Dynamics of Classical and Quantum Fields: An Introduction

Prof. Girish S. Setlur
Department of Physics
Indian Institute of Technology, Guwahati

Coherent State Path Integrals
Lecture - 46
Fermionic Coherent States

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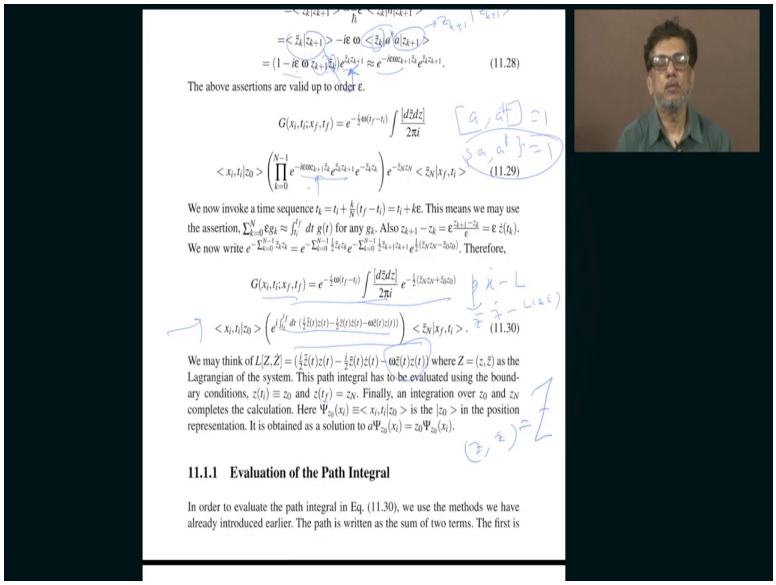

So, let us continue our discussion of Coherent State Path Integrals. So, if you remember what is coherent state path integrals, basically it is about wanting to study quantum mechanical systems using Lagrangians. But rather than think of the Lagrangian as involving you know the usual generalized coordinates like the position and the velocity that is q and q dot that is how you normally think of your classical Lagrangian.

Basically quantum mechanics can be rederived or basic it can be extracted from a classical Lagrangian by saying that you see not all paths are. So, in other words usually what happens is that, if somebody tells you the classical Lagrangian that is L of Q comma Q dot; then you know that if there is one path which obeys the Euler Lagrange equations. So, that is precisely the classical path; but a quantum particle is not going to always select the classical path.

So, what you do to study quantum mechanics is using Lagrangians is you say that, you see all paths are allowed; but each path comes with some weight and that weight is basically proportional to e raise to i by h bar times the action, the action is basically the time integral of the Lagrangian, Lagrangian integrated over time from some initial to final time. So, basically the weight itself is a complex number of unit modulus.

So, that is the funny thing about the path integral approach to quantum mechanics; basically it does not tell us that, in fact superficially it seems like all paths are equally probable because you see.

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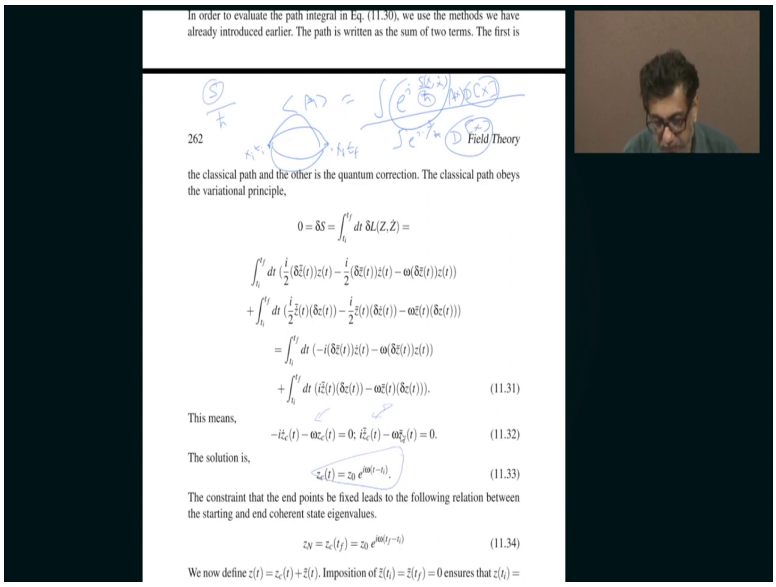
In order to evaluate the path integral in Eq. (11.30), we use the methods we have already introduced earlier. The path is written as the sum of two terms. The first is

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the classical path and the other is the quantum correction. The classical path obeys the variational principle,

$$0 = \delta S = \int_{t_i}^{t_f} dt\, \delta L(Z, \dot{Z}) =$$

$$\int_{t_i}^{t_f} dt \left(\frac{i}{2}(\delta\dot{\bar{z}}(t))z(t) - \frac{i}{2}(\delta\bar{z}(t))\dot{z}(t) - \omega(\delta\bar{z}(t))z(t)\right)$$

$$+ \int_{t_i}^{t_f} dt \left(\frac{i}{2}\dot{\bar{z}}(t)(\delta z(t)) - \frac{i}{2}\bar{z}(t)(\delta\dot{z}(t)) - \omega\bar{z}(t)(\delta z(t))\right)$$

$$= \int_{t_i}^{t_f} dt \left(-i(\delta\dot{\bar{z}}(t))z(t) - \omega(\delta\bar{z}(t))z(t)\right)$$

$$+ \int_{t_i}^{t_f} dt \left(i\dot{\bar{z}}(t)(\delta z(t)) - \omega\bar{z}(t)(\delta z(t))\right). \qquad (11.31)$$

This means,

$$-i\dot{z}_c(t) - \omega z_c(t) = 0;\ i\dot{\bar{z}}_c(t) - \omega\bar{z}_q(t) = 0. \qquad (11.32)$$

The solution is,

$$z_c(t) = z_0\, e^{i\omega(t-t_i)}. \qquad (11.33)$$

The constraint that the end points be fixed leads to the following relation between the starting and end coherent state eigenvalues.

$$z_N = z_c(t_f) = z_0\, e^{i\omega(t_f - t_i)} \qquad (11.34)$$

We now define $z(t) = z_c(t) + \bar{z}(t)$. Imposition of $\bar{z}(t_i) = \bar{z}(t_f) = 0$ ensures that $z(t_i) =$

So, basically what path integral approach says is that, if you want to find the average of any expectation value of anything; what you have to do is, basically rewrite this in terms of the action and divide by the. So, you integrate over all paths and then you divide by the appropriate normalization. So, this presumably depends on the path. So, this will depend on x x dot whatever it is; it can depend on x x dot; but bottom line is you are integrating over all paths, all x's.

So, if you have some x 1 t x i t i is your initial and x f t f as your final, then there will be a whole bunch of possible paths. So, what this says is that, all of them are allowed in quantum mechanics; see unlike in classical mechanics only one path is allowed, all other

paths are strictly forbidden, only one path is allowed. But in quantum mechanics all paths are allowed, but each comes with a weight.

And the funny thing about this weight is it is a complex number of unit modulus; so that means that if you just look at the absolute value of this weight, it is 1. So, implying therefore, that it seems like the probability of the particle choosing any path is pretty much the same, which seems rather counterintuitive and paradoxical. Because we expect classical paths to be slightly more favored even in a quantum system.

But the reason why that is you know, I mean the reason why this approach is not wrong is because you see that, your intuition that the classical path has to be slightly favored is recovered by realization that the Planck's constant is the one that is very small. So, when it is very small, the this is a phase term it is going to oscillate rapidly and most of the paths kind of they cancel themselves out; because as you move along the path, the phase oscillates rapidly because of the smallness of the Planck's constant. So, the only paths that are likely to contribute substantially are those which minimize the action.

So, you see because h is already minimized and it is in the denominator and you had better minimize the numerator also, otherwise the ratio is going to be infinite. And if it is infinite e raise to i times infinity kind of rapidly fluctuates between plus and minus 1 and averages out to 0. So, if you want to prevent that from happening, you should ensure that the Planck's constant which is very small will also imply that the action is as minimum as possible.

So, to try to keep pace with the catch up with the smallness of Planck's constant. So, bottom line is that, that is how you recover classical physics from this path integral approach to quantum mechanics. So, now, this coherent state path integral is not about studying this path integral using this approach, where this approach is about integrating our paths and your action depends on the position and generalized position and the generalized velocity.

But I want to study, I want to write down a Lagrangian not in terms of the usual generalized position, generalized velocities; but I want to write it in terms of the classical analog of creation and annihilation operators. Because see generalized position,

generalized velocities are classical versions, I mean the classical variables. But however, you see in quantum mechanics when you rewrite your Hamiltonian in terms of creation and annihilation operators, you get A and A dagger, which then these are complex.

So, if you want to now study the same system, which is now expressed in terms of this creation and annihilation operators which are complex operators; then you are forced to invoke the notion of a complex eigenvalue of these operators, which you can then use to construct a Lagrangian in terms of the eigenvalues of A and A dagger. So, which is precisely what we have done here, this is what that is.

This is basically the e raise to i by h bar; again I keep forgetting the h bar that is there. So, it is e raise to i by h bar integral of the action and this is my action for the harmonic oscillator in terms of the coherent state in terms of the eigenvalues of the creation and annihilation operators. So, now, as usual you see this is if I wanted to calculate the expectation value of some, I mean some Green's function; but let us only focus on this ok, because I have to integrate over end points and all that, so I have done, but let us focus only on this.

So, now, if I want to evaluate a path integral how did we do it in the case of harmonic oscillator when we were studying it in the x and x dot language; that means in terms of the original position and generalized velocity language. So, the way we did that is we looked at the classical solution of the extremum the path which extremis the action, so that we can then go ahead and expand our path around that extremum.

And so, in order to find the extremum we look at the variation of the action and set it equal to 0 and following very standard methods, we get these equations. So, this is the classical path that comes out.

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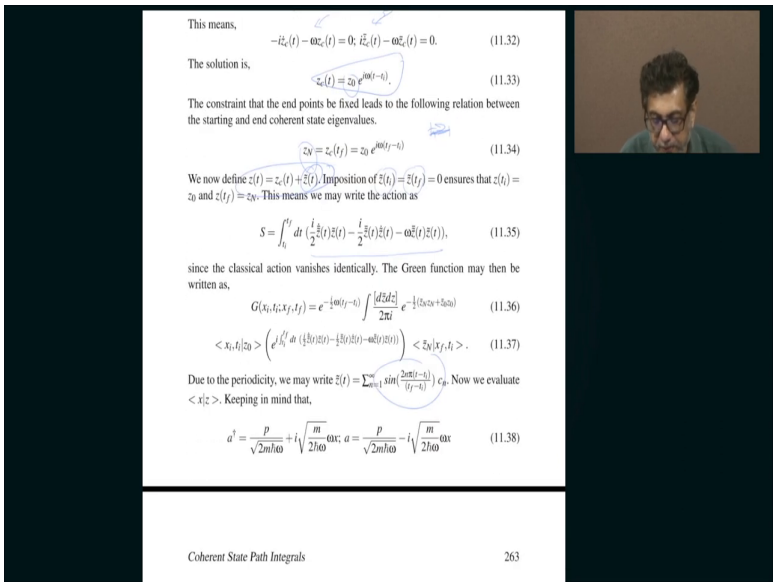

Now, you see we have to also ensure that the; so if the starting path is I mean the z value when it starts off at t equal to t i z 0, then z when it at the end point t equal to t f we postulate that the z value is z N. So, that therefore, the z N and z 0 are related in this way, they are related to t f and t y t I; again I keep listen I mean I have put h bar equals 1, otherwise it is h bar omega into, oh sorry you know now it is still omega, omega is frequency t is time that is fine ok, here it is fine, but there I should have introduced it.

Now, but nevertheless bottom line is. So, now, what you do is, you re rewrite your path as the classical path plus deviation from the classical path. So, this is this z tilde basically deviation from the classical path and because all paths start and end at the same point, the deviation should become 0 at the starting and ending points, ok. So, now, if you substitute that you will see that, all cross terms drop out and you will get a classical answer multiplied by this quantum fluctuation, ok.

So, the point is that you see the advantage of you might think why do we always split it up this way; why do we split up the path in terms of a classical path plus a quantum fluctuation? See the reason why we do that is because the quantum fluctuation now obeys periodicity; because the original path there is no periodicity, at t equal to t i z 0, at t equal to t f it is some unrelated z N.

So, there is no connection between z 0 and z N; but however by construction z tilde is 0 at t plus t i, it is also 0 at t equal to t f. So, that implies that this periodic function of its argument, where the period is basically t f minus t i. So, that is what I have done here. So, I have constructed a Taylor series, where the function vanishes at both t i and t f and it is periodic with respect to time with period t f minus t i. So, this is the most general way of doing that. So, now, you can go ahead and evaluate this particular integral, ok.

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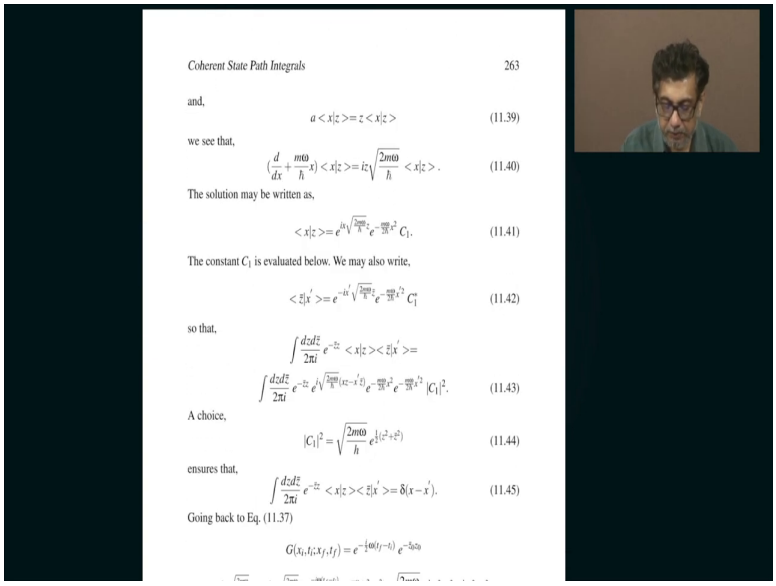

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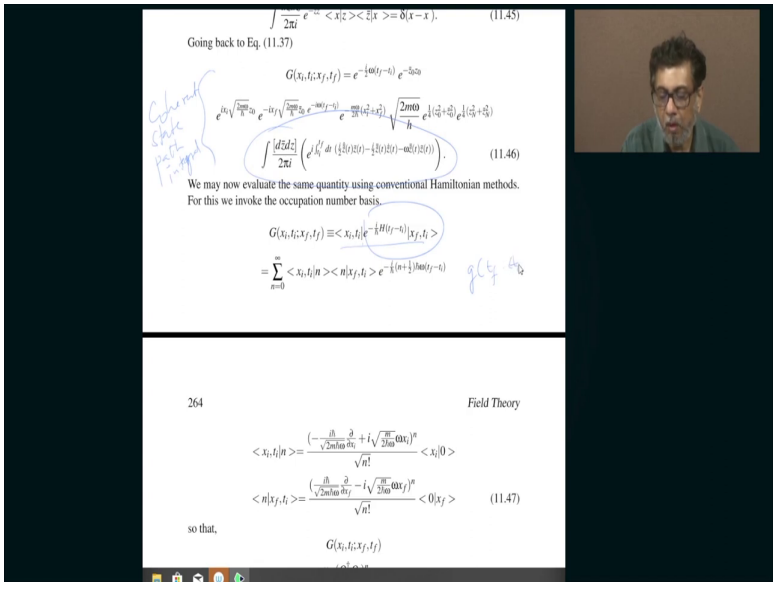

So, if you evaluate this particular integral; look I have used conventional methods now to evaluate this path integral. So, I have just pointed out that you could do it this way. So, rather I have evaluated the rest of it; I have evaluated this, this, this. So, you see when I do that, I end up with this. So, this is still there, this is the integration over quantum fluctuations; the rest of it has been evaluated, yeah because finally, we are going to be able to compare.

So, this is the coherent state path integral, this is the coherent state path integral version of the Green's function. So, it is basically is the coherent state path integral version of the Green's function; but there is also the Hamiltonian approach to this calculating the Green's function if you recall. This is the traditional Heisenberg picture; well in this case it is yeah is the Heisenberg picture. Well, you can think of it as, it is actually Schrodinger picture; because see the states, this is a state which evolves with time. So, it has evolved from t i to t f.

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$$< n|x_f,t_f >= \frac{(\frac{i\hbar}{\sqrt{2m\hbar\omega}}\frac{\partial}{\partial x_f} - i\sqrt{\frac{m}{2\hbar\omega}}\omega x_f)^n}{\sqrt{n!}} < 0|x_f > \quad (11.47)$$

so that,

$$G(x_i,t_i;x_f,t_f)$$

$$= e^{-\frac{1}{2}\omega(t_f-t_i)} \sum_{n=0}^{\infty} \frac{(Q_f^{\dagger}Q_i)^n}{n!} < x_i|0 >< 0|x_f >$$

$$= e^{-\frac{1}{2}\omega(t_f-t_i)}\, exp[Q_f^{\dagger}Q_i] < x_i|0 >< 0|x_f >$$

$$= e^{-\frac{1}{2}\omega(t_f-t_i)}\, exp[Q_f^{\dagger}Q_i] < x_i|0 >< 0|x_f > \quad (11.48)$$

where,

$$Q_i = e^{i\omega t_i}(-\frac{i\hbar}{\sqrt{2m\hbar\omega}}\frac{\partial}{\partial x_i} + i\sqrt{\frac{m}{2\hbar\omega}}\omega x_i) \quad (11.49)$$

$$Q_f^{\dagger} = e^{-i\omega t_f}(\frac{i\hbar}{\sqrt{2m\hbar\omega}}\frac{\partial}{\partial x_f} - i\sqrt{\frac{m}{2\hbar\omega}}\omega x_f). \quad (11.50)$$

We may use the so-called Hubbard-Stratanovich transformation to write,

$$e^{Q_f^{\dagger}Q_i} = \int \frac{d\alpha d\bar{\alpha}}{2\pi i} e^{-\bar{\alpha}\alpha} e^{\alpha Q_f^{\dagger}} e^{\beta Q_i} \quad (11.51)$$

$$G(x_i,t_i;x_f,t_f) =$$

$$e^{-\frac{1}{2}\omega(t_f-t_i)} \int \frac{d\alpha d\bar{\alpha}}{2\pi i} e^{-\bar{\alpha}\alpha}$$

$$e^{\beta e^{i\omega t_i}i\sqrt{\frac{m}{2\hbar\omega}}\omega x_i - \bar{\alpha}e^{i\omega t_i}\frac{i\hbar}{\sqrt{2m\hbar\omega}}\frac{\partial}{\partial x_i}} \Psi_0(x_i)$$

$$e^{-\bar{\alpha}e^{-i\omega t_f}i\sqrt{\frac{m}{2\hbar\omega}}\omega x_f + \alpha e^{-i\omega t_f}\frac{i\hbar}{\sqrt{2m\hbar\omega}}\frac{\partial}{\partial x_f}} \Psi_0^*(x_f). \quad (11.52)$$

Here, $\Psi_0(x) \equiv < x|0 >$ is the ground state wavefunction of the harmonic oscillator in the position space. We now use the Zassenhaus formula to write $e^{cx+c'\frac{d}{dx}} = e^{cx}e^{c'\frac{d}{dx}}e^{\frac{1}{2}cc'}$. This means, $e^{cx+c'\frac{d}{dx}}f(x) = e^{cx}f(x+c')e^{\frac{1}{2}cc'}$. Thus we may write,

So, you can go ahead and evaluate that, ok.

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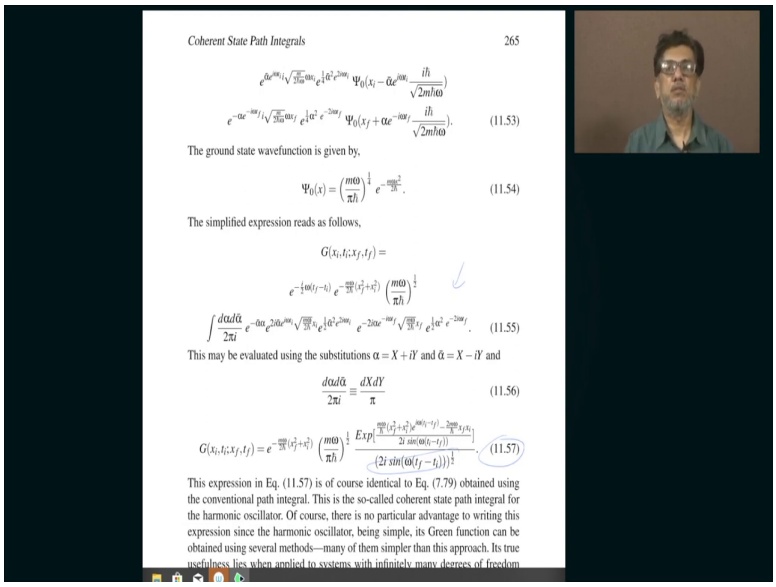

So, if you evaluate that you will end up with this result; I mean it is a little bit of a tedious algebra, ok. So, you will end up with this result, ok. So, now, this is the, this is the result that you have obtained and this is equal to whatever we got way back. So, basically what this approach tells you is, it tells you how to evaluate. So, remember that this was some constant.

So, this was some constant which was unknown; it was one of those G t f minus t i types. So, that is what it was. So, bottom line is that yeah it was something like that. So, that is what it was g t f minus t i and the rest of it gets evaluated; yeah, so it is a lot of tedious algebra and you have to go through it. So, most of the interesting dependencies are already contained here, this is that G of t bar t f minus t i, ok.

So, bottom line is that with some effort you can convince yourself that the coherent state path integral approach for the harmonic oscillator gives you the same Green's function as you would get if you did the conventional path integral for the quantum harmonic oscillator using position, coordinates and generalized velocities and so on.

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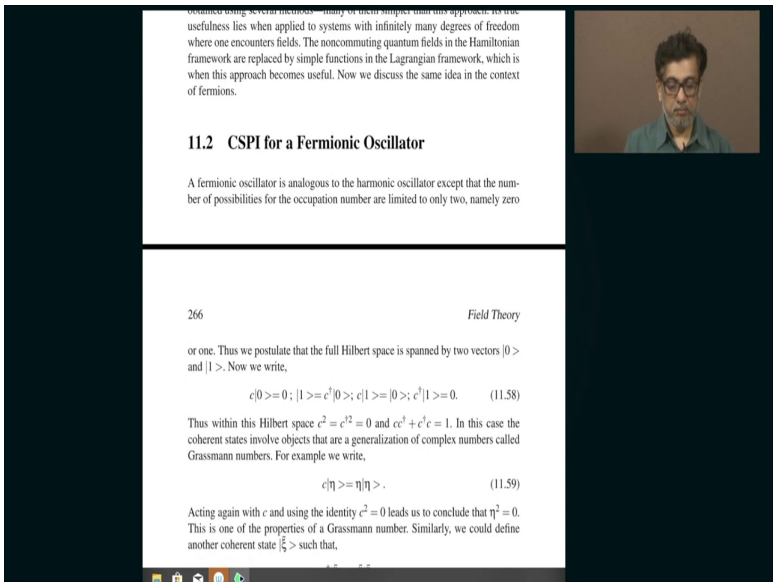

So, that is important and it is important for you to understand that; because you will see that in the end I am going to be able to generalize these two fields, because after all these the systems that we are studying of quantum fields. So, till now I have only introduced point particles, so that is one must tied to one spring and I am studying that quantum mechanically using a whole bunch of approaches.

You know whether it is starting from the original Schrodingers approach of wave functions Hermite polynomials then writing in terms of creation annihilation and studying in terms of those ladder operators and so on and so forth. Or studying the quantum harmonic oscillator Green's function using the conventional path integral which is, which involves position generalized position generalized velocity, in this case x x t and x dot t.

So, and lastly I studied the classical counterpart of A dagger A plus half into h bar omega. So, that is basically the coherent state path integral approach. So, you construct the adjoint of the Hamiltonian is Lagrangian and the Lagrangian will be basically classical; because it will be classical because Z and Z dash are complex numbers which are commuting with each other. And you integrate over all possible such complex number paths and then you get the same Green's function as you would if you had done conventional things.

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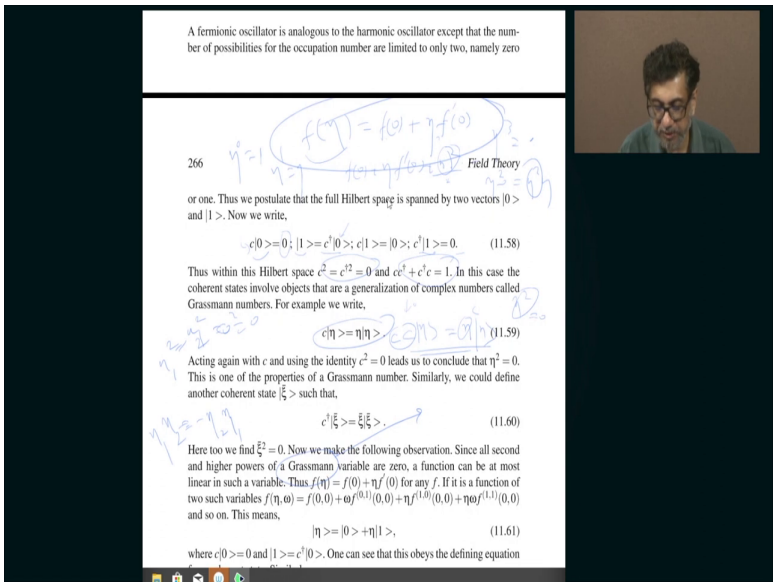

So, now, the question is how do you. So, this was all the harmonic oscillator a dagger a plus half; so that means the commutator of a and a dagger is 1, but then the commutator is 1 means basically you are studying bosons. But in nature you know that there are other types of particles called fermions. So, you should be able to study fermions also; because after all you know electrons are fermions, all these quarks leptons these are all fermions and the bosons are all the force carriers like photons, gluons, W and Z bosons and whatever. So, those are all bosons.

So, typically bosons of force carriers and material particles of fermions and since both exist in nature we should be able to study both quantum mechanically. And both are thought of as excitation of some field. So, just like quarks and leptons are excitations of a suitable matter field, quark field or lepton field.

So, similarly photons are excitation of the electromagnetic field and W Z bosons are excitations of the electro weak field. Basically, you know Weinberg, Salam and Glashow unified the big force with electromagnetic force. So, that is called the electro weak theory.

So, the fields there correspond to basically excitation of the electro weak field, so that could be either photons or the W boson or Z bosons. So, then you have strong forces that

which is responsible for holding the nucleus together. So, the strong nuclear force the material particles of quarks, but the force carriers are gluons. So, basically you need a theory which describes not only gluons like a dagger a would correspond to; because gluons bosonic particles, so they correspond to a dagger a type of thing.

But fermions will be also important, because you see the material particles of fermions. So, we should be able to do, we know how to do the conventional type of quantum mechanics using creation and annihilation operator for fermions that is quite easy; because after all in if you have a finite system, then if you have a suppose you have a state, you want to create a fermion, you cannot create one more unless that fermion comes attached with some label like spin up or spin down.

So, if it does not come attached with any label, then you can create either one fermion in that level or no fermions, yeah. So, the question is the following. So, how do you study fermions using path integrals? That is an important question; because you see in order to study anything using path integrals, you should first construct a classical Lagrangian. So, even though you are studying quantum mechanics, you have to first construct a classical Lagrangian and then you construct the classical action.

But then you do not, then that is where you stop, then you do not go ahead and write down the Euler Lagrange equation; rather what you do is you insert the classical action in the exponent of some weight, so that means you construct a weight of the form e raise to i by h bar times the action. And that weight is what tells you how much weight a certain quantum mechanical path; basically how much weight a path has when a quantum particle traverses along that path.

So, that weight is basically proportional to e raise to i by h power s. So, therefore, in order to do path integral approach to quantum mechanics, it appears that you really need a classical Lagrangian. But the funny question now is, how can you construct the classical theory of a fermion; because that is what it seems to imply. Because you see there is a classical theory of a mass tied to a spring this is basically the classical harmonic oscillator.

Because the quantum particles or the excitations of mass tied to a spring manifest themselves as bosons, because a and a dagger they have the commutative property, that is commutator of A and A dagger is 1. However, there are particles in nature that do not manifest themselves as boson, they manifest themselves as fermions.

So, now the question is, are there objects in nature that correspond to classical analogs of fermions? Because you know that there is a mass and there is a spring and you know tie them together that is what classical analog of a boson would look like; meaning in some sense the boson comes out by quantizing a mass tied per spring. So, the question you can naturally ask is what classical system when quantized gives you fermions?

Is it mass tied to a spring? No, it will give you bosons; then what tied to what we will when quantized give you a fermion? So, the answer to that is basically nothing that, there is no classical analog, at least that is the conventional answer. So, there is no classical analog of a fermion; so that means there is nothing which when quantized gives you a fermion. The fermion is already quantum, basically there is usually you are given the impression that quantum mechanics is kind of not possible unless you have a classical description to begin with.

That is how you are taught quantum mechanics and of course, with good reason; because most of the systems do have classical analogs. So, it makes perfect sense like the electromagnetic field; it has a classical analog and with the classical Maxwell equations and then when you quantize it, you get photons. So, the question is now which classical set of equations which when quantized will give you something like an electron; which is a quantum particle?

So, the answer is unfortunately nothing, there is no classical system which when quantized will give you an electron. An electron is already quantum and there is nothing classical about it, that there is no classical version of an electron; there is a classical version of a boson, which is must tied to a spring, there is no classical version of a fermion.

So, we have to learn how to, so but then. So, now, we are stuck; because we need a classical Lagrangian to do path integrals, but then there is no classical system as such.

So, we have to you know cook up something. So, that cooking up something will involve using some very strange mathematics. So, that strange mathematics tells us that the eigen values of the creation operator of say if suppose c is your annihilation of a fermion.

So, suppose you have a. So, if you have a state, you can either have one fermion or no fermion; because it does not come with any other label. So, if I take that state which does not contain any fermion and I try to annihilate it, I am going to get 0. Now, I can construct a state with one fermion by acting the creation operator on that state with no fermions; then if I try to annihilate that, I get a state with no fermions

But more importantly if I try to create one more fermion in a state that already has a fermion, I get 0 immediately, because of Pauli principle. So, you see now what we want is this Pauli principle is the anti commutation rule. So, the point is what we want to do is, we want to create. So, in other words we want to construct the eigen states of the fermionic annihilation operator.

So, because the eigen states and eigen values of the bosonic annihilation operators are simply complex numbers, any complex number can be. So, basically you pick a complex number; you can construct a state labelled by that complex number, which will automatically be by suitable construction an eigenstate of the bosonic annihilation operator. But now if you ask the question, can I do that for a fermionic annihilation operator?

Well, formally you can always write this; because you can say let eta be that eigenvalue which corresponds to the eigen value of the fermionic annihilation operator and this is what it is. But now you see keep in mind that c squared is 0; that is if you try to annihilate twice, you will get 0. So, now, suppose you act this supposed eigen state equation, eigen value equation by another annihilation.

What will you get? This will become c squared which is 0, but then this will give me a c times eta, which will give me eta squared; because c times eta is another eta times the state eta, so the eigenvalue times the state. So, there is already the eigen value there. So, if I multiply the two eigen values, I get eta squared; but this is 0, c c is 0, so eta squared should also be 0, because the state itself is not 0.

So, what that means, is these eigen values are nonzero; because obviously they are nonzero, they are supposed to label some non trivial eigenvalue, but their square is 0. So, obviously, they cannot be ordinary complex number numbers; because there is no ordinary complex number whose which is not 0, but whose square is 0. But in mathematics, there are objects that have these properties and they are called Grassmann numbers.

They are called Grassmann is name of some mathematicians. So, it is Grassmann variable Grassmann numbers. So, Grassmann numbers are numbers which have this property that if eta 1 is a Grassmann number and eta 2 is another Grassmann number; eta 1 into eta 2 is minus eta 2 into eta 1 and eta 1 squared equals eta 2 squared equals 0, ok. So, these are the properties of Grassmann numbers and moreover you can show that any function of some Grassmann number at most will involve just you know.

Just so if you do a Taylor series, see if you do a formal Taylor series what is this; is f 0 plus eta f dash 0 plus eta squared, but eta squared is 0, because eta is Grassmann number. But eta cubed is also 0; because eta square eta cubed is eta squared into eta, but eta squared is 0, so eta cubed is also 0. So, everything is 0, except eta to the power 0 which is 1 and eta to the power 1 which is eta and all higher powers are 0; if all higher powers are 0, any function of the Grassmann variable is linear. So, that makes some enormous simplification there.

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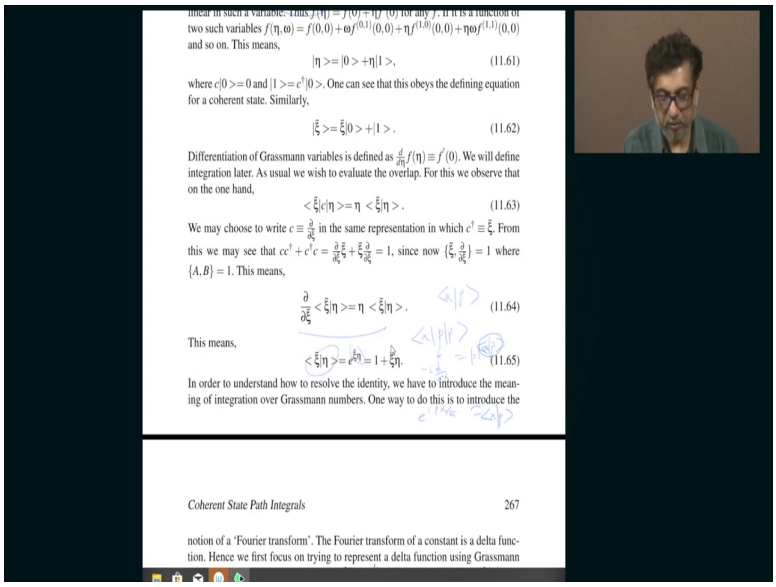

So, basically what will happen is that, you can now go ahead and construct coherent states with these types of properties. So, now, you can show that basically the Grassmann variables have these funny properties like this; because usually you think that you know if you use your usual approach like x and p type of thing you know that you see what is this is.

So, if you act p x p like this. So, on the one hand it is the eigen value p into x p, but on the other hand this is basically minus i h bar d by d x. So, therefore, you will get an equation. So, if you solve you will get p x p by h bar as your overlap. So, x overlap p is e raise to i p x by h bar. So, similarly here also you will get the overlap between this and this is something like that.

But then you see because these are Grassmann numbers; if you Taylor series only if zeroth order and first order term survive, all higher order terms are 0, because eta squared is 0 and eta dash squared is 0 etcetera, etcetera.

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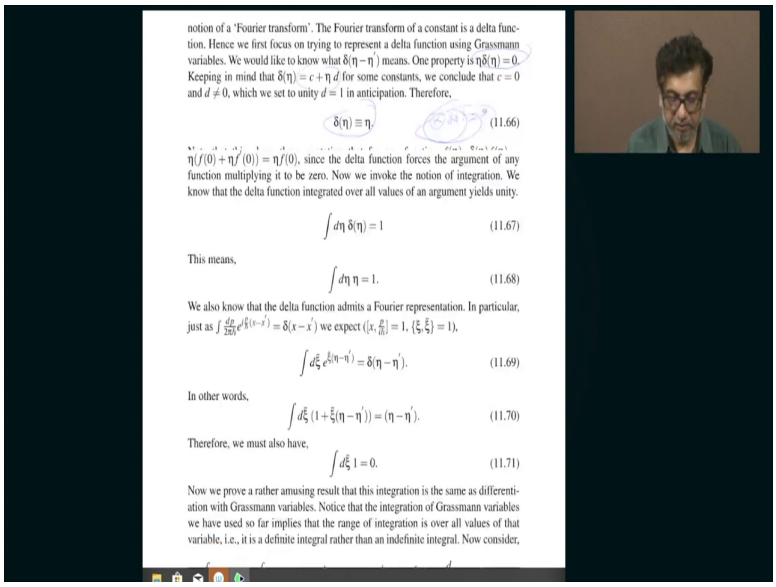

So, similarly the Dirac delta function has this funny property that; because we expect Dirac delta to be x delta x is 0, because you see the Dirac delta this whole thing is 0 if x is not 0, but if x is 0, it is still 0 because the coefficient is 0. So, the entire generalized function is 0. So, if you think of this x as now a Grassmann number, so now, you can see that because any function can only be written like this.

So, it is therefore, mandatory that the Dirac delta function is the Grassmann itself. So, and if you integrate over Dirac delta, you will get 1. So, integral of the Grassmann is 1. So, it is basically something like a definite integral.

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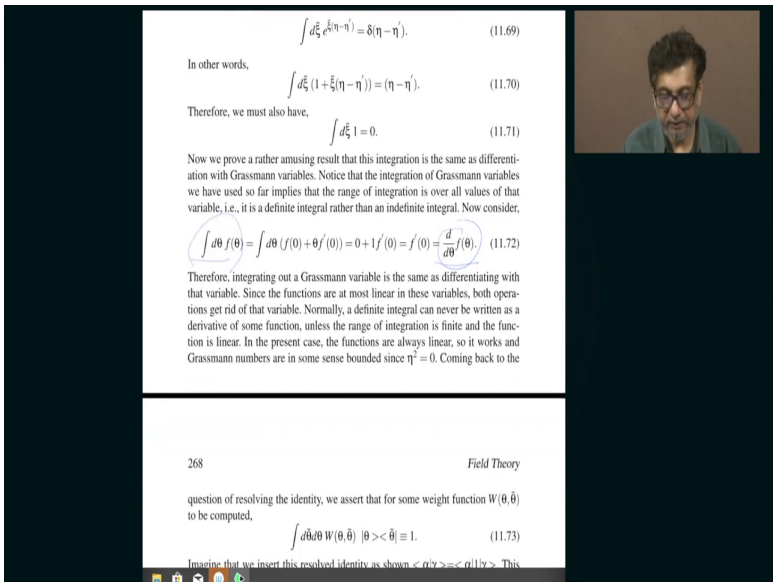

$$\int d\xi\, e^{\xi(\eta-\eta')} = \delta(\eta-\eta'). \quad (11.69)$$

In other words,

$$\int d\xi\, (1+\xi(\eta-\eta')) = (\eta-\eta'). \quad (11.70)$$

Therefore, we must also have,

$$\int d\xi\, 1 = 0. \quad (11.71)$$

Now we prove a rather amusing result that this integration is the same as differentiation with Grassmann variables. Notice that the integration of Grassmann variables we have used so far implies that the range of integration is over all values of that variable, i.e., it is a definite integral rather than an indefinite integral. Now consider,

$$\int d\theta\, f(\theta) = \int d\theta\, (f(0)+\theta f'(0)) = 0+1f'(0) = f'(0) = \frac{d}{d\theta} f(\theta). \quad (11.72)$$

Therefore, integrating out a Grassmann variable is the same as differentiating with that variable. Since the functions are at most linear in these variables, both operations get rid of that variable. Normally, a definite integral can never be written as a derivative of some function, unless the range of integration is finite and the function is linear. In the present case, the functions are always linear, so it works and Grassmann numbers are in some sense bounded since $\eta^2 = 0$. Coming back to the

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question of resolving the identity, we assert that for some weight function $W(\theta,\bar{\theta})$ to be computed,

$$\int d\bar{\theta} d\theta\, W(\theta,\bar{\theta})\ |\theta><\bar{\theta}| \equiv 1. \quad (11.73)$$

Imagine that we insert this resolved identity as shown $< q|\gamma> = < q|1|\gamma >$. This

So, like that you can construct many many such and the integration is same as, that is the most bizarre thing about Grassmann. So, the thing is that if you integrate; if you integrate the Grassmann variable, so this is some kind of a definite integral. So, it is the same as differentiating the; because once you differentiate this whole thing becomes a constant, because f theta is anyway linear in theta. So, if you differentiate, it becomes a constant.

So, definite integral of a function over the Grassmann variable is same as its derivative, which is something very hard to believe, but it is true

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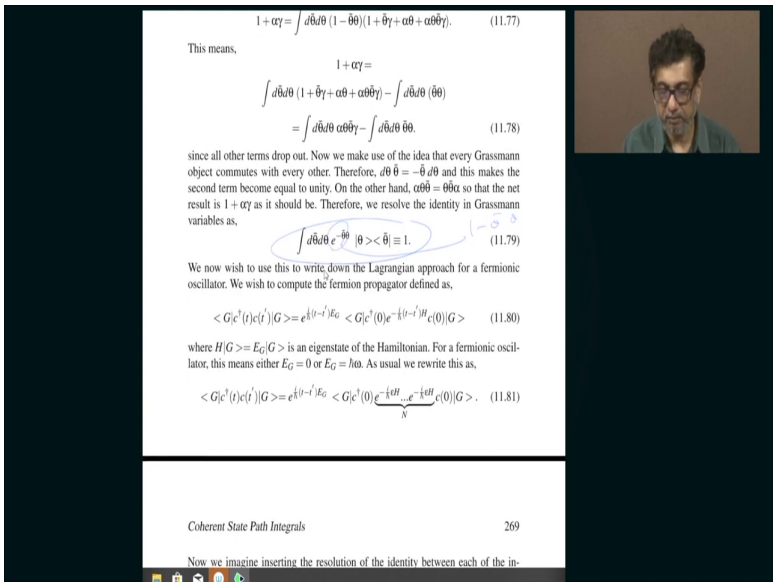

$$1+\alpha\gamma=\int d\bar{\theta}d\theta\,(1-\bar{\theta}\theta)(1+\bar{\theta}\gamma+\alpha\theta+\alpha\theta\bar{\theta}\gamma). \quad (11.77)$$

This means,

$$1+\alpha\gamma=$$

$$\int d\bar{\theta}d\theta\,(1+\bar{\theta}\gamma+\alpha\theta+\alpha\theta\bar{\theta}\gamma)-\int d\bar{\theta}d\theta\,(\bar{\theta}\theta)$$

$$=\int d\bar{\theta}d\theta\,\alpha\theta\bar{\theta}\gamma-\int d\bar{\theta}d\theta\,\bar{\theta}\theta. \quad (11.78)$$

since all other terms drop out. Now we make use of the idea that every Grassmann object commutes with every other. Therefore, $d\theta\,\bar{\theta}=-\bar{\theta}\,d\theta$ and this makes the second term become equal to unity. On the other hand, $\alpha\theta\bar{\theta}=\theta\bar{\theta}\alpha$ so that the net result is $1+\alpha\gamma$ as it should be. Therefore, we resolve the identity in Grassmann variables as,

$$\int d\bar{\theta}d\theta\,e^{-\bar{\theta}\theta}\,|\theta><\bar{\theta}|\equiv 1. \quad (11.79)$$

We now wish to use this to write down the Lagrangian approach for a fermionic oscillator. We wish to compute the fermion propagator defined as,

$$<G|c^{\dagger}(t)c(t')|G>=e^{\frac{i}{\hbar}(t-t')E_G}<G|c^{\dagger}(0)e^{-\frac{i}{\hbar}(t-t')H}c(0)|G> \quad (11.80)$$

where $H|G>=E_G|G>$ is an eigenstate of the Hamiltonian. For a fermionic oscillator, this means either $E_G=0$ or $E_G=\hbar\omega$. As usual we rewrite this as,

$$<G|c^{\dagger}(t)c(t')|G>=e^{\frac{i}{\hbar}(t-t')E_G}<G|c^{\dagger}(0)\underbrace{e^{-\frac{i}{\hbar}tH}\ldots e^{-\frac{i}{\hbar}tH}}_{N}c(0)|G>. \quad (11.81)$$

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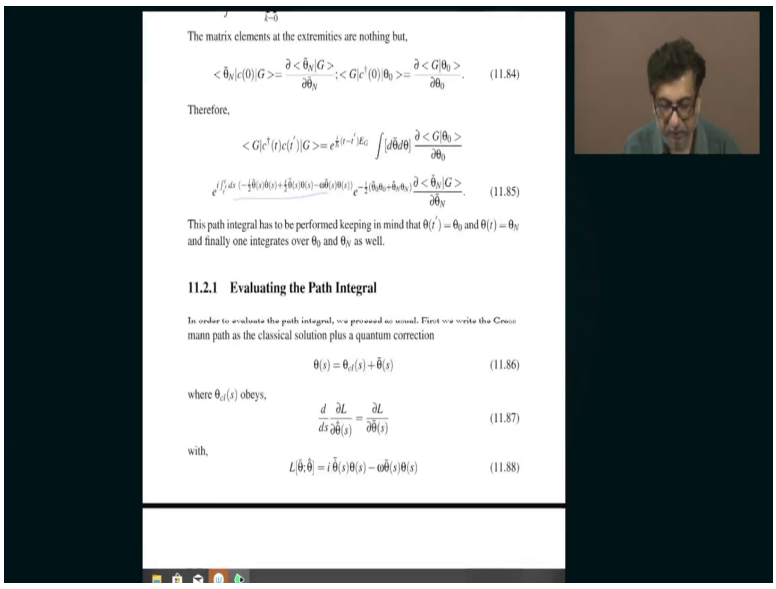
Now we imagine inserting the resolution of the identity between each of the in-



The matrix elements at the extremities are nothing but,

$$<\bar{\theta}_N|c(0)|G>=\frac{\partial<\bar{\theta}_N|G>}{\partial\bar{\theta}_N};<G|c^{\dagger}(0)|\theta_0>=\frac{\partial<G|\theta_0>}{\partial\theta_0}. \quad (11.84)$$

Therefore,

$$<G|c^{\dagger}(t)c(t')|G>=e^{\frac{i}{\hbar}(t-t')E_G}\int[d\bar{\theta}d\theta]\frac{\partial<G|\theta_0>}{\partial\theta_0}$$

$$e^{i\int_{t'}^{t}ds\,(-\frac{1}{2}\dot{\bar{\theta}}(s)\theta(s)+\frac{1}{2}\bar{\theta}(s)\dot{\theta}(s)-\omega\bar{\theta}(s)\theta(s))}e^{-\frac{1}{2}(\bar{\theta}_0\theta_0+\bar{\theta}_N\theta_N)}\frac{\partial<\bar{\theta}_N|G>}{\partial\theta_N}. \quad (11.85)$$

This path integral has to be performed keeping in mind that $\theta(t')=\theta_0$ and $\theta(t)=\theta_N$ and finally one integrates over θ_0 and θ_N as well.

11.2.1 Evaluating the Path Integral

In order to evaluate the path integral, we proceed as usual. First we write the Grassmann path as the classical solution plus a quantum correction

$$\theta(s)=\theta_{cl}(s)+\tilde{\theta}(s) \quad (11.86)$$

where $\theta_{cl}(s)$ obeys,

$$\frac{d}{ds}\frac{\partial L}{\partial\dot{\bar{\theta}}(s)}=\frac{\partial L}{\partial\bar{\theta}(s)} \quad (11.87)$$

with,

$$L[\bar{\theta},\dot{\theta}]=i\,\dot{\bar{\theta}}(s)\theta(s)-\omega\bar{\theta}(s)\theta(s) \quad (11.88)$$

So, now, you can just go ahead and so, just like case of bosonic coherent state; you have this over completeness, here also you will have similar over completeness. But keep in mind that this is very simple, this is same as 1 minus theta dash theta. So, it is not really, I mean it is overkill to write it like that. So, same procedure, we follow exactly the same procedure.

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And when you do you will end up some with something very similar, but except that now you have to evaluate the path integral.

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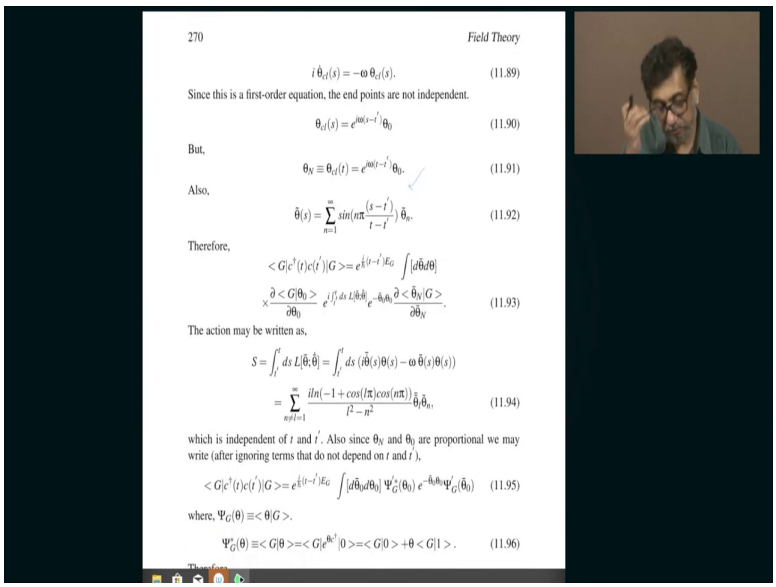

And even that is also very similar, ok.

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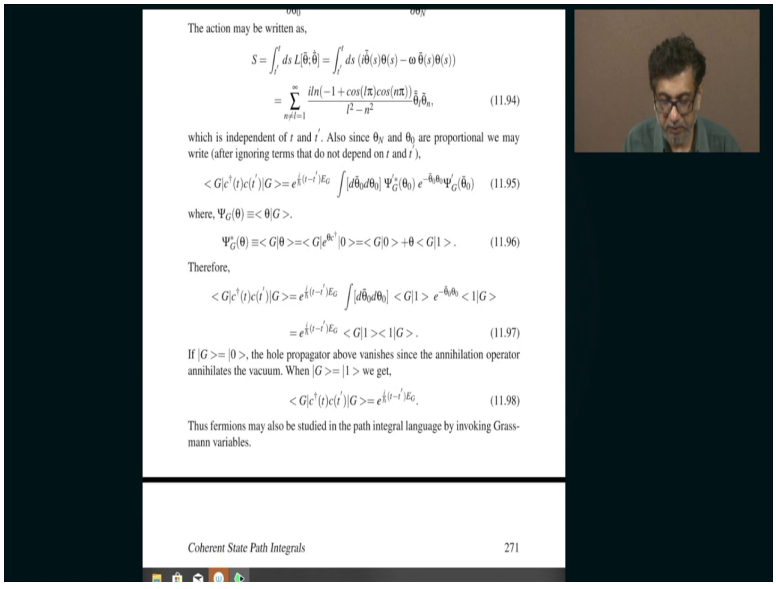

So, but when you evaluate it, you will get something much simpler than what you will get in the case of bosons. Because there you can have as many bosons as you want in a state, but fermions you can have either 0 or 1 and you will get a Green's function that is

incredibly simple like this. So, I will allow you to look through these steps; because I would not spend I probably would not even ask you these questions in the assignments, because it is important if you want to specialize in particle physics.

I think condensed matter people do not use Grassmann variables that much; but particle physics people use it quite a bit, at least the field theory crowd of particle physics.

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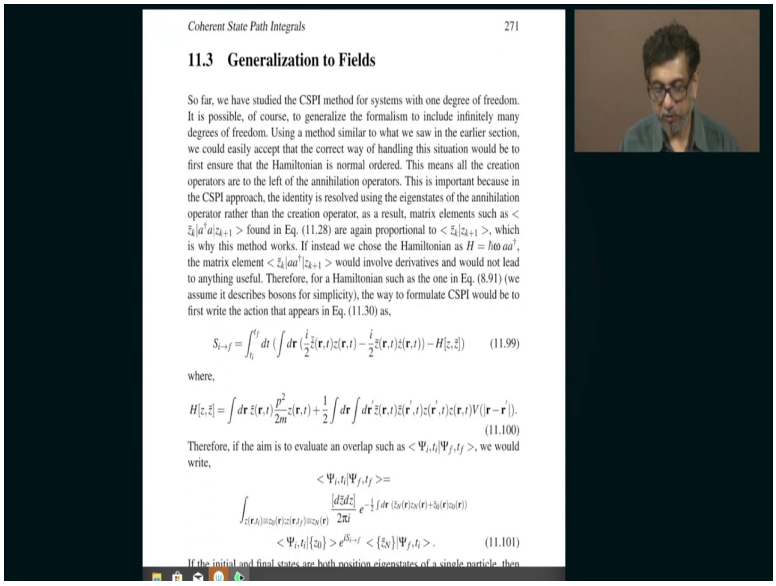

So, it is kind of worthwhile knowing it to some extent. So, now, you can generalize this to field. So, you see if you have an electron field, you can say what it looks like.

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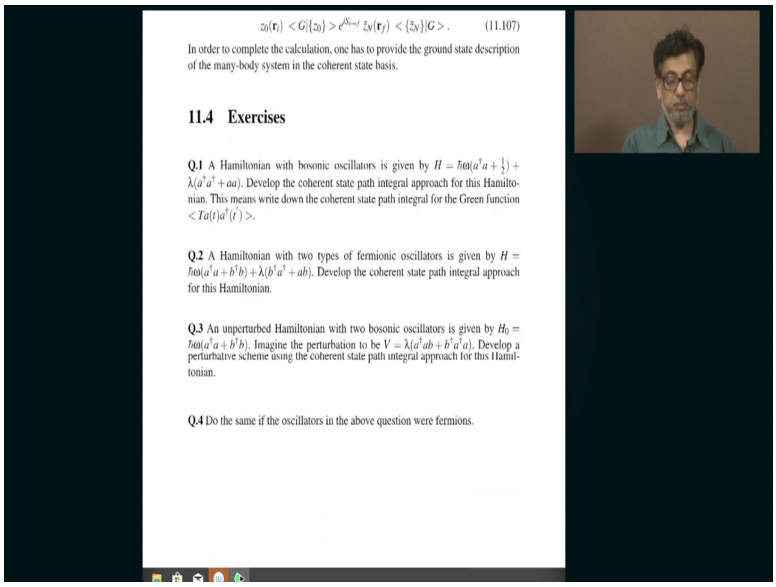

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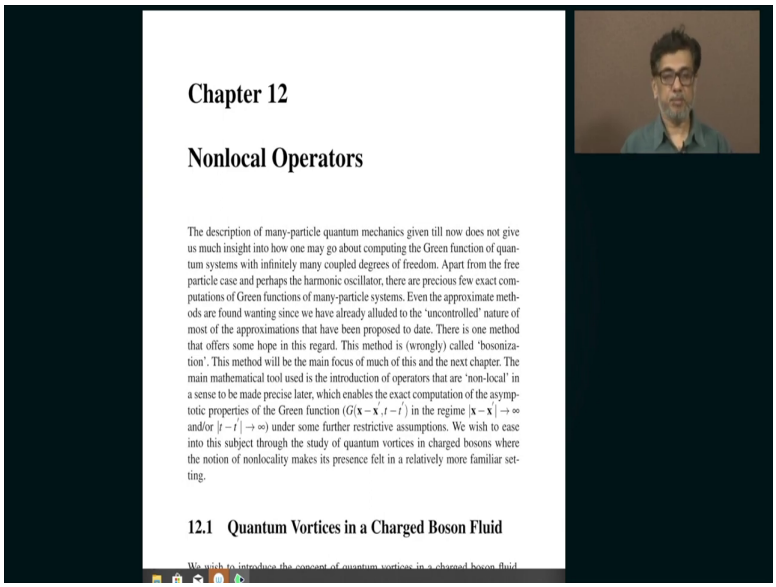

So, the coherent state path integral of electrons in a solid. Because now electrons being fermions, you have the fermionic coherent state path integral of a fermi gas, you know in a solid; I mean, so you can imagine that that could potentially be useful. So, I am going to stop here as far as path integrals are concerned, especially coherent path integrals; I may not have done full justice to this subject, especially the fermionic coherent state path integral, but it is there in the book, so you should look it up.

So, I am going to spend the next two lectures which will conclude this course discussing my own research. So, I have developed certain non local operators in quantum many body theory, which I believe are extremely important; but they are very under developed, in the sense that they show a lot of promise, but they are extremely technical and very hard to manipulate.

But I believe that if you put in the effort and learn how to utilize them properly; they will shed very deep light on various aspects of quantum many body theory. So, I want to discuss those objects and they go by the name of non local operators in quantum many body theory. So, that is the last topic of this NPTEL MOOCS course and I will probably spend maybe a maximum of two or maybe three lectures, very likely two lectures explaining that.

So, after that I am considering this course as more or less done. So, I continue to encourage you to ask questions over email, over live sessions; try to ask specific technical questions after reading the text and listening to the YouTube videos. Ask me specific technical questions; do not ask me vague questions like I did not understand this subject. So, ask me a specific question from the chapter and I will answer it, ok.

Thanks for going along for the ride with me. So, I hope even though you may not realize it now, you will find this; especially if you decide to specialize in theoretical physics, you will definitely sooner or later find whatever I have explained quite useful, even though it may not be apparent to you right now. So, I am going to conclude in the next two lectures.

Thanks for listening.