Dynamics of Classical and Quantum Fields: An Introduction

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Nonlocal Operators
Lecture - 47
Nonlocal particle hole operators - Bosons

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Single-particle Green functions in exactly solvable models of Bose and Fermi liquids

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Based on a class of exactly solvable models of interacting Bose and Fermi liquids, we compute the single-particle propagators of these systems exactly for all wavelengths and energies and in any number of spatial dimensions. The field operators are expressed in terms of Bose fields that correspond to displacements of the condensate in the Bose case and displacements of the Fermi sea in the Fermi case. Unlike some of the previous attempts, the present attempt reduces the answer for the spectral function in any dimension in both Fermi and Bose systems to quadratures. It is shown that when only the lowest-order sea-displacement terms are included, is the random-phase approximation in its many guises recovered in the Fermi case, and Bogoliubov's theory in the Bose case. The momentum distribution is evaluated using two different approaches, exact diagonalization and the equation of motion approach; the novelty being, of course, the exact computation of single-particle properties including short-wavelength behavior. [S0163-1829(98)00624-9]

I. INTRODUCTION

Recent years have seen remarkable developments in many-body theory in the form of an assortment of techniques that may be loosely termed bosonization. The beginnings of these types of techniques may be traced back to the work of Tomonaga[1] and later on by Luttinger[2] and by Lieb and Mattis.[3] The work of Sawada[4] and Arponen and Pajanne[5] in recasting the Fermi-gas problem in a Bose language has to be mentioned. Arponen and Pajanne recover corrections to the random-phase approximation (RPA) of Bohm and Pines [6] in a systematic manner. In nuclear physics, bosonization is widely used to study collective properties, for an introduction see the book by Iachello and Arima.[7] In the 1970s an attempt was made by Luther[8] at generalizing these ideas to higher dimensions. Closely related to this is the work by Sharp *et al.*[9] in current algebra. More progress was made by Haldane[10] which culminated in the explicit computation of the single-particle propagator by Castro-Neto and Fradkin[11] and by Houghton *et al.*[12] and also by Kopietz *et al.*[13] Rigorous work by Frohlich *et al.*[14] is also along similar lines. Also the work of Frau *et al.*[15] on algebraic bosonization is relevant to the present article as the authors have considered effects beyond the linear dispersion in that article. The exactly solvable models of Calogero and Sutherland are of relevance here as well, the exact propagators of these models have been computed by various authors.[16] Recently, these types of models have been generalized to more than one dimension by Ghosh.[17]

the study of He-II by Sunakawa *et al.*[18] However, the authors are not aware of a rigorous study of the meaning of this object, in particular, an explicit formula for the canonical conjugate of the density operator has to the best of the authors' knowledge never been written down in terms of the field operators. The work by Sharp *et al.*[9] comes close to what we are attempting here.

The concept of a sea displacement is a generalization of the traditional approach used for bosonizing one-dimensional (1D) systems such as the Tomonaga-Luttinger[1,2] models. There, one introduces Bose fields that correspond physically to displacement of the Fermi surface (in 1D, Fermi points). These Bose fields have simple forms relating them to number-conserving products of Fermi fields. The field operator is obtained by exponentiating the commutation rule between the surface-displacement operator and the field operator. By analogy, we generalize these ideas, so that one is no longer restricted to be close to the Fermi surface. The way this is done is to postulate the existence of Bose fields that correspond to displacements of the Fermi sea rather than just the Fermi surface. From this it is possible to write down formulas for the number-conserving product of two Fermi fields in terms of the Bose fields. A similar construction is possible when the parent fields are bosons, but here, we find that instead of sea displacements, we have to introduce operators that correspond physically to displacements of the condensate. Actually, the Bose case is much simpler and a mathematically rigorous formulation of this correspondence is possible. This is a boon, since we use this fact and make

Ok. So, today, let us move to a new topic and this topic will conclude the present MOOCs course. And this particular topic is basically based on largely or nearly only on my own research that I have been doing over the last several years. So in fact, it way back in 1997, I published a paper with my PhD guide from University of Illinois, where I set out an agenda to express the properties of interacting quantum particles in terms of operators that correspond to particle hole excitations.

So, this kind of an idea has a long history. And it is only in the mid-90s that people actually figured out how to do it in a way that is practically useful substantial way. So, in fact, the ideas of Duncan Haldane and others, who used this basic ideas of quantizing the Fermi surface that was very crucial in many of these developments. So, the work that I did is kind of also pretty much inspired and so to some extent even borrowed from those ideas. But there are some substantial differences.

So, I will not be dwelling on the real literature on the subject. I mean to be honest if I have to be fair to the other authors, I have to spend a lot of time discussing literature survey what others have done and so on. But that is not the purpose of the present course anyway. It is not to be, I mean it is not meant to be an honest discussion of what others have done in some chronological order or something. So, it is just meant to you know highlight important topics that students should then make an effort to learn and fill all those gaps.

Namely, they should make an effort to see who has done what when and so on and so forth. So, in order for them to do that I have to tell them some resources. So, for example, this paper that I published in physical review in 1998, even if you dislike large parts of it, in fact, in hindsight the fermion part of this paper is rather I mean it is kind of well-motivated, but it is in the end rather incomplete.

But the boson part of this paper is quite ok. But still it motivates this the introduction of the so called non-local operators in a very nice way. But the most important positive aspect positive quality of this paper is that it has a large number of very relevant references which normally a reader will be hard pressed to find if it was not you know listed explicitly that way. So, I urge you to read this paper, if not for any other reason, at least for the references. But I want to convince you that it is worth reading even for the contents, ok.

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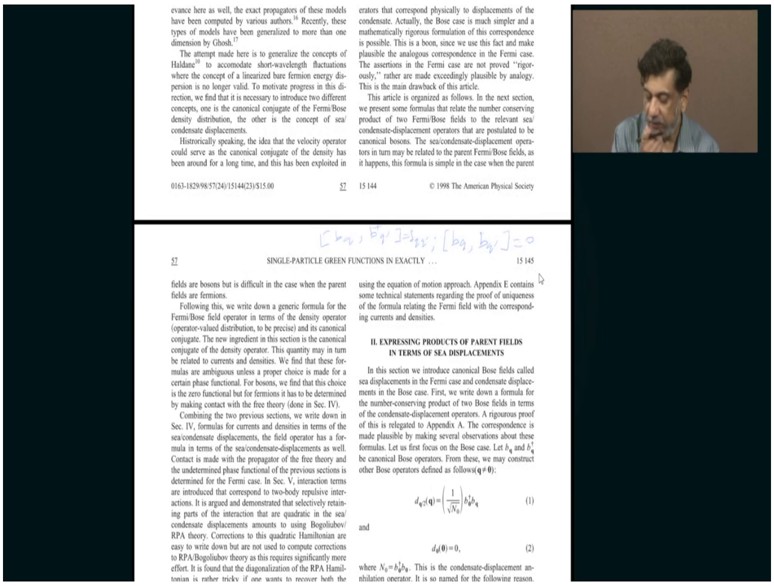

evance here as well, the exact propagators of these models have been computed by various authors.[16] Recently, these types of models have been generalized to more than one dimension by Ghosh.[17]

The attempt made here is to generalize the concepts of Haldane[10] to accomodate short-wavelength fluctuations where the concept of a linearized bare fermion energy dispersion is no longer valid. To motivate progress in this direction, we find that it is necessary to introduce two different concepts, one is the canonical conjugate of the Fermi/Bose density distribution, the other is the concept of sea/condensate displacements.

Histrorically speaking, the idea that the velocity operator could serve as the canonical conjugate of the density has been around for a long time, and this has been exploited in erators that correspond physically to displacements of the condensate. Actually, the Bose case is much simpler and a mathematically rigorous formulation of this correspondence is possible. This is a boon, since we use this fact and make plausible the analogous correspondence in the Fermi case. The assertions in the Fermi case are not proved "rigorously," rather are made exceedingly plausible by analogy. This is the main drawback of this article.

This article is organized as follows. In the next section, we present some formulas that relate the number conserving product of two Fermi/Bose fields to the relevant sea/condensate-displacement operators that are postulated to be canonical bosons. The sea/condensate-displacement operators in turn may be related to the parent Fermi/Bose fields, as it happens, this formula is simple in the case when the parent

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$[b_q, b^\dagger_{q'}] = \delta_{qq'}$; $[b_q, b_{q'}] = 0$

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fields are bosons but is difficult in the case when the parent fields are fermions.

Following this, we write down a generic formula for the Fermi/Bose field operator in terms of the density operator (operator-valued distribution, to be precise) and its canonical conjugate. The new ingredient in this section is the canonical conjugate of the density operator. This quantity may in turn be related to currents and densities. We find that these formulas are ambiguous unless a proper choice is made for a certain phase functional. For bosons, we find that this choice is the zero functional but for fermions it has to be determined by making contact with the free theory (done in Sec. IV).

Combining the two previous sections, we write down in Sec. IV, formulas for currents and densities in terms of the sea/condensate displacements, the field operator has a formula in terms of the sea/condensate-displacements as well. Contact is made with the propagator of the free theory and the undetermined phase functional of the previous sections is determined for the Fermi case. In Sec. V, interaction terms are introduced that correspond to two-body repulsive interactions. It is argued and demonstrated that selectively retaining parts of the interaction that are quadratic in the sea/condensate displacements amounts to using Bogoliubov/RPA theory. Corrections to this quadratic Hamiltonian are easy to write down but are not used to compute corrections to RPA/Bogoliubov theory as this requires significantly more effort. It is found that the diagonalization of the RPA Hamiltonian is rather tricky if one wants to recover both the using the equation of motion approach. Appendix E contains some technical statements regarding the proof of uniqueness of the formula relating the Fermi field with the corresponding currents and densities.

II. EXPRESSING PRODUCTS OF PARENT FIELDS IN TERMS OF SEA DISPLACEMENTS

In this section we introduce canonical Bose fields called sea displacements in the Fermi case and condensate displacements in the Bose case. First, we write down a formula for the number-conserving product of two Bose fields in terms of the condensate-displacement operators. A rigourous proof of this is relegated to Appendix A. The correspondence is made plausible by making several observations about these formulas. Let us first focus on the Bose case. Let $b_{\mathbf{q}}$ and $b^\dagger_{\mathbf{q}}$ be canonical Bose operators. From these, we may construct other Bose operators defined as follows($\mathbf{q} \neq \mathbf{0}$):

$$d_{\mathbf{q}/2}(\mathbf{q}) = \left(\frac{1}{\sqrt{N_0}}\right) b^\dagger_{\mathbf{0}} b_{\mathbf{q}} \quad (1)$$

and

$$d_{\mathbf{0}}(\mathbf{0}) = 0, \quad (2)$$

where $N_0 = b^\dagger_{\mathbf{0}} b_{\mathbf{0}}$. This is the condensate-displacement annhilation operator. It is so named for the following reason.

So, let me start off by explaining what I am talking about. You see the idea is that remember that we have been discussing Hamiltonians, we have been describing Hamiltonian of many particle systems in terms of creation and annihilation operators.

So, remember that I told you that if you have a system of free fermions moving in space you can introduce something all the annihilation field operator which is psi bracket r comma t, which annihilates a quantum particle it could be a boson or a fermion at position r at time t. And I also told you that if there are two body interactions between the particles, then the Hamiltonian becomes a quartic.

That means that the Hamiltonian will involve two creation and two annihilation operators, when you are referring to the term that corresponds to interaction between particles. See, on the other hand, the kinetic energy term will only involve one creation and one annihilation operator. But now if you sit back and think about it, you will see that you see the fact that the Hamiltonian consists of these pairs of one creation and one annihilation makes you suspect that it might be possible to give that operator a name.

So, instead of calling psi dagger r psi r dash as whatever I just called it, namely psi dagger r psi r dash is better to give it some kind of different name. So, what it does is basically it is a particle hole creation operator. So, it kind of it annihilates a particle somewhere and creates a particle somewhere else. So, in other words, it is creates a particle hole pair. It creates a hole and then it creates a particle. So, that is called a particle hole creation operator.

So, it is better to give that a name as some you know b b dagger or something. So, let me tell you what I mean perhaps. So, ok let me actually display it then. So, you see if I am talking about see the underlying particles are bosons, right. So, then your underlying bosons in momentum space will have this property. So, they will have a commutation properties.

So, b q, q is your usual the translationally invariant, for a translationally invariant system the good quantum numbers of momenta. So, you have b b dagger commutator is Kronecker delta and b b commutator is 0.

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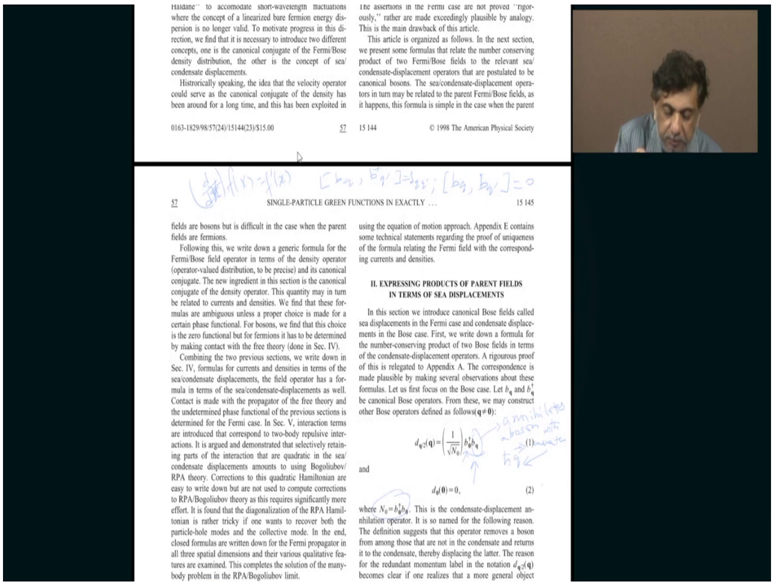

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and

$$d_{\mathbf{0}}(\mathbf{0}) = 0, \quad (2)$$

where $N_0 = b^\dagger_{\mathbf{0}} b_{\mathbf{0}}$. This is the condensate-displacement annhilation operator. It is so named for the following reason. The definition suggests that this operator removes a boson from among those that are not in the condensate and returns it to the condensate, thereby displacing the latter. The reason for the redundant momentum label in the notation $d_{\mathbf{q}/2}(\mathbf{q})$ becomes clear if one realizes that a more general object

So, now I am going to define something which is somewhat peculiar, but it is basically a particle hole creation operator. See, what this does is it annihilates a boson with momentum h bar q, right. So, q is your wave vector. So, this annihilates a boson specifically a boson, ok with momentum and then it creates a boson with momentum 0, ok.

So, you might be wondering why did I define it in this peculiar way, and there is a reason for that. But more than just that, you see I am also going to be forced to for reasons I will tell you later. But basically I am going to then multiply this by 1 by square root of N 0, where now N 0 is actually now itself an operator. See, this is what makes this subject so difficult because these are what are called non-local operators. So, that you know you have operators that appear in the denominator. I can just give you some simple flavor of what non-local operators can be.

See, for example, you see if I say d by dx. So, this is an operator. So, it takes a; why is this an operator? Because you can act it on some function and you will get some other. So, if you fix x, it will act on the function and it will produce some number which is basically the derivative. So, d by dx is called an operator for that reason. But then you see if I want to make sense out of this.

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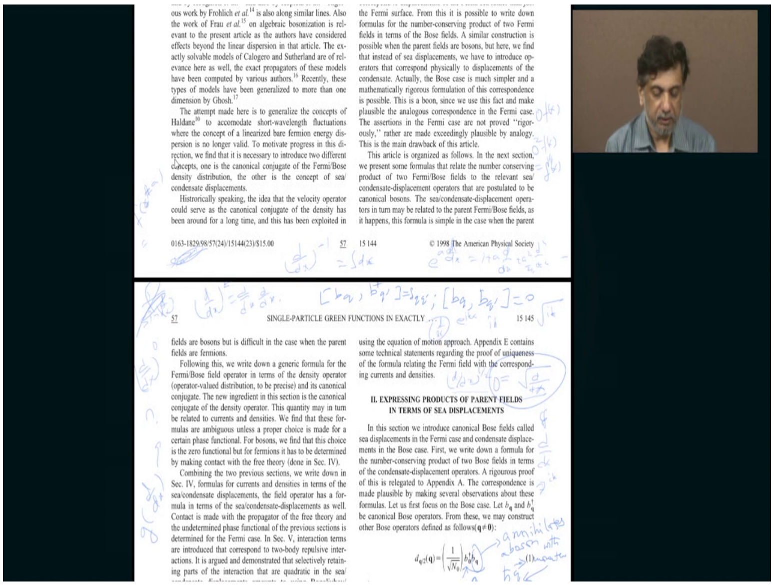

Suppose, I want to make sense out of some function of d by d x. So, what does this mean? So, this function if it is very simple like whole squared, so then d by dx whole squared is basically means d by dx then again d by dx, that is what this means. So, now, if it is whole squared, this is perfectly fine, cubed it is fine like that. But d by dx to the power 0 is also fine. That is same as not doing anything. But, what is d by dx to the power minus 1? So, this is where things start to get a little funny.

So, this is basically the inverse of the derivative which we know as integration. So, if you have integer powers you can at least make sense out of it this way. So, d by dx whole squared is differentiating twice, d by dx whole cube differentiating 3 times, d by dx raise to the power minus one is integrating once, d by dx raise to the power minus 2 is integrating twice.

So, then you can make sense out of things like you know d by e raise to a, d by d x. So, this is perfectly valid because then I can expand this in 1 plus a d by dx plus a squared d square by dx square by 2 factorial etcetera. So, this because d by dx makes sense; d by dx whole squared is basically d by dx acting twice. So, all of this makes sense and addition makes sense.

So, similarly, you can write even crazier things, like you know you can write, sin of you know 1 by whatever yeah you can write crazier things like this. So, if you want to write 1 by sin d by dx, that is basically inverse of sin of d by d x. So, you can make a lot of sense whenever Taylor series is possible.

But things become even funnier when you ask questions like what is d by dx to the power half, what is square root of d by dx. So, these are what are called non-local operators. So, in fact, d by dx to the power minus 1 is already non-local because you see yeah, ok, let us not go there.

Bottom line is that this is certainly non-local. So, because you have to first make sense of this. See, what does this mean? This mean this is an operator which has the property that if you act this operator on some function, twice it is the same as differentiating it once. So, that is what this operator does.

So, if you call this operator O. So, O acting, O acting on f of x means the same as it is that operator when you act twice is same as differentiating it once. So, the question is you might say that like what if I just want to know how it acts once, I mean I know it if it acts twice as same as differentiating it once.

So, in order for you to do that I will I will not be able to spend a lot of time explaining, but basically you can make sense out of this through Fourier transform. So, you express f of x in terms of some plane waves as a linear combination of plane waves. And basically, if you do that then d by dx is nothing but it gets replaced by i k. So, if your basis states is i k x, then d by dx is same as multiplying doing d by dx is same as multiplying by i k. So, therefore, doing square root of d by dx is same as multiplying by square root of i k.

So, basically your Fourier components gets multiplied by square root of i k. And then, when you do the transform again you will get the meaning of that. Now, you see that will, so the meaning of that will means that the this operator square root of d by dx acting on f of x will not depend upon f of x only or f dash of x or f double dash of x, basically it will depend on all the derivatives of f of x at x. So, basically it is non-local in that sense.

So, non-local means is same as this. If I take x f of x plus a this is non-local because the answer for what this is does not depend on how this function behaves close to x equal to a. It depends on how it behaves far away from because a can be anything. So, f of x plus a to know, what is f of x plus a it is not enough to know how this behaves close to f of x. That means, you have to know how f of, you have to know that function at all points. It is not enough to know what it is close to x. So, as a function of x its non-local, because it depends on how the function is at a point far away from x.

So, in fact, you can translate that into this other language, I told you about. You can Taylor series this in powers of a and when you do that you will get all derivatives of f of x. So, in other words, these two descriptions are equivalent. So, saying that this function depends on what this argument is far away from x is same as saying that it depends on all derivatives of f at that value x, ok. So, that is basically what typically one means by non-local. And in some similar sense this N 0 is also non-local, ok.

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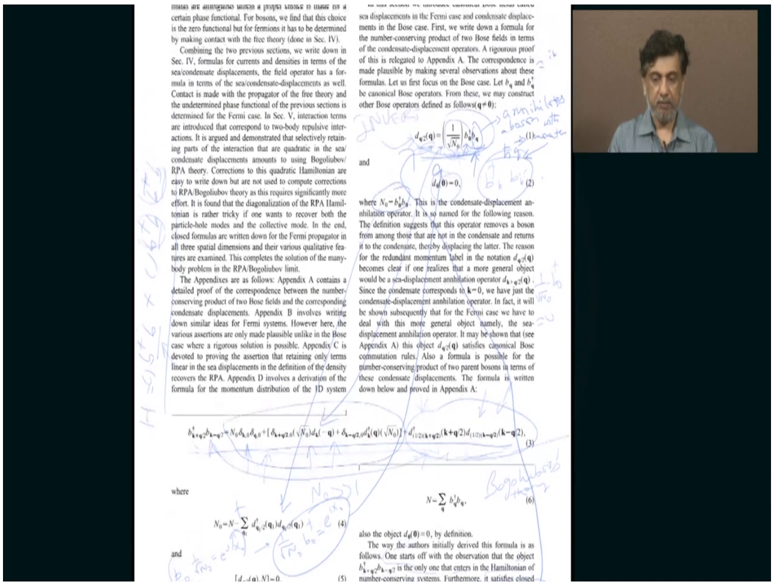

So, this is non-local for a similar reason that there is an operator and a square root of the operator and that operator is in the denominator. So, in that sense, it is also non-local. So, now, you see if q is 0, I choose to not define it using this formula. So, that definition is this. So, now so the question is this is certainly a some kind of a particle hole creation operator because it creates a hole because b q annihilates a boson therefore, it creates a hole and b dagger 0 and creates a boson, but at the value q equal to 0.

So, therefore, it is particle hole creation. But then its non-local particle hole creation because this d is non-local because of this extra term there. So, you might be wondering why I selected b dagger 0. So, there is a specific reason for that because you see for non-interacting bosons the ground state is basically a condensate. So, that means, that all the bosons are sitting at q equal to 0. So, this is in some sense a kind of an operator that takes a boson outside the condensate and leaves a hole behind in the condensate, or it does the reverse.

If there is some bosons sitting outside the condensate and a large number of bosons in the condensate it brings it back to the condensate. So, it is kind of; so, d actually does that, d brings back a Boson from outside the condensate into the condensate. Whereas, d dagger does the reverse, it takes a boson from the condensate and puts it outside the condensate.

So, these are basically particle hole creation operator for bosons in the situation where you have a condensate. So, now, mathematically, so you do not have to necessarily interpret that way. You can just say this is the mathematical definition. Now, the important question is there are two important questions, first is you see the b satisfy simple commutation rules they are after all exact bosons. Namely that b commutator with other b is 0, b commutator with b dagger is Kronecker delta.

So, now the question is what about the ds. So, the commutator is it ds are also bosons yeah that is the important question. So, you will see that because I have defined it this way, it so happens that the ds are also exact bosons. So, in fact, that is a very fascinating. So, just like these are exact boson these are exact bosons only if so long as I include this very funny non-local factor. If I do not have that non-local factors then these are not exact bosons. So, that is the important thing.

So, now, that is one facet that is very important aspect. So, which is what makes is likely to make it useful, it is only if your particle hole creation operators are also bosons. See, the reason is because you see the Hamiltonian will be writable as something involving b dagger b plus if you are in interacting interaction between bosons it will have b dagger b b dagger b something, some something into b dagger b plus something else into b dagger b b dagger b. So, this would correspond to the two body interactions between bosons.

Now, the point is that you see if you are successful in rewriting these I mean these products of particles in terms of, so these are products of bosons, but then this peculiar combination of products of bosons are also bosons. But they are individual annihilation operators. So, these are products of creation annihilation operators, but they are peculiar non-local combinations. But then, the end product is still a above annihilation of a of a different boson.

So, now if you are successful in doing that then, you can suspect that there is some sense in which this Hamiltonian will then become basically quadratic plus perhaps a linear term. So, there will be a quadratic term. In fact, this will not be linear for reasons I will tell you later, this will also be quadratic, ok. So, bottom line is that this whole thing will become quadratic in the new bosons, even though your original Hamiltonian was quartic in the original bosons. So, this is basically the fundamental property which makes this so called bosonization technique useful.

So, you might think that why am I calling this bosonization. Everything is still a Boson. bosonisation means you are turning something that is not a boson into a boson, but the bs were bosons, the ds are bosons. So, there is nothing mean. So, in other words, in this context bosonization means turning one kind of bosons into other kinds of bosons.

But later on you will find that the more interesting version of bosonization is to turn objects which are fermions into bosons which is really remarkable. But I will not be fully successful in doing that. I will be success I will be well let me get there. But bottom line is that for Bosons it is an exact thing that, this non-local correspondence immediately gives you a canonical bosons, yeah.

So, now the question is that what should I do with this. Why is this useful? See, this is useful so long as I am able to express some general operator b k, b k dash, in terms of these ds. So, if I am able to write; in other words, if I am able to invert this I want to invert. So, if I can invert this, then write b k dagger in terms of these ds then I am likely to be able to use it. Because after all my Hamiltonians are of this sort. So in fact, it so happens that you can do that and this is how you do that.

So, I invite you to verify this. All you have to do is insert this formula here and show that it is an identity. So, you have b dagger k plus q by 2 b k minus q by 2 is an exact result. This is an exact result. See it is if both are 0, it is N 0. If one of them is 0, it is either d or d dagger, but if none of them are 0, yeah. So, if one of them is 0, anyway by definition this is supposed to be 0. So, if one of them is 0, it is either d or d dagger. And if none of the k plus q 2 or k minus q, none of them are 0, then this is same as this.

These two are same which is quite amazing. So, this is something you have to verify on your own. And N 0 can also just like N 0 can be written like this it can also be written like this. So, the N 0 original definition involved the bs, but you can also write the N 0 in terms of the ds, where N is the total number of particles which is fixed. So, N 0 is the number of bosons in the condensate, yeah.

So, so in other words, any Hamiltonian which involve involves the bs because they are always necessarily going to involve b dagger bs sort of thing. So, you can always insert this formula into your Hamiltonian and then start studying it. So, in fact, you see there is some sense in which; so, if you have a situation where your condensate is very large. So, if your N 0 is very large, you can suspect that this is the dominant term, this is the next dominant term, this is the sub dominant term. So, you might as well decide to work with this.

In fact, if you approximate b dagger b with this, you are essentially doing what is called Bogoliubov's theory, so which is well known in solid state physics. So, that is effectively saying that the condensate is very large. So, you are looking at small excitation. So, small number of particles can get excited from the condensate and small number can which are already excited can return to the condensate. So, there can be small fluctuations of the condensate.

So, if ignoring this will amount to studying Bogoliubov's theory. So, Bogoliubov's theory basically allows you to, so this becomes exactly solvable in that limit because N 0 is very large you can treat it as a number. So, its fluctuations are small. So, you ignore its fluctuations, and this theory becomes exactly solvable and because there are b. Remember the bs were exact bosons, but the ds are also exact bosons. So, and the your Hamiltonian is purely quadratic in the ds. So, it is exactly solvable.

So, the Bogoliubov's theory gives you a; so, basically it gives you all the eigen values of the of the excitation and so on. So, you will get the Bogoliubov's spectrum and so on. So, that is the interesting reason why we decide to rewrite properties of interacting quantum particles. Not in terms of creation and annihilation of particles themselves, but rather in terms of creation and annihilation of particle hole pairs because that is how the Hamiltonian in all the condensed matter problems manifest themselves.

They manifest themselves as particle hole pair operators. So, you do not have many situations where the number of particles in your system is not conserved a priori. So, yeah, so it is basically a Hamiltonian conserves the number of particles most of the time.

So, the question is how would you deal with this? So, now, the question is how would you generalize this to fermions? You see the or a the bs are all bosons.

So, if they are bosons, it is really fortunate that you can rewrite this in terms of other operators which are also bosons, which is quite a remarkable accident because it does not, it need not have been that way. In fact, the reason why it is that way is because this is something like a unitary operator. So, in fact, you can show that this behaves like a unitary operator. So, this is just a unitary operator. So, it is just unitary operator times b is d. So, that is clearly. So, whatever commutation rule b obeys d also obeys.

So, bottom line is that this is very you know it is like suspiciously easy, and it did not have to be that way that there is no reason why this should have been exact bosons. And what is even more surprising is that if you invert this corresponding the pondance, there is no reason why b dagger b should be writable in this rather simple way. So, there is no reason why there should not have been you know infinitely many terms after this plus dot dot dot dot dot, but there is not. This is all there is to it.

So, you see, so these are the two surprising aspects of this sort of transformation. So, on the one hand these are exact bosons. But then it did not have to be that way. And then secondly, when you invert this correspondence and write the number conserving products, you get just a finite number of terms, ok. So, the next important thing that well I think I have not done it here, but it is worth pointing out and that, is there a possibility you can write just the b itself in terms of the ds.

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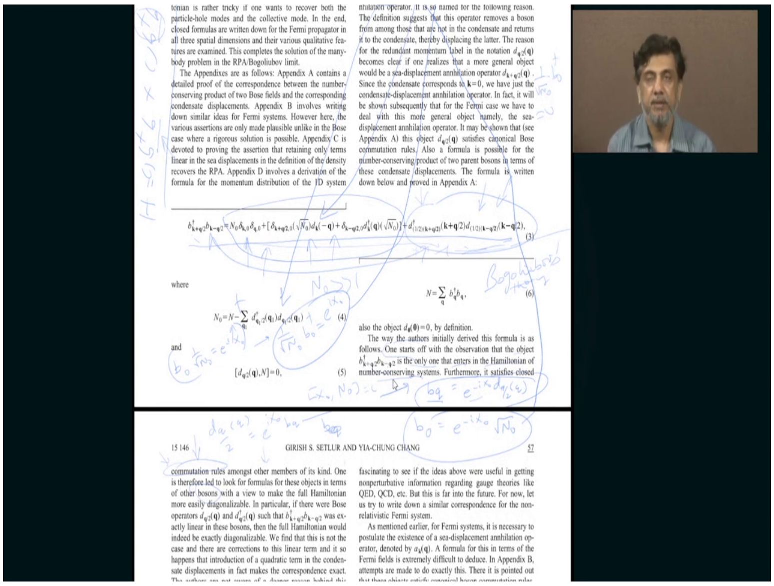

So, is there a way I can write b itself? So, fortunately the answer is yes? Because you see remember you can always write b 0 as e raise to minus X 0 times square root of N 0 see where X 0 N 0 is i. So, basically, X 0 is canonically conjugate to N 0. So, this is called density phase transformation. So, in fact, you can convince yourself that this basically because of this, so if you use this idea and you insert it here, so you insert it here. So, you can write this operator as, so you see this is nothing but b 0 1 by square root of N 0 is e raise to minus i X naught.

So, if I take dagger on both sides you get if I take dagger on both sides this becomes 1 by square root of N 0 b 0 dagger equals e raise to plus i X 0. So, now, if I insert this here, so if I insert it there, so this becomes e raise to plus i X 0 times b q. So, therefore, you see it is nice to know that b q itself can be written as e raise to minus i X naught into d q by 2 into q, I mean bracket q. So, you see b q, just the annihilation operator itself has a formula in terms of the ds, provided you also invoke this one. So, this is the canonical conjugate to your the number of bosons in the condensate.

So, it is nice to know you can do things like this. But the really important useful version of this is when the bs that I am talking about here are not bosons, but the original particles of bermions. But I still want the number conserving products to be actually bosons. So, you might be wondering why that is. Because, after all you see if bs are bosons, then d being boson is not that surprising. But what I am trying to imply is that if bs are fermions, ds are still some version of ds which I am going to define later. They will still be bosons or I want them to be bosons.

But the question is why do I expect such a eventuality or a possibility? Why do I think that number conserving products of fermions should have anything to do with bosons? So, the reason is given in this sentence here. One starts off with the observation that the object b dagger b with different momentum levels is the only one that enters in the Hamiltonian of number conserving systems.

Furthermore, it satisfies closed commutation rules. So, regardless of whether the bs are bosons or fermions, the number conserving products of these operators away closed commutation rules among other members of its kind. One is therefore, led to look for formulas for these objects in terms of other bosons, not see because they obey closed commutation rules.

So, the number conserving products of two operators see whether regardless of whether the original underlying particles of bosons or fermions. The number conserving products of those objects will obey closed commutation rules, even though the original annihilation operators may have obeyed anti commutation rules with their creation and annihilation counterparts.

So, even though they may have obeyed anti-commutation or commutation, the number conserving products will obey commutation rules among other members of its kind, with other members of its kind. So, therefore, we are led to look for formulas for these objects in terms of other bosons because they obey commutation rules rather than anti-commutation rules. So, that is the that is the motivation for looking for a Fermionic version of this correspondence which is incredibly hard to deduce.

Well, this is already hard if you think about it is. I mean once I explain it to you it does not seem that hard, but it took me a long time to figure this out. And the point is that the Fermionic version I figured out after a gap of several decades, maybe a yeah almost a decade, from here to took me that long. So, maybe even more than a decade.

And in fact, it took me a long time to even understand that rigorous Fermionic analog of this might be needed because I was able to circumvent, in fact, as I have explained in this paper and my later papers that a lot of physics can be extracted even by circumventing these technical issues. And so, that kind of put the mathematically rigorous analysis of these transformations on the back burner. So, for a long time I did not do it.

So, it is only when referees of various journals insisted, started thinking about it. And finally, I did it and its part of chapter 12 of my of that textbook that you are using right now. So, it is already there in print. Namely, it is there in my text book in chapter 12. So, that is the Fermionic version of whatever I have just described which is only to be found in this paper that I wrote in 1997, as part of my PhD thesis.

Next, I am going to explain to you the Fermionic version of whatever I have explained and then I will explain to you why that is so incredibly interesting and important. So, and that would conclude this MOOCs course on Dynamics of Classical and Quantum Fields. So, this is the most advanced topic in this course. And it has a kind of a research flavor.

So, a lot of the topics, I mean a lot of questions that I am going to pose in the very last lecture which is the next one, will have no easy answers. And I strongly urge those of you who want to specialize in quantum field theory and many body theory and so on to give it a serious thought and see if you can contribute to the research literature by thinking about those issues, ok. I am going to stop now. So, let us meet for the final lecture.