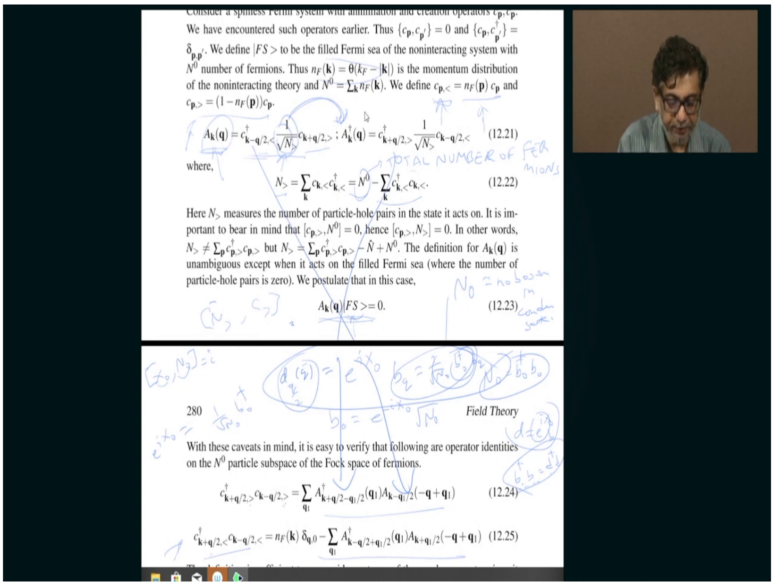

So, that is what we showed that basically because, in the case of bosons it was possible to write an operator called d q. There are two subscripts mainly because you see the implication is that for fermions it will be k plus means it the k equal to 0 corresponds to. So, k is always 0 because for bosons the k equal to 0 is special. So, this becomes q by 2. So, there is a redundant momentum label to emphasize the fact that it is possible to have a more general type of object with two separate momentum levels for fermions.

That is what we are doing today, but I am just refreshing your memory about bosons. So, it has you might be wondering why there is a redundant momentum label that is just to remind you that for fermions these two need not be related; they can be independent. So, that is because you see the condensate is a single point in momentum space which is k equal to 0 whereas, the ground state of a collection of fermions is an extended object.

It is the entire Fermi c you see it is all fermions up to a certain momentum value which is k f ok. So, the point was that I could define, if I chose to define it like this where you see a b 0 is nothing, but e raised to minus X 0 N 0 right and N 0 is b dagger 0 b 0. So, this is the number of bosons in the condensate and X 0 is basically canonically conjugate to N 0.

So, it is one of those funny operators which is canonically conjugate I mean, the these are all very formal issues you know there are some very basic mathematical issues that you have to be aware of for example, you know there is a well-known theorem that says that since N 0 is positive definite; that means, it cannot be negative.

So, there it is you can rigorously show that X 0 cannot be self adjoint. So, you cannot simply postulate that let there be a self adjoint X 0 such that X 0 commutator N 0 is i. Because you can show that when N 0 is non negative you cannot force X 0 to be self adjoint. So, you can force it to be canonically conjugate, but it will not be self adjoint. So, there are all these issues ok.

So, this is like a physicist version of this proof. So, I will have to work with a situation where N 0 is actually nearly infinity. Then only this makes sense ok and that is typically the case in an actual condensate ok. So, N 0 is macroscopically large. So, I am talking about ground state this is not like finite temperature. This is basically the ground state of a boson system. So, if that is the case you see then this is nothing, but what is e raised to i X 0.

It is basically given as 1 by square root of N 0 into b dagger 0 which is what that is. So, that is how I define it as like this. So, its b dagger 0 b q. So, I want to do something similar. So, this you see I showed that this is an exact boson and the point is that remember that if you have products if you have number conserving products of quantum particles which create and annihilate quantum particles.

See that number conserving products will obey closed commutation rules regardless of whether those creation and annihilation are creating fermions or boson.

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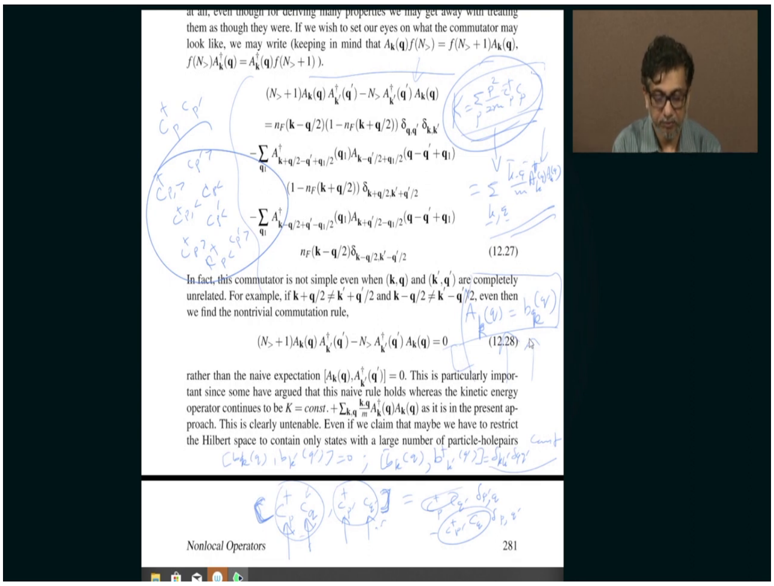

So, suppose you have C p C q and C p dash C q dash. So, this is always going to be equal to C dagger p C q dash delta p dash q minus C dagger p dash C q delta p q dash. So, this is regardless of whether these are bosons of fermions. So, I could be Cs can be bosons or fermions, but the commutator of this will. So, even if the Cs are fermions it is still the commutator that is. So, in other words the commutator of number conserving two number conserving products is a linear combination of number conserving products.

So, regardless of whether the individual creation or annihilation in regardless of whether they are fermions or bosons. So, that even if the fermions it is the commutator not the anti-commutator. So, that is the reason why I want to be able to write down you know some version of this a non-local number conserving product in such a way that number conserving product corresponds to an exact boson ideally speaking.

But I will not be able to do that at least in this lecture; I am going to give you a hint that it is possible to do it, but it is a two-step process. So, first you have to introduce a non-local operator which is almost entirely analogous to this, but then unfortunately for the case of fermions it will not be a boson. It will still obey complicated closed commutation rules, but it will not be a boson.

So, the question is that then you can you have to further re express that in terms of bosons which is still an unfinished research problem which I am still working on. So, those of you are interested in you know collaborating with me on this to finish this research agenda. So, which is basically to make section 12.2 practically useful, you please contact me, but let me explain to you what is 12.2.

So, it is basically the fermionic counterpart of what I discussed in the early previous lecture. So, you see. So, there is no condensate for fermions. So, instead there is a filled Fermi c that is the ground state. So, the momentum distribution; that means, the all the fermions below k f are occupied and all of them above are not occupied. So, this is the Fermi Dirac distribution at zero temperature.

So, now you define something called the an annihilation operator in momentum space which annihilates fermion below the Fermi surface. So, this annihilates a fermion inside the Fermi surface and this annihilates outside the Fermi surface. So, you can imagine that this fermion is. So, when this fermion acts on some typical element of your fox space; that means, if you imagine yours eigen states to consist of the ground state of the system that is basically a system where all the fermions are below the Fermi surface and none are above the Fermi surface.

And then you can add a few more states for example, you can take away some fermions near the Fermi surface and which were below the Fermi surface initially and then you elevate them to some momentum states above the Fermi surface. So, thereby you will be creating excited states. So, basically by acting particle hole creation operators on the ground state you can create excited states.

So, if you have a Hilbert space of states of that nature then you can imagine that most of the time you see if you act this operator that is the operator that tries to take away fermion below the Fermi surface, it will usually be successful; whereas, if you try to take away a fermion above the Fermi surface you will mostly be unsuccessful unless you hit upon the rare state where you know there is an excited electron just at that position.

But whatever it is I mathematically I am perfectly at liberties to define these operators. So that means, I can define an operator which corresponds to taking away an electron below the Fermi surface or taking away an electron above the Fermi surface. So, having done that similarly creation also. So, having done that I define this capital A subscript k q.

So, you see this is the reason why now you have two different momentum labels because there is no notion of condensate. So, I have to deal with the fact that I can create and annihilate you know at two different momentum points. So, here you see the definition is of this nature and remember there was this non local square root of N, here also there is a square root of N, but here this N has the N 0, it was there in the case of bosons.

In the case of fermions this corresponds to yeah this N 0 you should not confuse; this is the total number of particles which is a constant. Total number of fermions unfortunately I have not been very consistent with notation. See here N below 0 means its number of bosons and condensate which is yesterday last lecture earlier lecture. So, whereas, N superscript 0 is total number of fermions of today's lecture.

So, you see the idea is that this N greater basically counts the number of particle hole pairs. It basically tells you that so, this is the number of fermions remaining below the Fermi surface and this is the total number of fermions. So, the difference is basically the number of fermions above the Fermi surface which is basically the total number of particle hole pairs because every time there is a fermion above the Fermi surface there is a hole below the Fermi surface.

So, that is also counts the number of particle hole pairs. So, basically you see this is the whole point. So, this is a non-local particle hole creation operator which keeps track of the number of particle hole pairs that have been created. So, this makes perfect sense on all states except the states where there are no particle hole pairs because then this denominator becomes 0.

So, then I postulate that in that case this; so this is in situations where I do not act it on I act it on any state which has more than one at least one particle hole pair. In that case N greater will never be equal to 0. But in case N greater is 0; that means, I am talking about the ground state of the non-interacting system then I postulate that A k q acting on that is 0.

So, now with these types of assumption this all matrix elements of this make perfect sense; you know the matrix elements of this mathematically well-defined meaning. So, if they have well defined meaning, In fact, you can show again remember that it was very easy to show that in the case of bosons because you see it was just simply d was just some unitary times the b.

So, it was easy to show that b dagger b dot, dot, dot, dot means different different momentum level same d dagger d because, this will cancel out I mean, this e raised to i X dot; if you pretend X dot is Hermitian it will cancel out. But then this was easy in that case, but it is not so easy here, because not only you have summation also; you have to actually prove this.

So, I am not going to prove this here it is left as one of the exercises what you have to do is just insert this here and insert this here, insert this here and prove that it I mean just do the sigma and show that it is an identity because, the right hand side will involve A, but A is defined in terms of the c s. So, if you insert and do sigma you have to get back this answer.

In fact, you can show that you will get back ok. So, here also you will show that it you can get back. So, basically you see this is the inversion.

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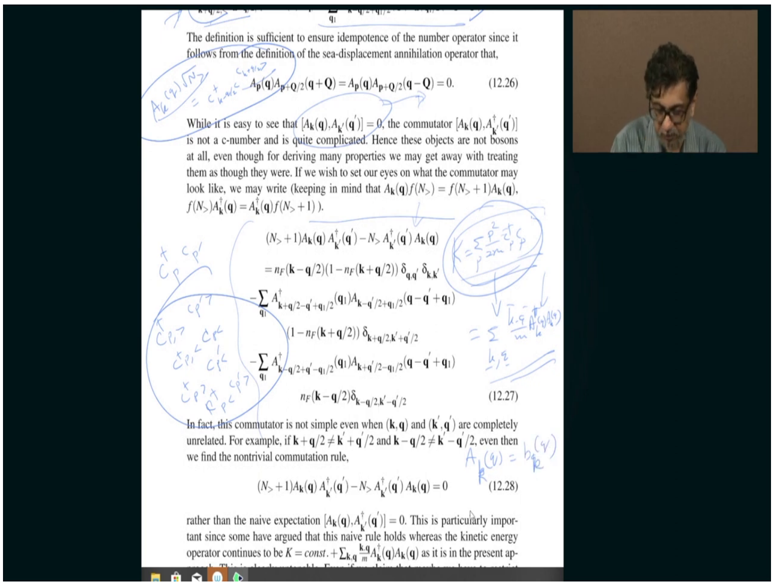

So that means, I have inverted this and re expressed; so, you can re express. So, the you see after all if you have a you have c dagger p c p dash. There only these things. So, in the case of creation or annihilation above or below the Fermi surface there will be these four possibilities either you do both above or both below or you do one above one below or you do one above one below in the reverse way.

So, these are the only four possibilities. So, that exhausts all the possibilities when it comes to creating a particle hole pair. So, you see the if you both are above the Fermi surface, it is expressible in terms of the as this way; if both are below its expressed, but if one is above one is below it is simple.

In fact, you see N greater is going to commute with c greater. So, you might as well put this on this side; because there is no see, N greater paradoxically will commute with c greater because N greater is in terms of c less. So, it will commute with c greater, but N greater will not commute with c less.

So, it will commute with c greater which is something you would not have expected. So, you can therefore, put this on this side and then multiply from the right on with respect to A. So that means, so basically what will happen is that you see A k q multiplied by N greater is nothing, but c dagger k minus q by 2 less c k plus q by 2 greater.

So, you see that is the thing. So, there are four possibilities; one is greater-greater less-less both are expressed already. So, the greater-greater, less-less I have expressed in terms of the A's and less greater is expressed in terms of A's this way and if you take dagger of this you will get greater less also expressed in terms of the A. Then it will become square root of N greater A dagger k q.

So, that exhausts the ways in which you can express the number conserving products of fermions in terms of these funny A's. So, you might be wondering why am I doing all this. First of all, you see the commutation rules obeyed by the A's are not simple they are actually not bosons. So, the thing is that why am I doing this? The reason why I am doing this is because, you can show that you see the kinetic energy of free fermions which is writable as p squared by 2 m into c dagger p c p can actually be shown to be the same as sigma k comma q k dot q divided by m A dagger k q A k q.

So, in other words, you see this is the fundamental reason why this is useful. So, the kinetic energy which was diagonal in the fermions is also diagonal in these very funny complicated A operators. See these A operators are very non trivial because they are non-local see there is this N greater which is measures the number of particle hole pairs which is itself an operator and its square root appears in the denominator.

So, it makes it an ultra non local operator. So, nevertheless even though it looks that bizarre because it looks that bizarre this is possible. See, because I have defined it in that peculiar way, the kinetic energy of free fermions which is diagonal in the original fermions is also diagonal in these operators, but it would have been amazingly beneficial if this A's if these A's were in fact, exact bosons sadly they are not.

So, what you have to do in order to make the scheme truly useful is you have to find exact bosons. So, you see first of all the A's I have been able to show rather easily that this yeah this is always obeyed. So, you see if A's were bosons we expect this at the very least and in fact, even though unfortunately A A dagger commutator is not Kronecker delta k k dash q q dash which is what we expect if A's were bosons.

But at least A commutator A with different k's and q's or at least these this is identically 0. That is something we really like it is a kind of a relief, but; however, the fact that A commutator A dagger is not Kronecker delta it is very unfortunate. So, but however, in case of the analogous construction in bosons it was already boson, but here it is not. So, you have to work harder.

So in fact, the commutator the version of the I mean the what resembles the commutator actually looks like this. So, the question is. Firstly, how do you do that? So, that I will try to postpone and explain to you later; that is how do you know construct exact bosons which you can then relate A 2; that means, you should be able to write A k q in terms of some exact bosons called b k q perhaps.

So, these are exact bosons. So that means, these bosons will have the property that b k q b k dash q dash is 0 and also b k q b dagger k dash q dash is equal to delta k k dash delta q q dash times something some constant ok some constant involving k q and all that. But basically it is this is the right hand side is proportional to the identity operator.

So, if you can find b’s like this then you are all set; because you can do lots with this. Because, if you do not find this you cannot it is very it is a big struggle because the A’s are very non trivial. So, the question is how do you write this and this is the topic of great urgency now. We have to actually do it and I am trying this on my own right now; those of you who are listening to this lecture and are interested in collaborating with me please contact me yeah.

So, this is very crucial now. But however, even in the absence of this very crucial ingredient namely even though we are not been successful in writing these A k q in terms of exact bosons you can still go ahead and this I am going to skip I have just told you yeah.

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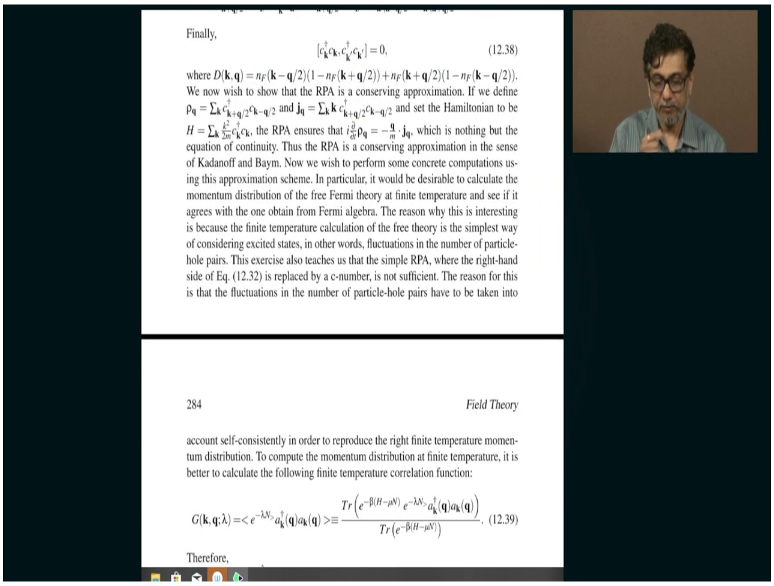

So, you can go ahead and write down the momentum distribution, you can derive the Fermi Dirac distribution in terms of these operators in terms of A k q’s.

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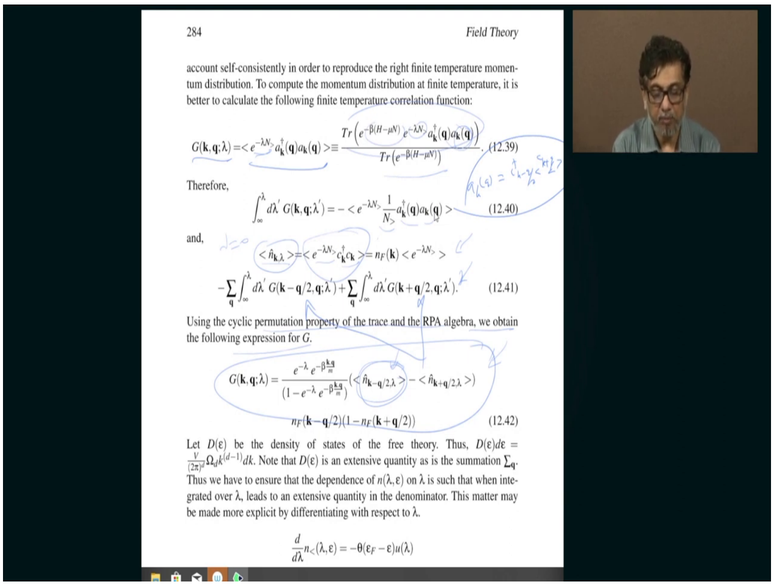

So, the question is how do you do that. So, I first define small letter a k q as basically this. So, I define small letter a k q as basically c dagger k minus q by 2 less c dagger c k plus q by 2 greater. So, this is small letter a k q. So, now, you see now I am going to try and calculate this average. So in fact, you can ok let us show this. So, this is the definition of this g that I am trying to calculate.

So, this is the definition of A k q. So, A k q is basically this capital A k q without that square root of N ok. So, just without that square root of N. So, it is k k minus q by 2 less c dagger k minus q by 2 less c k plus q by 2 greater. So, you see now I am going to try and calculate this. So, this is nothing, but in grand ensemble it is this ok.

So, now, you see the device of introducing this exponential is that when you do an integral over this lambda prime parameter you will bring down an N dagger ok; N greater dagger ok and I also choose to define something called the lambda parameterized average of N k which is basically before taking the average of c dagger k c k you multiply by this.

So, the momentum distribution of fermions will correspond to putting lambda equal to 0. So, now, you can show that this is nothing, but this. So, this is again a it is a bit of an algebra. So, I will allow you to think about it. So, basically the momentum I mean the lambda parameterized momentum distribution which is defined like this is expressible in terms of the G's in this way ok.

So, now if you are successful in finding this one using the properties of the A's not the fermions in terms of the properties of the A's then you would have successfully derived the Fermi Dirac distribution using an algebra that is reminiscent of bosons rather than fermions. Because, you see after all what are A's they are number conserving products of fermions.

And I have repeatedly told you that number conserving products of fermions are very close commutation rules; they do not know obey anti commutation rules. So, you are trying to express number conserving products of fermions in terms of objects that more closely resemble bosons alright. So, it says you can find this trace using the cyclic permutation and the commutation rules obeyed by a k q and the Hamiltonians and N dagger.

And you can convince yourself that this is what it is ok. So, this it comes also the G k q is also related to this lambda parameterized momentum distribution. So, now, you see so therefore, this kind of indirectly tells you the so this is some kind of an integral equation. So, if you insert this into this equation, it gives you an integral equation for this lambda parameterized momentum distribution.

So, which when having obtained that you simply said lambda equal to 0 that will give you the Fermi Dirac distribution. So, it is a lot of work for deriving something we know how to do using the simple Fermi algebra. But you see this is this has been derived using an algebra that is more reminiscent of bosons because see the these objects a k q's the number conserving products of fermions which obey closed commutation rules rather than anti commutation rules.

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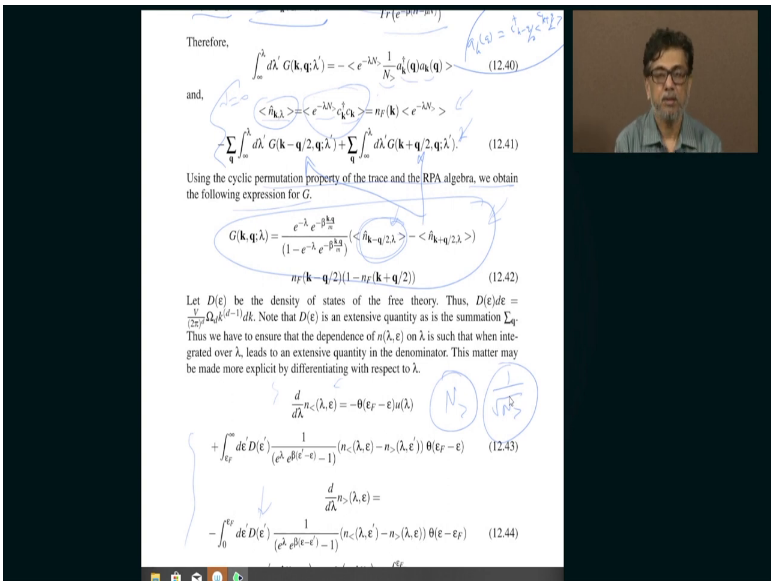

Now, you can convert this integral equation into a differential equation of this sort ok. So, you will have two types of N's one is when k is below the Fermi surface and when k is above the Fermi above and I have you know exchange k for because it is an isotropic system I have replaced k with the energies and density of states and so on. So, this is a standard solid state physics approach.

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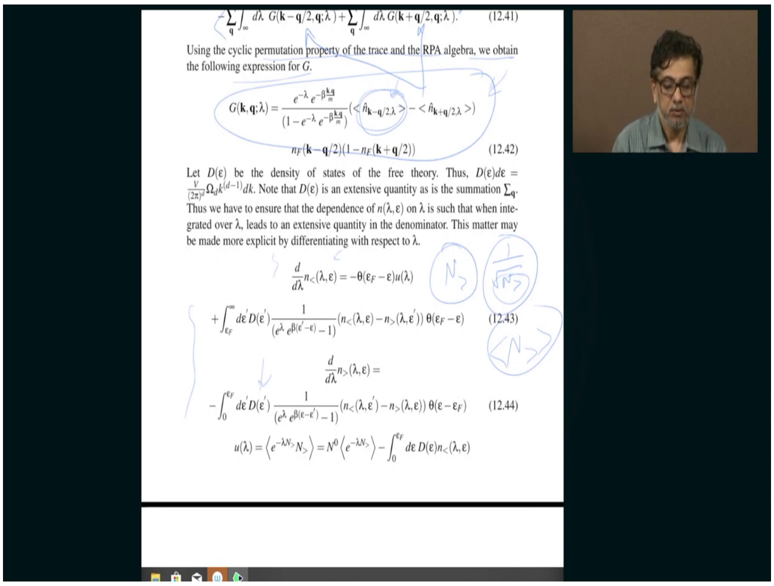

So, when you do that you can solve these two coupled equations by this sort of an answers and when you again a lot of algebra.

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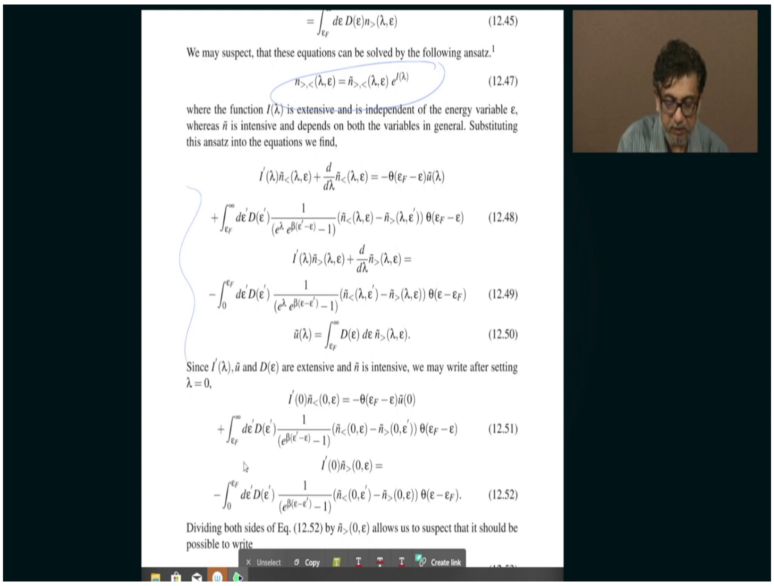

So, once you do that you can show that in the end you get this result ok. So, you can show that basically that when you put lambda equal to 0 which is what this is you get the Fermi Dirac distribution ok.

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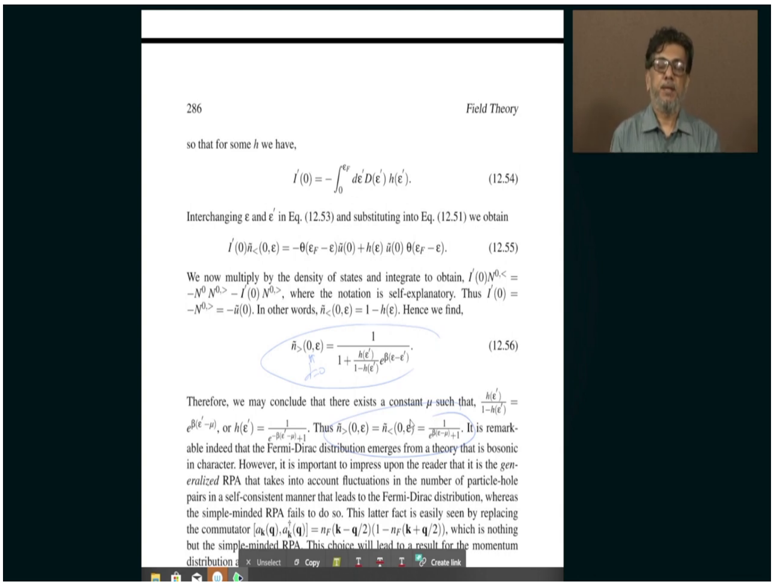

So, the bottom line here is that you see the moral of the story is that this would not have worked, if I had not consciously introduced. So, this is something you have to go through these steps in great detail to understand. So, if I had not introduced this quantity which is the one that fluctuates. See, studying a system of fermions at finite temperature is the simplest way in which you can study the effect of fluctuating number of particle hole pairs.

Because you see this 1 over square root of N greater is basically very sensitive to fluctuations in the number of particle hole pairs. So, the fact that this appears repeatedly in the formalism means that it is trying to tell you that you have to be conscious of this object and you have to treat it very carefully.

So, what this analysis shows is that studying this carefully and self consistently as we often like to say, that if you study itself consistently then you are going to correctly obtain the Fermi Dirac distribution. So, if you become a kind of naive and try to replace this by its expectation value or you know try to replace it by some average like we have instinctively try to do in physics whenever some nasty operator appears we replace it by its average.

So, if you try all kinds of those uncontrolled types of approximations you will never get the Fermi Dirac distribution. So, the moral of the story is that you have to self consistently and exactly take into account the fluctuations of the number of particle hole pairs in a self-consistent way, it is only then that you will actually be able to derive the Fermi Dirac distribution not otherwise ok.

So, the thing is that I do not want to spend too much more time because there is in much else I can say which makes any rigorous sense. So, the current state of the art with respect to constructing particle hole like operators for fermions which you can then use to study interacting Fermi theories is whatever I have explained just now.

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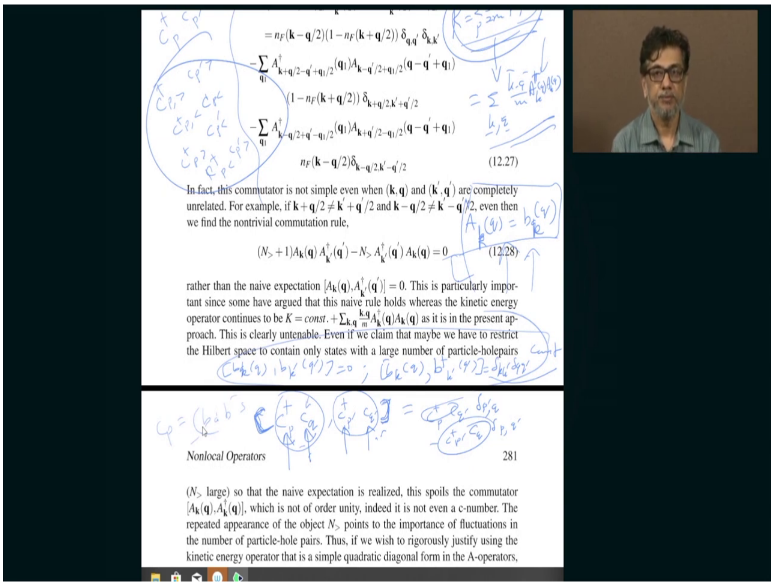

So, the next important agenda which is unfinished is extremely urgent which is to write the A k q's in terms of exact bosons. So, these b k q's like I was talking about. So, they have to obey this. So, yeah. So, those of you are interested please contact me we have to do this. So, having done this the last item of the agenda would be to write the annihilation operator alone in terms of the b k q's in terms of the b's and b daggers.

So, this would correspond to what is called bosonization; that means, you are expressing the fermion annihilation operator in terms of exact bosons. So, you see that is possible for chiral fermions in one dimension as it is well known in the community and I have a YouTube lectures on that separately which is also mentioned in my institute website.

So, you can have a look at that, but that is very peculiar to a very peculiar sort of fermion which are called chiral fermions. So, chiral fermions are fermions where the dispersion is completely linear and it is proportional to the energy its proportional to momentum and they there is yeah. So, the ground state of the system has infinitely many particles, even if the system size is finite because all negative energy states are occupied.

So, bottom line is that yeah. So, that is a very artificial and you know contrived sort of system, but in absence of anything more realistic people kind of make do with that and in one dimension that humongous amount of literature that uses that sort of idea to study one dimensional interacting fermions. So, that goes by the name of chiral bosonization. But what I am displaying here is basically a very much more ambitious way of studying fermions in not just one dimension it this could apply in one dimension as well.

But it could apply more realistically in two and three dimensions and that to for fermions of the usual garden variety kind not the chiral kind. So, the where the energy dispersion is basically p squared by 2 n. So, that is the whole idea. So, I am going to be able to do that. So, I am going to be able to write the Fermi field purely in terms of exact bosons, but then I have to first construct those exact bosons by expressing this capital A k q which I have so painstakingly written down in terms of exact boson.

So, that is likely to be an important first step. So, having done that we can go ahead and see, if we can write down the annihilation operator purely in terms of the exact bosons. After which we can try and make use of it so that is the whole point. So, ok I am going to stop here and this concludes the present MOOC's NPTEL course on dynamics of classical and quantum fields.

So, I hope you have enjoyed this course, if not I do not blame you because you see this subject is rather advanced and I know that many of you who are enrolled in this course, are somewhat like beginners and you have trouble following the course which has such a vast syllabus and which has gone at such a incredible pace.

But you do not have to necessarily learn everything all at once just you know go through the lectures carefully, slowly pause it and listen to it several times read the textbook and so on. And if you still have questions about various topics please send me a mail, I will be happy to answer any questions.

So, thanks for listening to all these lectures, I hope to hear from you ok.

Thank you.