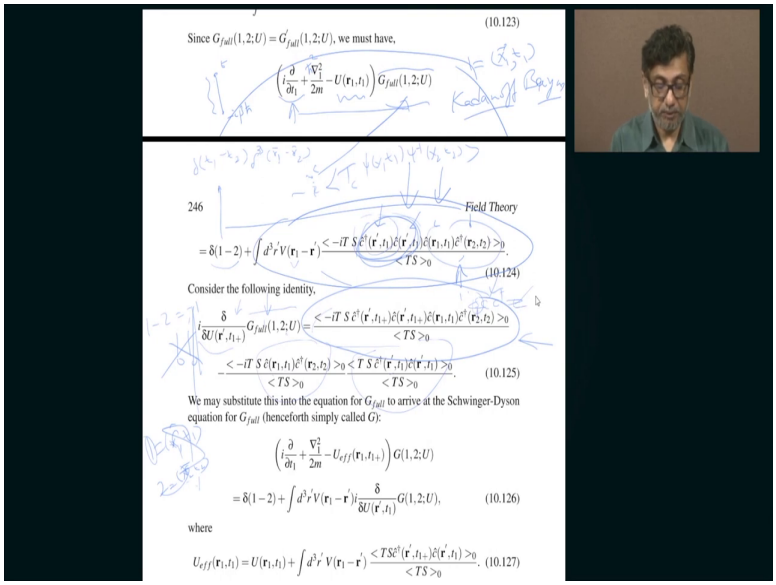
Dynamics of Classical and Quantum Fields: An Introduction

Prof. Girish S. Setlur
Department of Physics
Indian Institute of Technology, Guwahati

Coherent State Path Integrals
Lecture - 45
Bosonic Coherent States

(Refer Slide Time: 00:31)



Ok. So, let us continue where we left off. If you recall in the last class, I was telling you about how it is possible to introduce a source; that means, you pretend that there is an external potential which depends on time. But that time all those times are on the imaginary axis you know between t equal to 0 and t equals minus i beta h bar.

Then, what you do is that you define your Green's function along the time ordered Green's functional with times ordered along this imaginary axis in between within this interval. So, then, it is possible to write down the equation of motion that is basically that the time dependence, the how this green function changes with see one of those time. See after all, the Green's function depends on two times; it depends on the time of the first operator which is usually the annihilation operator.

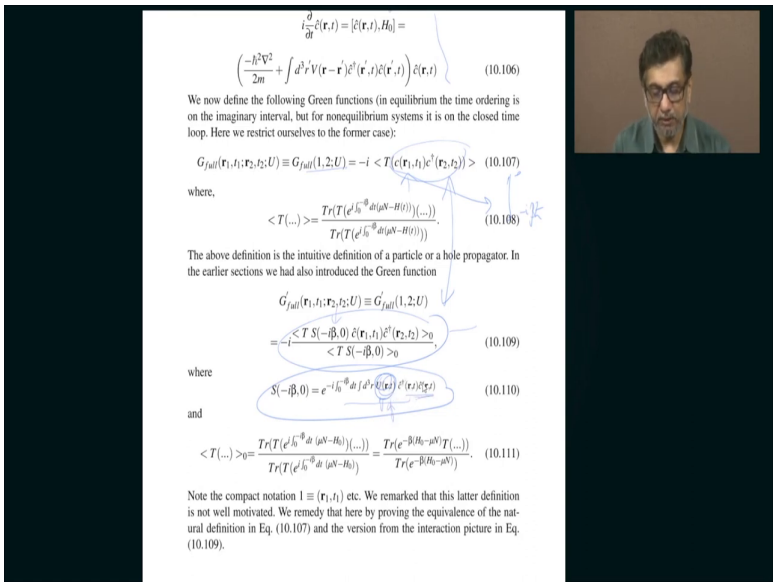
So, remember that what this Green's function is. It is basically the time ordered; it so its time ordered along the imaginary axis. So, I will put c there just to remind myself that the it is along the imaginary axis and this is what it is. So, I have just picked one of the times and I have decided to find the time evolution with respect to that particular time which is I have just selected; I could have selected t 2 also. But I have selected t 1. So, the point is that the equation obeyed by this Green's function is going to be this ok; yeah.

So, this typically comes with a minus i also ok. So, yeah because if it comes with a minus i, then this becomes you see this is shorthand for a spatial and time Dirac delta function multiplied together. This is I mean this is just shorthand of writing; it is not an the number 1 minus number 2, it is not like 1 minus 2 equals minus 1. I mean 1 means x 1 t 1; 2 means x 2 t 2. So, it is just short hand because I have to otherwise write x 1 comma t 1, x 2 comma t 2. It is very irritating, but here I have no choice, I have to explicitly write r 1 r 2 or whatever r 1, r 2.

So, I have to explicitly write that here ok; it is not x 1 I have selected r 1, r 2. r 1 comma t 1 is shorthand for that is 1; r 2 comma t 2 shorthand is 2. So, the point is that the equation of motion obeyed by the Green's function is basically this. So, now, you see this involves two creation and two annihilation operators; is not it? So, but then, I told you that the whole purpose of introducing this external time dependent potential which is anyway not there and you know trying to complicate matters by introducing something that is not there.

It had better have a valid benefit that is if it does not benefit me enormously, I have no business complicating matters by introducing something that dint exist before. So, you will soon see that it is in fact very very useful. So, the reason why it is useful is that if you take this Green's function and formally differentiate with respect to this source that I have introduced. So, what is going to happen is that you will see that you see after all the source is present only in the S matrix.

(Refer Slide Time: 04:33)



$$i\frac{\partial}{\partial t}\hat{c}(\mathbf{r},t) = [\hat{c}(\mathbf{r},t), H_0] =$$

$$\left(\frac{-\hbar^2\nabla^2}{2m} + \int d^3r' V(\mathbf{r}-\mathbf{r}')\hat{c}^\dagger(\mathbf{r}',t)\hat{c}(\mathbf{r}',t)\right)\hat{c}(\mathbf{r},t) \quad (10.106)$$

We now define the following Green functions (in equilibrium the time ordering is on the imaginary interval, but for nonequilibrium systems it is on the closed time loop. Here we restrict ourselves to the former case):

$$G_{full}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2;U) \equiv G_{full}(1,2;U) = -i < T(c(\mathbf{r}_1,t_1)c^\dagger(\mathbf{r}_2,t_2)) > \quad (10.107)$$

where,

$$< T(...) >= \frac{Tr(T(e^{i\int_0^{-i\beta} dt(\mu N - H(t))}(...)))}{Tr(T(e^{i\int_0^{-i\beta} dt(\mu N - H(t))}))}. \quad (10.108)$$

The above definition is the intuitive definition of a particle or a hole propagator. In the earlier sections we had also introduced the Green function

$$G^<_{full}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2;U) \equiv G^<_{full}(1,2;U)$$

$$= -i\frac{< T\, S(-i\beta,0)\, \hat{c}(\mathbf{r}_1,t_1)\hat{c}^\dagger(\mathbf{r}_2,t_2) >_0}{< T\, S(-i\beta,0) >_0}, \quad (10.109)$$

where

$$S(-i\beta,0) = e^{-i\int_0^{-i\beta} dt \int d^3r \hat{c}^\dagger(\mathbf{r},t)\hat{c}(\mathbf{r},t)} \quad (10.110)$$

and

$$< T(...) >_0 = \frac{Tr(T(e^{i\int_0^{-i\beta} dt\ (\mu N - H_0)}(...)))}{Tr(T(e^{i\int_0^{-i\beta} dt\ (\mu N - H_0)}))} = \frac{Tr(e^{-\beta(H_0-\mu N)}T(...))}{Tr(e^{-\beta(H_0-\mu N)})}. \quad (10.111)$$

Note the compact notation $1 \equiv (\mathbf{r}_1, t_1)$ etc. We remarked that this latter definition is not well motivated. We remedy that here by proving the equivalence of the natural definition in Eq. (10.107) and the version from the interaction picture in Eq. (10.109).

And if you recall this S matrix is defined like this. So, the source is only here. So, if I differentiate with respect to the source, I will bring down a density there which is precisely what this term which involves four fermions interaction is ok. So, basically you see it involves the density here.

So, it involves this is the density here; c dagger r dash t 1 c r dash t 1 that is basically the density. So, by differentiating with respect to the source of the Green's function, so this is the original Green's function r 1 t 1, r 2 t 2; r 1 t 1 is annihilation, r 2 t 2 is creation. But now, I want to see how I can bring down a density.

So, if it is just this, there is nothing there. So, I have to difference. So, if there is an S, there you see the S has the density in the exponent. If I differentiate with respect to the source, I will be bringing down a density. But then, you see the Green's function is the ratio of the S matrix on the numerator and S matrix also in the denominator. So, now if I differentiate with respect to the source, I will end up getting this what is called the correlation function, it is this average. But then, I have to subtract out the average when I pair up these operators in this way.

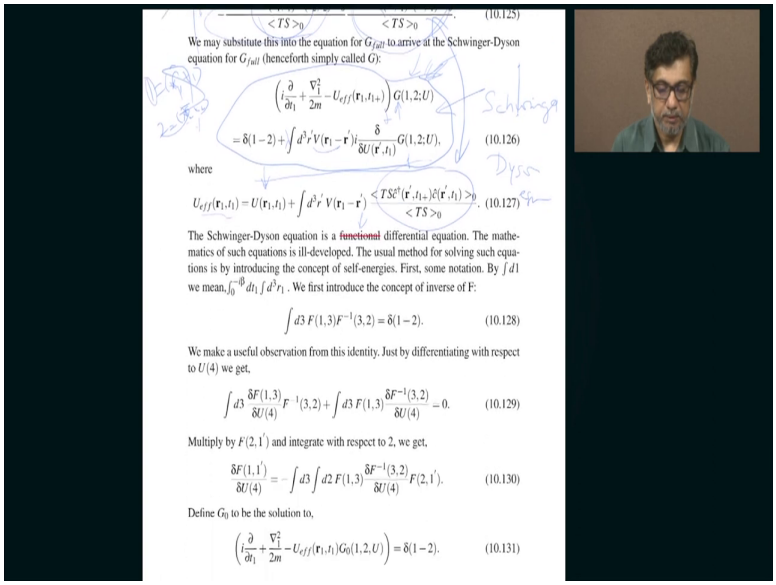
So, basically that is what it will amount to ok. So, now, so in other words, this term that I was trying to see if I can write in a way that involves only one particle Green's function

because after all this has two particle operator, it involves two creation and two annihilation. So, now, I want to be able to express this in terms of something involving only one particle. So, you see at the level of this equation its already clear that I have been successful in doing that.

Because after all what is this that I have circled here; it involves two creation and two annihilation. But here it involves only one creation, one annihilation because this is my original Green function which involves annihilation creation.

But this also still involves annihilation creation; there is also one particle Green's function times and some other version of the one particle Green's function. So, there are all these are one particle versions of some products or whatever it is of some versions of one particle Green's function. So, that the two particle Green's function has now been expressed purely in terms of some combination of one particle Green's functions.

(Refer Slide Time: 07:20)



And that is an enormous advantage because you see now you can go ahead and I am not sure who crossed this out; it is I mean this is not meant to be crossed out. I certainly dint cross it out ok; yeah. So, the point is that this differential equation. So, you see once you take this to this side, I can lump all this it becomes U plus this one times this Green's

function ok. So, it will just involve. So, this is the original Green's function this is basically G full and this will involve this sort of thing. So, this is same as this.

And this is the original G full; this is G full. So, I bring this to this side and I express only this term in terms of this plus this and this plus this is basically the effective potential. So, it is like the average. So, physically what this means is this is the average potential seen by a particle at r 1, t 1 not only due to the externally imposed source; but also due to all the other particles around it.

So, this is the effective potential seen by the particle at r 1 t 1, but you do not have to call it that. I mean formally it is just this expression. So, the point is that formally you see, now I have succeeded in writing down the equation of motion for the one particle Green's function which is which involves quantum particles that are not only have kinetic energy; but also a potential energy by way of mutual interaction between each other pairwise potential energy.

So, I have been successful in writing the Green's equation for the Green's function in the form of a functional differential equation, which only involves other versions of the one particle Green's function. So, in other words, it does not involve equations; it does not involve Green's functions where the more than two particles are involved.

So, bottom line is that the price I have to pay to do that, that is the price I have to pay to avoid introducing four particle rather two particle, three particle, four particles etcetera Green's function. So, the price I have to pay to ensure that the equation for the Green's function only involves one particle Green's functions or some version of the one particle Green's function.

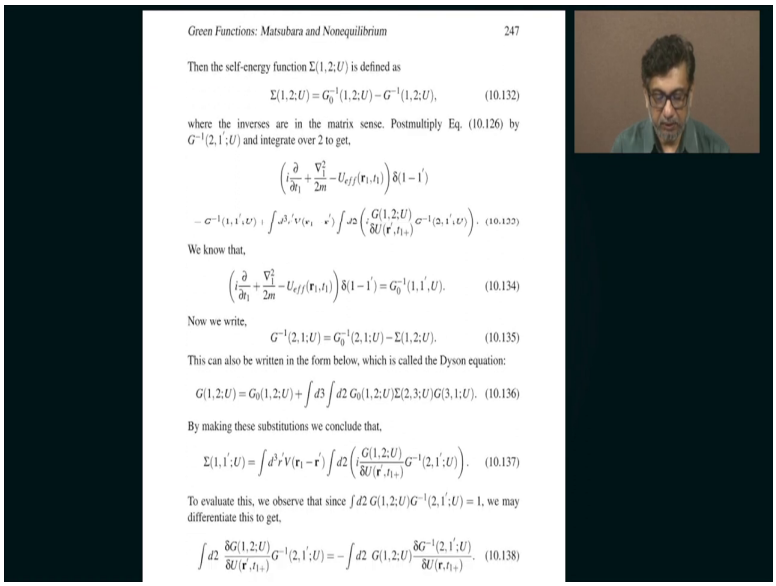
So, the price I have to pay is that I have to introduce artificially an external time dependent source that is defined along the imaginary axis. But then, you might say that given that in the end, it is not there, how do I get rid of it? So, obviously, the answer is that once you somehow are successful in solving this equation, by the way nobody knows even know, how to solve such equations. So, these are functional differential equations which not only involve partial derivatives in the usual sense, but they also involve functional derivatives.

So, there is no there is no well-developed mathematical theory to solve this. It is just you know writing this is useful because it allows you to develop perturbation scheme in a very systematic way because that is the only scheme we have at our disposal to solve this because we just end up expanding in powers of this. So, we pretend there is a lambda here and you just expand and you expand your G's in powers of lambda and substitute and compare both sides and that sort of thing. Pretty much see that is all you can do. So, this allows you to do that in a systematic way; that is it.

So, now, the question is how do I get rid of that external source? So, the answer is that if somehow you develop a perturbation scheme or whatever and truncate it in the mains and so on and so forth, then you go ahead and set that external source to 0 and then, you have gotten rid of it that way.

Yeah, so that that is the whole point. So, this is called the Schwinger Dyson equation; Schwinger Dyson equation ok. So, the rest of this chapter just tells you how to implement that perturbation scheme that I was talking about. So, basically you define something called the self-energy.

(Refer Slide Time: 12:19)



Green Functions: Matsubara and Nonequilibrium 247

Then the self-energy function $\Sigma(1,2;U)$ is defined as

$$\Sigma(1,2;U) = G_0^{-1}(1,2;U) - G^{-1}(1,2;U), \quad (10.132)$$

where the inverses are in the matrix sense. Postmultiply Eq. (10.126) by $G^{-1}(2,1';U)$ and integrate over 2 to get,

$$\left(i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U_{eff}(\mathbf{r}_1,t_1)\right)\delta(1-1') - \cdots = \delta(1-1') + \int d3\, v(\mathbf{r}_1 - \mathbf{r}')\int d2\left(\frac{G(1,2;U)}{\delta U(\mathbf{r}',t_{1+})}G^{-1}(2,1';U)\right). \quad (10.133)$$

We know that,

$$\left(i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U_{eff}(\mathbf{r}_1,t_1)\right)\delta(1-1') = G_0^{-1}(1,1';U). \quad (10.134)$$

Now we write,

$$G^{-1}(2,1;U) = G_0^{-1}(2,1;U) - \Sigma(1,2;U). \quad (10.135)$$

This can also be written in the form below, which is called the Dyson equation:

$$G(1,2;U) = G_0(1,2;U) + \int d3 \int d2\, G_0(1,2;U)\Sigma(2,3;U)G(3,1;U). \quad (10.136)$$

By making these substitutions we conclude that,

$$\Sigma(1,1';U) = \int d^3r' V(\mathbf{r}_1 - \mathbf{r}')\int d2\left(i\frac{G(1,2;U)}{\delta U(\mathbf{r}',t_{1+})}G^{-1}(2,1';U)\right). \quad (10.137)$$

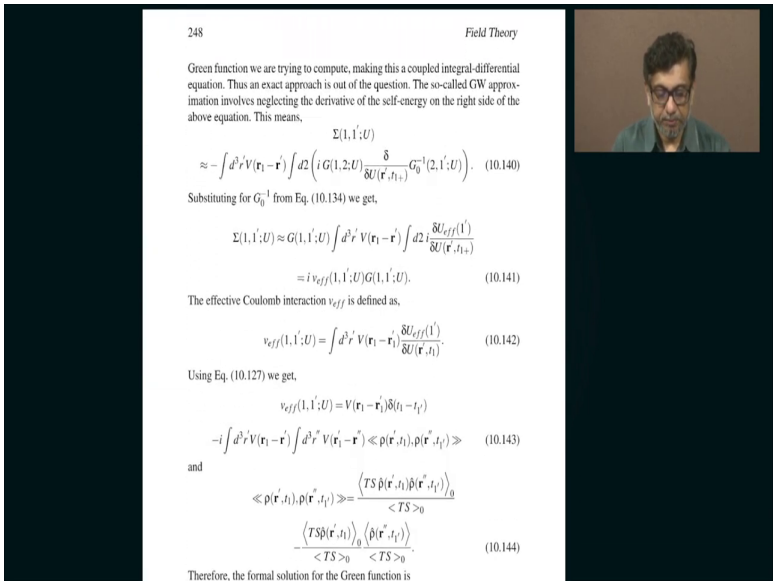
To evaluate this, we observe that since $\int d2\, G(1,2;U)G^{-1}(2,1';U) = 1$, we may differentiate this to get,

$$\int d2\, \frac{\delta G(1,2;U)}{\delta U(\mathbf{r}',t_{1+})}G^{-1}(2,1';U) = -\int d2\, G(1,2;U)\frac{\delta G^{-1}(2,1';U)}{\delta U(\mathbf{r},t_{1+})}. \quad (10.138)$$

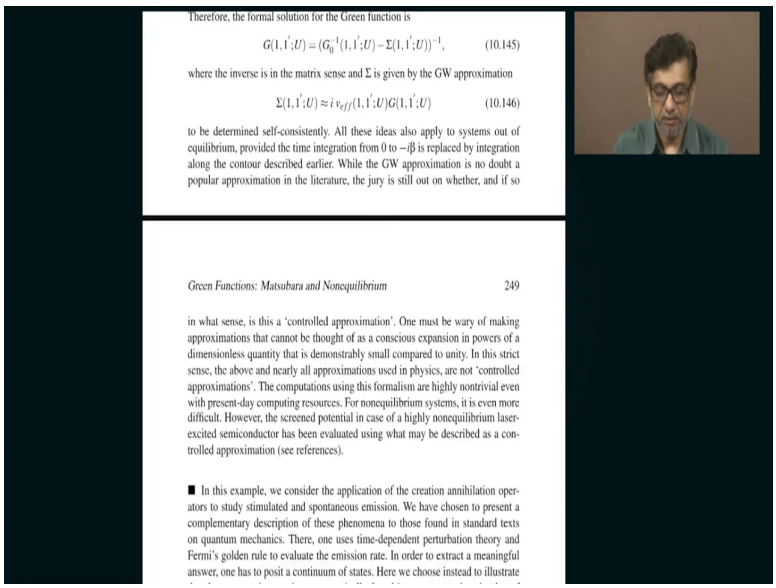
So, I think I will going to skip that because basically that is only if you are interested in actually carrying out that perturbation series in some systematic way. So, that you will

end up with something called the GW approximation to the leading order. So, your; self-energy can be expanded in powers of this coupling and the lowest order contribution is called the GW approximation which itself is pretty formidable and it cannot be done.

(Refer Slide Time: 12:46)



(Refer Slide Time: 12:49)



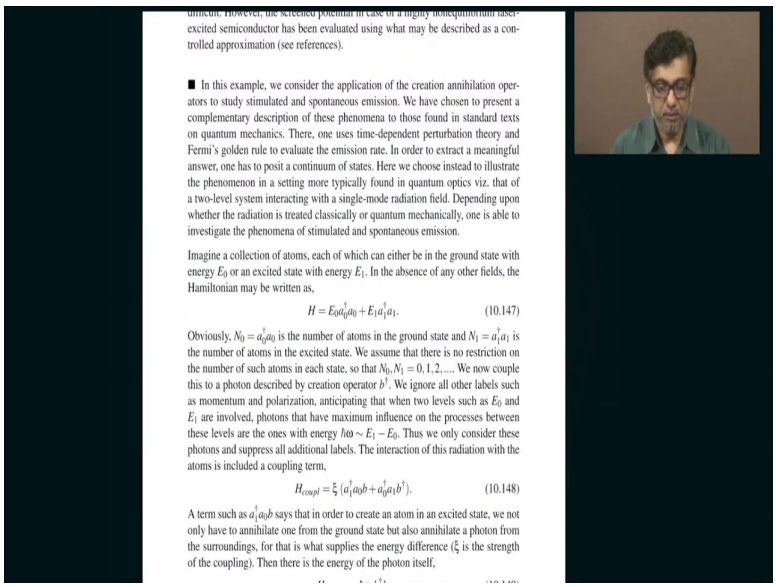
So, the whole thing is still non-linear even after. So, the functional derivative aspect goes away at the first order. The moment you truncate, you can get rid of the functional aspect of the problem. But then, you will still be forced to reckon with the non-linear partial

integro differential aspect of the problem. So, it will remain an integro partial differential equation.

Yeah, I mean it is kind of one runs out of adjectives to describe these sort of equations. So, bottom line is the incredibly hard to solve and nobody even with present day computers even the GW approximation cannot be solved in any convincing way alright.

So, I am going to stop here as far as this particular topic is concerned. So, basically this particular topic was the Schwinger Dyson equation. So, the Schwinger Dyson equation is a functional differential equation for the time the ordered Green's function of a system of particles interacting mutually and also, with an external source that is defined on the imaginary time axis. So, it only involves the, that Green's function the one particle time ordered Green's function and nothing else. So, that is as far as that topic is concerned.

(Refer Slide Time: 14:20)


difficult. However, the screened potential in case of a highly nonequilibrium laser-excited semiconductor has been evaluated using what may be described as a controlled approximation (see references).

■ In this example, we consider the application of the creation annihilation operators to study stimulated and spontaneous emission. We have chosen to present a complementary description of these phenomena to those found in standard texts on quantum mechanics. There, one uses time-dependent perturbation theory and Fermi's golden rule to evaluate the emission rate. In order to extract a meaningful answer, one has to posit a continuum of states. Here we choose instead to illustrate the phenomenon in a setting more typically found in quantum optics viz. that of a two-level system interacting with a single-mode radiation field. Depending upon whether the radiation is treated classically or quantum mechanically, one is able to investigate the phenomena of stimulated and spontaneous emission.

Imagine a collection of atoms, each of which can either be in the ground state with energy E_0 or an excited state with energy E_1. In the absence of any other fields, the Hamiltonian may be written as,

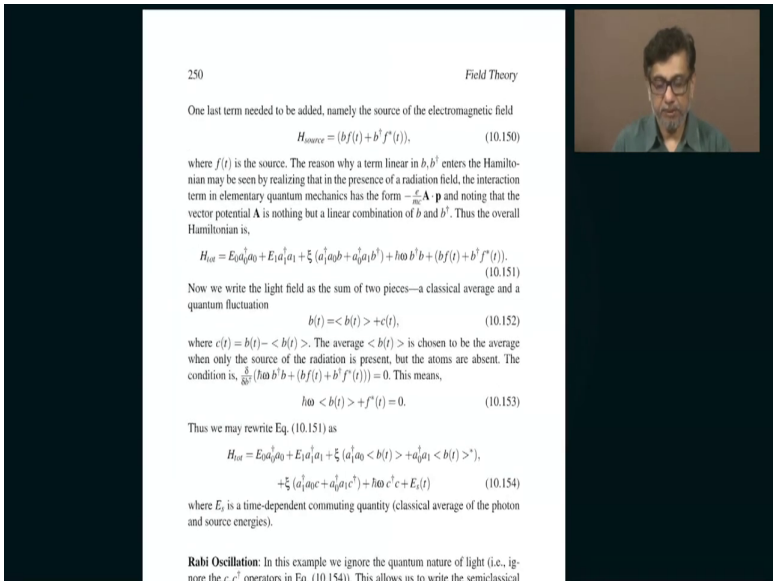
$$H = E_0 a_0^\dagger a_0 + E_1 a_1^\dagger a_1. \qquad (10.147)$$

Obviously, $N_0 = a_0^\dagger a_0$ is the number of atoms in the ground state and $N_1 = a_1^\dagger a_1$ is the number of atoms in the excited state. We assume that there is no restriction on the number of such atoms in each state, so that $N_0, N_1 = 0, 1, 2, \ldots$. We now couple this to a photon described by creation operator $b^\dagger$. We ignore all other labels such as momentum and polarization, anticipating that when two levels such as E_0 and E_1 are involved, photons that have maximum influence on the processes between these levels are the ones with energy $\hbar\omega \sim E_1 - E_0$. Thus we only consider these photons and suppress all additional labels. The interaction of this radiation with the atoms is included a coupling term,

$$H_{coupl} = \xi \left(a_1^\dagger a_0 b + a_0^\dagger a_1 b^\dagger \right). \qquad (10.148)$$

A term such as $a_1^\dagger a_0 b$ says that in order to create an atom in an excited state, we not only have to annihilate one from the ground state but also annihilate a photon from the surroundings, for that is what supplies the energy difference (ξ is the strength of the coupling). Then there is the energy of the photon itself,

(Refer Slide Time: 14:26)



250 *Field Theory*

One last term needed to be added, namely the source of the electromagnetic field

$$H_{source} = (bf(t) + b^\dagger f^*(t)), \tag{10.150}$$

where $f(t)$ is the source. The reason why a term linear in $b, b^\dagger$ enters the Hamiltonian may be seen by realizing that in the presence of a radiation field, the interaction term in elementary quantum mechanics has the form $-\frac{e}{mc}\mathbf{A}\cdot\mathbf{p}$ and noting that the vector potential $\mathbf{A}$ is nothing but a linear combination of b and $b^\dagger$. Thus the overall Hamiltonian is,

$$H_{tot} = E_0 a_0^\dagger a_0 + E_1 a_1^\dagger a_1 + \xi\,(a_1^\dagger a_0 b + a_0^\dagger a_1 b^\dagger) + \hbar\omega\, b^\dagger b + (bf(t) + b^\dagger f^*(t)). \tag{10.151}$$

Now we write the light field as the sum of two pieces—a classical average and a quantum fluctuation

$$b(t) = < b(t) > + c(t), \tag{10.152}$$

where $c(t) = b(t) - < b(t) >$. The average $< b(t) >$ is chosen to be the average when only the source of the radiation is present, but the atoms are absent. The condition is, $\frac{\delta}{\delta b^\dagger}(\hbar\omega\, b^\dagger b + (bf(t) + b^\dagger f^*(t))) = 0$. This means,

$$\hbar\omega < b(t) > + f^*(t) = 0. \tag{10.153}$$

Thus we may rewrite Eq. (10.151) as

$$H_{tot} = E_0 a_0^\dagger a_0 + E_1 a_1^\dagger a_1 + \xi\,(a_1^\dagger a_0 < b(t) > + a_0^\dagger a_1 < b(t) >^*),$$

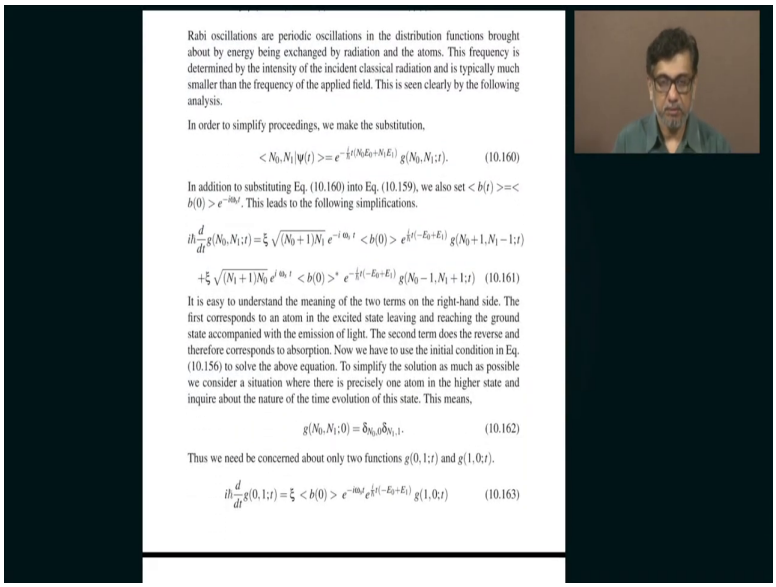
$$+\xi\,(a_1^\dagger a_0 c + a_0^\dagger a_1 c^\dagger) + \hbar\omega\, c^\dagger c + E_s(t) \tag{10.154}$$

where E_s is a time-dependent commuting quantity (classical average of the photon and source energies).

Rabi Oscillation: In this example we ignore the quantum nature of light (i.e., ignore the $c, c^\dagger$ operators in Eq. (10.154)). This allows us to write the semiclassical

So, the this the other topic that I have introduced in the next chapter is some other simpler versions of the non-equilibrium situation, where see earlier I was talking about a continuum problem, where the degrees of freedom are truly infinite. But here, I have reverted to a system with a finite in fact just two level systems.

(Refer Slide Time: 14:50)



Rabi oscillations are periodic oscillations in the distribution functions brought about by energy being exchanged by radiation and the atoms. This frequency is determined by the intensity of the incident classical radiation and is typically much smaller than the frequency of the applied field. This is seen clearly by the following analysis.

In order to simplify proceedings, we make the substitution,

$$< N_0, N_1 | \psi(t) > = e^{-\frac{i}{\hbar}t(N_0 E_0 + N_1 E_1)}\, g(N_0, N_1; t). \tag{10.160}$$

In addition to substituting Eq. (10.160) into Eq. (10.159), we also set $< b(t) > = < b(0) > e^{-i\omega_0 t}$. This leads to the following simplifications.

$$i\hbar\frac{d}{dt}g(N_0, N_1; t) = \xi\sqrt{(N_0+1)N_1}\, e^{-i\,\omega_0 t} < b(0) > e^{\frac{i}{\hbar}t(-E_0+E_1)}\, g(N_0+1, N_1-1; t)$$

$$+\xi\sqrt{(N_1+1)N_0}\, e^{i\,\omega_0 t} < b(0) >^*\, e^{-\frac{i}{\hbar}t(-E_0+E_1)}\, g(N_0-1, N_1+1; t) \tag{10.161}$$

It is easy to understand the meaning of the two terms on the right-hand side. The first corresponds to an atom in the excited state leaving and reaching the ground state accompanied with the emission of light. The second term does the reverse and therefore corresponds to absorption. Now we have to use the initial condition in Eq. (10.156) to solve the above equation. To simplify the solution as much as possible we consider a situation where there is precisely one atom in the higher state and inquire about the nature of the time evolution of this state. This means,

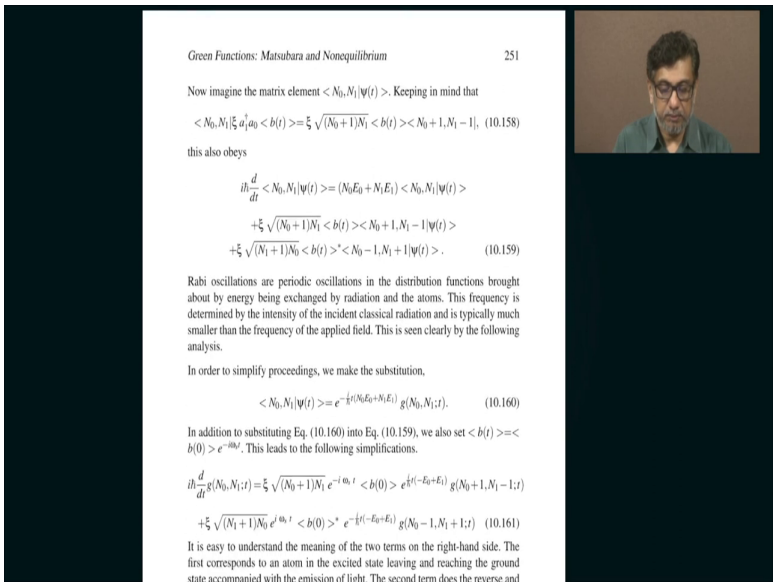
$$g(N_0, N_1; 0) = \delta_{N_0,0}\delta_{N_1,1}. \tag{10.162}$$

Thus we need be concerned about only two functions $g(0,1;t)$ and $g(1,0;t)$.

$$i\hbar\frac{d}{dt}g(0,1;t) = \xi < b(0) > e^{-i\omega_0 t} e^{\frac{i}{\hbar}t(-E_0+E_1)}\, g(1,0;t) \tag{10.163}$$

Just to tell you that if you get rid of the field aspect of the question; that means, after all the title of this course is dynamics of classical and quantum fields. So, strictly speaking this part of the topic should not even have been there in the textbook.

(Refer Slide Time: 15:07)



Green Functions: Matsubara and Nonequilibrium 251

Now imagine the matrix element $< N_0, N_1|\psi(t) >$. Keeping in mind that

$$< N_0, N_1|\xi\, a_1^{\dagger} a_0 < b(t) >= \xi \sqrt{(N_0+1)N_1} < b(t) >< N_0+1, N_1-1|, \quad (10.158)$$

this also obeys

$$i\hbar \frac{d}{dt} < N_0, N_1|\psi(t) >= (N_0 E_0 + N_1 E_1) < N_0, N_1|\psi(t) >$$
$$+\xi \sqrt{(N_0+1)N_1} < b(t) >< N_0+1, N_1-1|\psi(t) >$$
$$+\xi \sqrt{(N_1+1)N_0} < b(t) >^{*} < N_0-1, N_1+1|\psi(t) > . \quad (10.159)$$

Rabi oscillations are periodic oscillations in the distribution functions brought about by energy being exchanged by radiation and the atoms. This frequency is determined by the intensity of the incident classical radiation and is typically much smaller than the frequency of the applied field. This is seen clearly by the following analysis.

In order to simplify proceedings, we make the substitution,

$$< N_0, N_1|\psi(t) >= e^{-\frac{i}{\hbar}t(N_0 E_0 + N_1 E_1)}\, g(N_0, N_1; t). \quad (10.160)$$

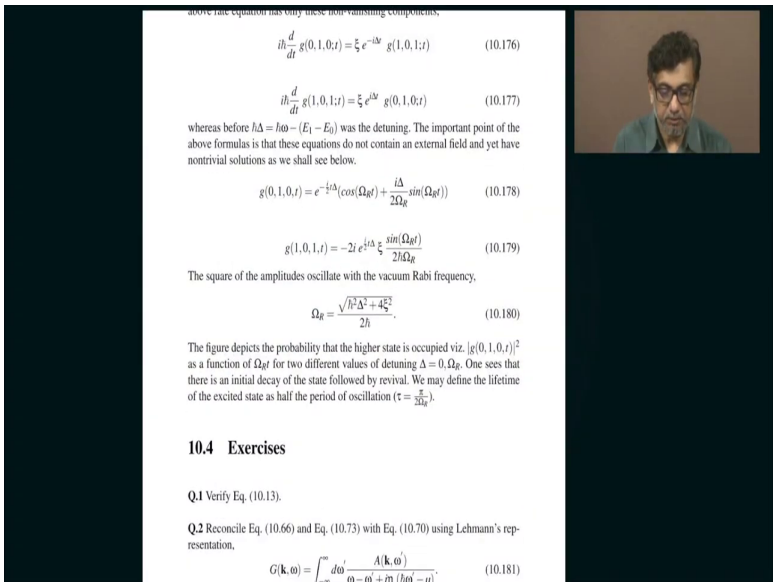
In addition to substituting Eq. (10.160) into Eq. (10.159), we also set $< b(t) >=< b(0) > e^{-i\omega_0 t}$. This leads to the following simplifications.

$$i\hbar \frac{d}{dt} g(N_0, N_1; t) = \xi \sqrt{(N_0+1)N_1}\, e^{-i\,\omega_0\, t} < b(0) > e^{\frac{i}{\hbar}t(-E_0+E_1)}\, g(N_0+1, N_1-1; t)$$
$$+\xi \sqrt{(N_1+1)N_0}\, e^{i\,\omega_0\, t} < b(0) >^{*} e^{-\frac{i}{\hbar}t(-E_0+E_1)}\, g(N_0-1, N_1+1; t) \quad (10.161)$$

It is easy to understand the meaning of the two terms on the right-hand side. The first corresponds to an atom in the excited state leaving and reaching the ground state accompanied with the emission of light. The second term does the reverse and

But it is there only because it tries to tell you that if already you know with, in systems are not in equilibrium, there is lot of interesting physics even at the level of finite number of degrees of freedom. So, there is something called Rabi oscillation and so on and so forth.

(Refer Slide Time: 15:28)



above rate equation has only these non-vanishing components,

$$i\hbar \frac{d}{dt} g(0,1,0;t) = \xi\, e^{-i\Delta t}\, g(1,0,1;t) \quad (10.176)$$

$$i\hbar \frac{d}{dt} g(1,0,1;t) = \xi\, e^{i\Delta t}\, g(0,1,0;t) \quad (10.177)$$

whereas before $\hbar\Delta = \hbar\omega - (E_1 - E_0)$ was the detuning. The important point of the above formulas is that these equations do not contain an external field and yet have nontrivial solutions as we shall see below.

$$g(0,1,0,t) = e^{-\frac{i}{2}t\Delta}\left(cos(\Omega_R t) + \frac{i\Delta}{2\Omega_R} sin(\Omega_R t)\right) \quad (10.178)$$

$$g(1,0,1,t) = -2i\, e^{\frac{i}{2}t\Delta}\, \xi\, \frac{sin(\Omega_R t)}{2\hbar\Omega_R} \quad (10.179)$$

The square of the amplitudes oscillate with the vacuum Rabi frequency,

$$\Omega_R = \frac{\sqrt{\hbar^2\Delta^2 + 4\xi^2}}{2\hbar}. \quad (10.180)$$

The figure depicts the probability that the higher state is occupied viz. $|g(0,1,0,t)|^2$ as a function of $\Omega_R t$ for two different values of detuning $\Delta = 0, \Omega_R$. One sees that there is an initial decay of the state followed by revival. We may define the lifetime of the excited state as half the period of oscillation ($\tau = \frac{\pi}{2\Omega_R}$).

10.4 Exercises

Q.1 Verify Eq. (10.13).

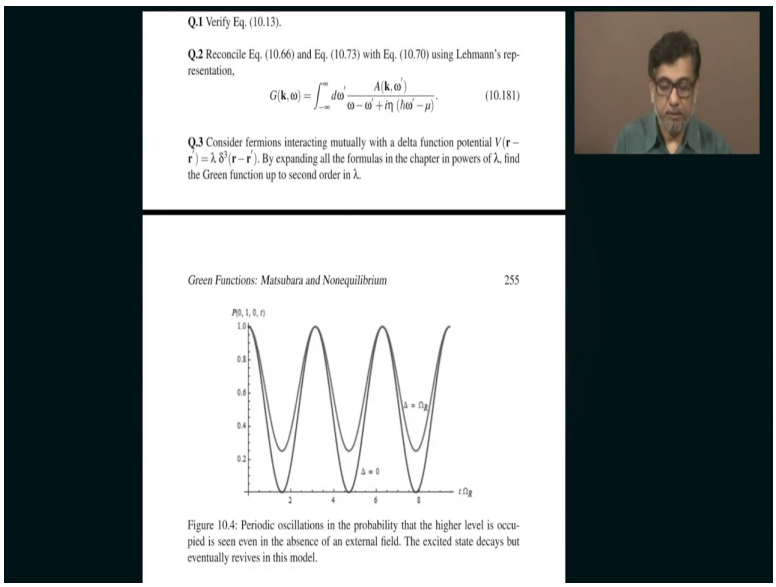
Q.2 Reconcile Eq. (10.66) and Eq. (10.73) with Eq. (10.70) using Lehmann's representation,

$$G(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} d\omega' \frac{A(\mathbf{k}, \omega')}{\omega - \omega' + i\eta\,(\hbar\omega' - \mu)}. \quad (10.181)$$

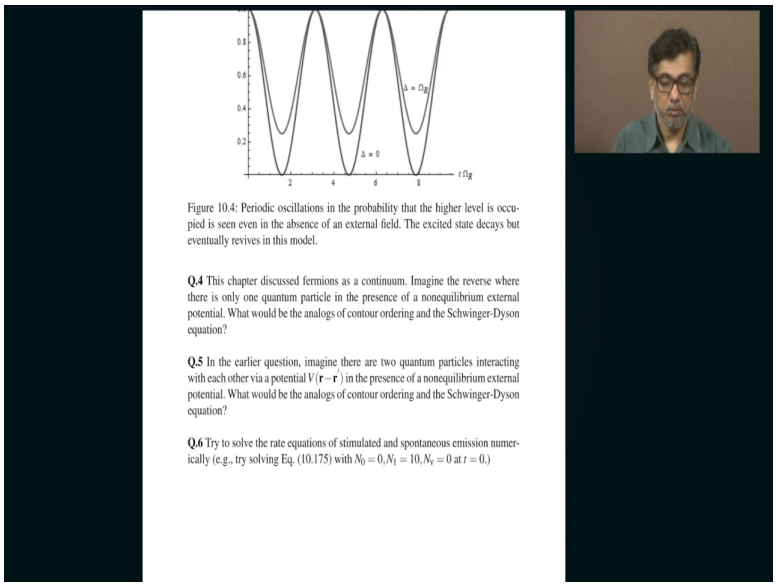
So, you can have spontaneous stimulated emission etcetera. I am just introducing that just to highlight the fact that non-equilibrium systems are already reasonably

complicated, even when you choose to study a system with finite number of degrees of freedom or oscillators with just few labels.

(Refer Slide Time: 15:51)

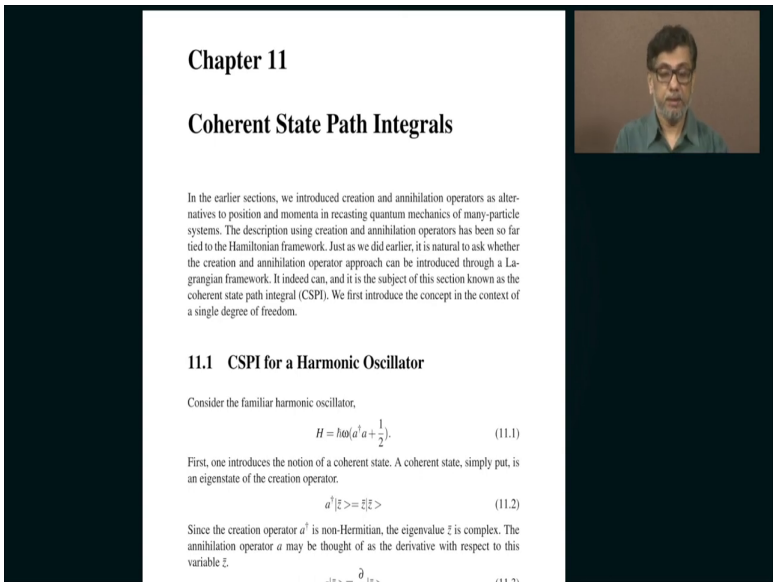


(Refer Slide Time: 15:53)



But then, that would not be consistent with the title of the textbook which is dynamics of classical and quantum fields. So, let us get back to fields.

(Refer Slide Time: 16:02)



So, if I decide to get back to fields, the other topic I have to next discuss is basically the idea of coherent state path integrals. But as usual you know before I get to fields, I have to introduce the version of this for system with finite number of degrees of freedom. So, usually that is how in fact not usually, that is the only way to do it. You introduce it for a finite discrete number of degrees of freedom and then, pretend that that index is now continuously large ok.

So, what is this coherent state path integrals? So, if you remember, we studied the quantum harmonic oscillator using the creation annihilation language; but that involves that still involves dealing with the Hamiltonian. So, you see the Hamiltonian involves creation and annihilation operators, but remember the spirit of the path integral approach. So, the spirit of the path integral approach is that you should be able to do quantum mechanics not merely with always with Hamiltonians, but you should also be able to do quantum mechanics using Lagrangians.

So, now, the question is you see now I have in front of me the harmonic quantum harmonic oscillator in terms of creation and annihilation operators. So, the question is can I now write down the Lagrangian of the quantum harmonic oscillators not in terms of position and momentum, but again still in terms of creation and I mean some. So, you see I want to de quantize the creation and annihilation operators because after all you see

in quantum mechanics, the annihilation operators basically an operator which is in some sense complex. Because it is the complex linear combination of two self-ad joint operators. So, the question is you see the de quantized version of the annihilation operator would simply be a complex number.

So, now the question is I want to be able to see if I can introduce a path integral approach to this problem in terms of the eigen values of the annihilation operator. So, that is the peculiar point of view; but it is a it is a valid question to ask. You see there is the Hamiltonian is now expressed in terms of creation and annihilation operators.

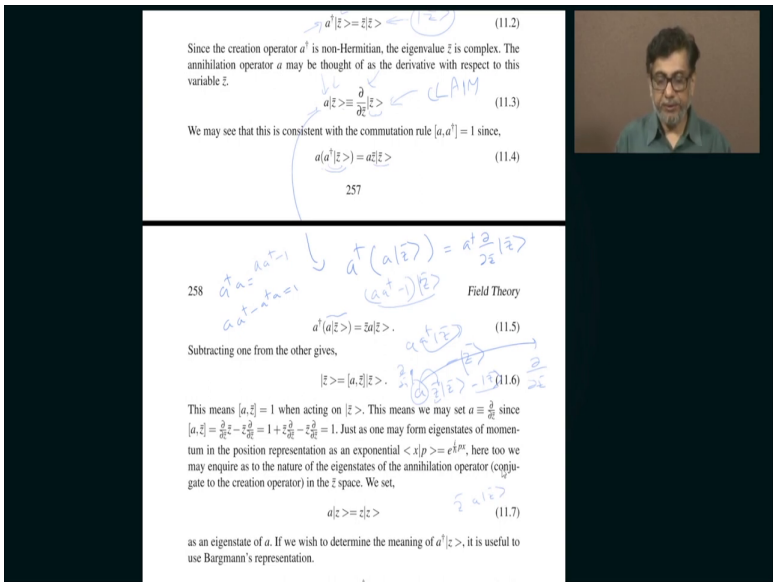
Now, remember how it is in the original path integral. So, if the original Hamiltonian was position and momentum and they were operators the quantum mechanics of such a system using Lagrangians will involve actually the classical Lagrangian.

That means, it will involve e raise to i h bar integral d t of l, where l is the classical Lagrangian, where you have to sum over all the classical paths. That means, the all the classical paths connecting some initial and final end points. So, those classical paths will have some classical positions and classical velocities. So, similarly, here you see I have a quantum Hamiltonian involving creation and annihilation operators.

So, now, if I want to study this using not the Hamiltonian picture, but using Lagrangians, I will now be writing this you know in terms of the path integral of the form e raise to i by h bar integral d t of l, where now l is the Lagrangian in terms of the de quantized versions of a and a dagger. Just like in the case of path integrals, I had to write down the Lagrangian in terms of x and x dot, which are x is basically the de quantized version of the position operator because now it is now a classical object.

So, similarly m x dot is the de quantized version of the momentum operator p. So, similarly, here I want to be able to write down a de quantized version of a and a dagger. So, that I can express my Lagrangian in terms of those objects, which would then correspond to the classical paths that the system is taking.

(Refer Slide Time: 21:02)



$$a^\dagger|\bar{z}>=\bar{z}|\bar{z}> \quad (11.2)$$

Since the creation operator $a^\dagger$ is non-Hermitian, the eigenvalue $\bar{z}$ is complex. The annihilation operator a may be thought of as the derivative with respect to this variable $\bar{z}$.

$$a|\bar{z}>\equiv\frac{\partial}{\partial\bar{z}}|\bar{z}> \quad (11.3)$$

We may see that this is consistent with the commutation rule $[a,a^\dagger]=1$ since,

$$a(a^\dagger|\bar{z}>)=a\bar{z}|\bar{z}> \quad (11.4)$$

257

258 *Field Theory*

$$a^\dagger(a|\bar{z}>)=\bar{z}a|\bar{z}>. \quad (11.5)$$

Subtracting one from the other gives,

$$|\bar{z}>=[a,\bar{z}]|\bar{z}>. \quad (11.6)$$

This means $[a,\bar{z}]=1$ when acting on $|\bar{z}>$. This means we may set $a\equiv\frac{\partial}{\partial\bar{z}}$ since $[a,\bar{z}]=\frac{\partial}{\partial\bar{z}}\bar{z}-\bar{z}\frac{\partial}{\partial\bar{z}}=1+\bar{z}\frac{\partial}{\partial\bar{z}}-\bar{z}\frac{\partial}{\partial\bar{z}}=1$. Just as one may form eigenstates of momentum in the position representation as an exponential $<x|p>=e^{ipx}$, here too we may enquire as to the nature of the eigenstates of the annihilation operator (conjugate to the creation operator) in the $\bar{z}$ space. We set,

$$a|z>=z|z> \quad (11.7)$$

as an eigenstate of a. If we wish to determine the meaning of $a^\dagger|z>$, it is useful to use Bargmann's representation.

So, clearly, those are necessarily just like the de quantized version of the position operator is just the position eigen value. So, similarly, the de quantized version of the creation operator is just the eigenvalue of the creation operator which in this case happens to be complex. So, if I decide to introduce this, then I postulate that there has to be some state labelled by its eigenvalue z with a bar on top ok. So, that is supposedly the complex eigenvalue of the creation operator.

So, the claim is that if I decide to define the state in this way that is this is in directly the definition of the state basically tells you how the state comes about. It is the eigenstate of the creation operator. So, now, you can easily convince yourself that the eigenstate of annihilation operator is similarly given I mean it is given by in this case it is given by multiplying that state by some z; I mean its eigen value. So, basically it is the now an eigen state. So, if this is the eigenstate of creation operator, there is absolutely no reason why it should be the eigenstate of the annihilation operator also. So, in fact it is not.

So, if you take the annihilation operator and act it on the state, you will get a different state. But it so happens that that different state is obtained by simply differentiating that state with respect to the eigen value. So, of course, you might think that this seems kind of completely out of the blue; how do I know this? So, I know this because if I go ahead and apply this claim to this state rather than some other state, if I apply to this state. So,

remember I have claimed that a time this is basically this. Now, what is this? This is after all the eigen states. So, I can write it like this. So, first I do this.

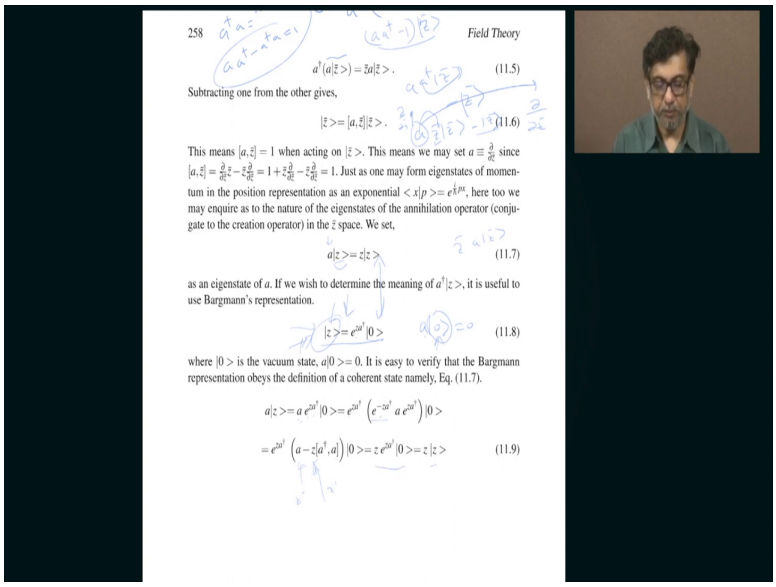
So, I first take a act it on a dagger z; but then what is a dagger z? It is z with a bar times z state z with a bar. But if I do the reverse, so if I do the reverse, clearly this particular state is yeah. So, I will have to justify this ok. So, yeah. So, this is not a very nice way of writing this. So, the bottom line is that you see what is the claim? The claim is that if I act look just take this here, you act this is the claim; claim.

So, if I act this on a, what is this supposed to be? It is supposed to be this ok; but then, what is a dagger a? It is basically 1 plus a dagger, rather it is minus see a dagger minus a dagger a is 1 that is the commutator of a and a dagger is 1. So, it is basically. So, basically a dagger a is a dagger minus 1.

So, this is basically a dagger minus 1 acting on z state z ok. So, the point is that if I take this. So, what is this? This is basically a acting a dagger acting on state z minus state z, but what is this one equal to? This is basically a acting on z minus state z ok. So, now, if I say that the action of this on this state is same as d by d z, then you see clearly yeah. So, this will clearly give me what I am looking for. So, a the action of a on this state is same as acting this on that state.

So, what is d by d z bar acting on this state? First, it will act on this it will give me z, but that is getting cancelled by this. So, then, this a will go and sit in the middle here. So, that is basically it will tell me it is z a times z ok. So, basically what it is saying is that this particular state.

(Refer Slide Time: 26:03)



So, in other words, this is it is consistent with this identity ok. So, the idea is that these two are consistent; but you can also do it the other way. So, you can introduce a coherent state with respect to the eigen values of the annihilation operator rather than the creation operator ok.

So, in that case, you can go ahead and write this operator in terms of this state which is annihilated by a and you can show that this is these two are consistent. So, I am going to allow some of these statements to be proved in the exercises perhaps because it is kind of confusing for me to explain everything verbally.

So, I think you just have to work it out. So, for example, here I have worked it out. So, if you see the claim is that the. So, these are called coherent states; for reasons, I will again explain later on what is. So, coherent about it. So, the point is that right now for our purposes, it is just a state, which is an eigenstate of the annihilation operator for example this one. See the earlier one was eigenstate of the creation operator. So, these two are you know in some sense dual to each other. So, the point is that this is the eigenstate of the annihilation operator.

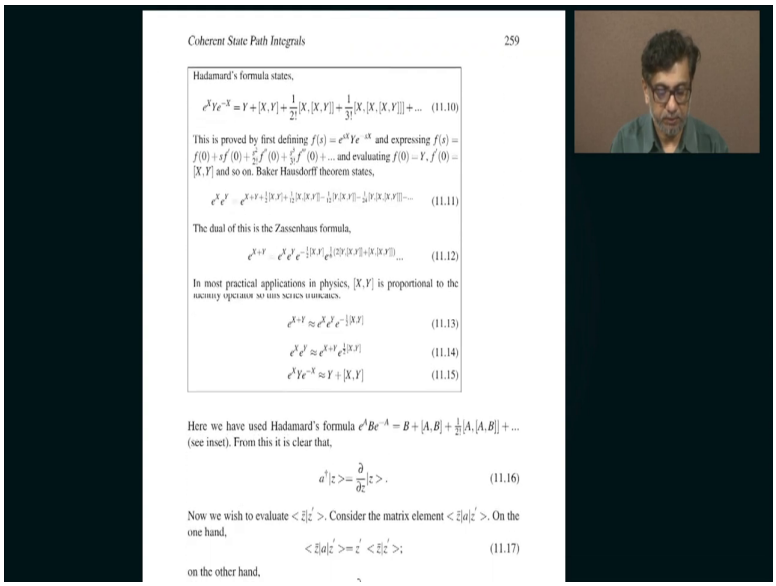
So, the claim is that this is this state can also be explicitly constructed this way. So, see we all know how to what this means. This is the ground state of the harmonic oscillator.

So, it is annihilated by the annihilation operator. So, what this says is that you know formally expand this out in powers of z and act it on the ground state of the harmonic oscillator and whatever state you get is the coherent state. So, whatever state you get will be an eigenstate of the annihilation operator with this complex eigen values z. So, the question is how do we show that?

So, you simply write this in terms of a times this and then, you do your normal means you multiply and divide by this operator and then, you just expand in powers of z, you will see that all terms drop out except the linear term. There was 0th power z to the power 0 survives z to the power one survives all others drop out because you know the commutator of.

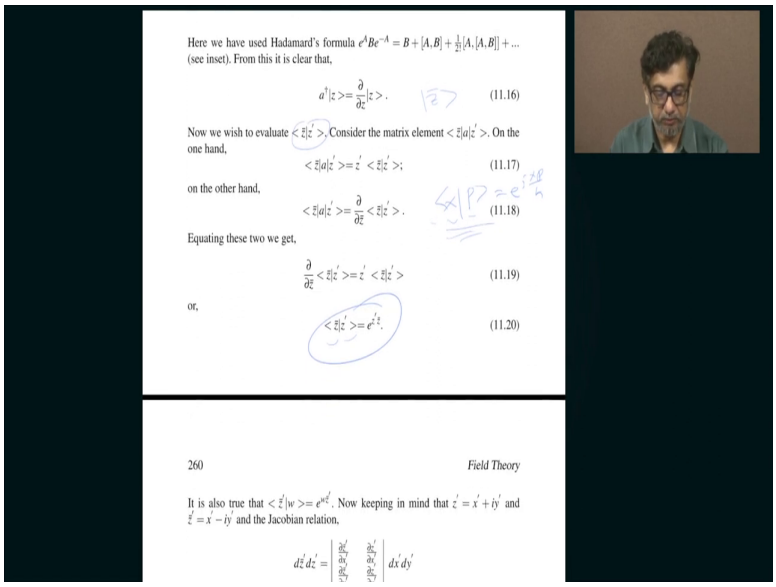
So, if you have a commutator of a and a dagger, it is just proportional to identity. So, all further commutators become 0. So, it is z to the power 0, z to the power 1. So, now, if you work this out, it is simply this. So, that proves; it is very easy to prove this.

(Refer Slide Time: 28:55)



So, basically this is the eigenstate of the annihilation operator. So, it is the coherent state.

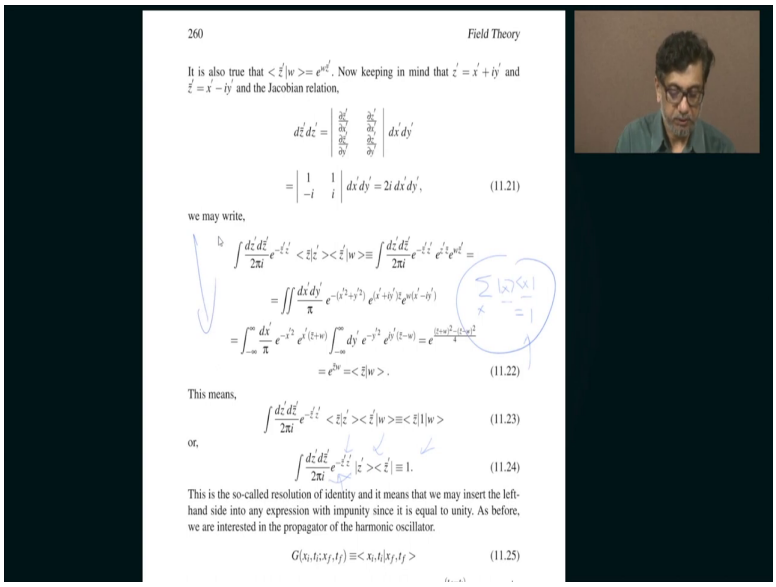
(Refer Slide Time: 28:57)



So, you can also work out this overlap. You see I told you that this state is the eigenstate of the creation operator. So, how are they related? So, they are related in a similar way to how x and p states; remember how this is its e raise to i x p by h bar. So, if you have eigenstate of momentum and eigenstate of position, these are somehow conjugates to each other and their overlap is basically exponential of the two Eigen, product of the two eigen values.

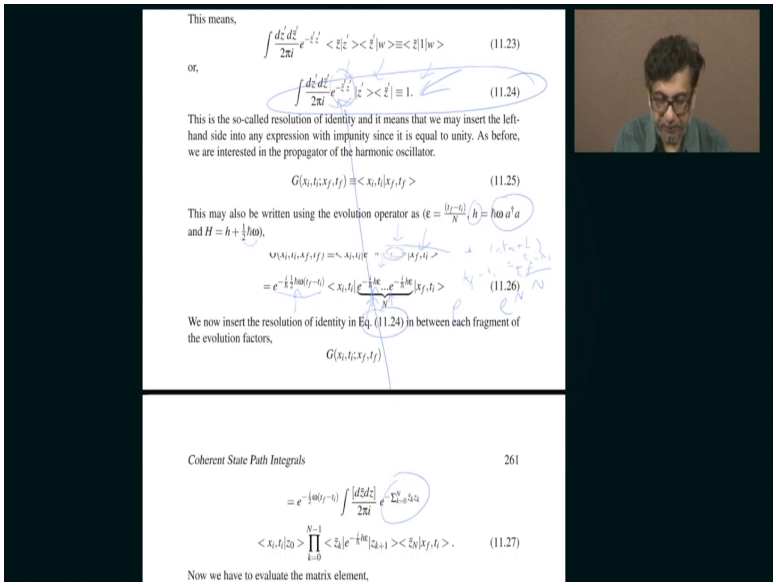
So, here also you get something similar. So, if you work this out, you will see that the overlap of the eigenstate of the annihilation operator and the overlap of the eigenstate of the position, I mean creation operator are very similar to overlap of position and momentum operators that is basically the exponential of the product of the eigen values.

(Refer Slide Time: 29:56)



But you can also go ahead and prove complete. So, yeah. So, there is something remember that there is something called completeness. So, if you have a set of if you have Hermitian operators you can show that the eigenstates of self-adjoint operator are complete. In the sense that you can write down basis in terms of those eigenstates of such a self-adjoint operator; but here, a and a dagger are not self adjoint. So, they are actually complex.

(Refer Slide Time: 30:37)



So, in some sense that contain more information than a self-adjoint because self-adjoint operators are in some sense real operators. So, here the it is a complex operator which has real and imaginary parts. So, the eigenstates of this are probably there is probably lot of you can suspect that there will be a lot of duplicacy; that means, that there will be unnecessary duplication of the states that they will not all be linearly independent and in fact that is in fact true. So, in fact, the eigenstates of these operators are basically called over complete.

So, if you want to introduce a resolution of identity, the way you write like this. So, remember you write this sort of thing. So, if x is the position operator and this is the eigenstate of position you can go ahead and write like this resolution of identity. So, you can do similar things with this.

But then that means, that you have to introduce a weight which compensates for the fact that these states are over complete. So, you can show that this is how you have to resolve the identity. So, I will allow you to work this out and perhaps the exercises or if I do not get around to that you have to simply follow these steps. So, just follow all these steps.

So, now, you see the original question I wanted to ask was answer was how do I study the Lagrangian version of the quantum particle using not x and t, but using creation and annihilation operators. See I know how to study the quantum harmonic oscillator using the quantum versions of x and p. I know how to study the quantum harmonic oscillator using the classical versions of x and p, but in a path integral form. That means, you see I want to study only the quantum harmonic oscillator, I do not want to study classical harmonic oscillator.

But if I want to study quantum harmonic oscillator using x and p operators, then of course, I have to use the Hamiltonian approach. But if I want to study the quantum harmonic oscillator using Lagrangians, then I should use the classical Lagrangian, but I have to remember to do a path integral. That means, that it is not no longer the Euler Lagrange equation, it is the sum over all paths, with each path being weighted by an appropriate factor, which is proportional to the exponential of the action multiplied by imaginary unit divided by h bar.

So, similarly, here now suppose I want to study the quantum harmonic oscillator using creation and annihilation operators, I would certainly be using the Hamiltonian approach. So, that has the advantage of immediately giving me the eigen values for free because it is just Hamiltonian is h bar omega into a dagger a plus half and a dagger a is integer eigenvalues.

So, it is n plus half. So, that is easy. So, but the question is suppose I want to study the quantum harmonic oscillator using the Lagrangian obtained from the Hamiltonian which is written down in terms of creation annihilation operator, not in terms of x and p. So, now. So, in other words, the de quantized version of a and a dagger. So, how do I do that? So, that is what we are trying to answer.

So, the answer to that is we have to develop a path integral approach which now involves the de quantized versions of the creation annihilation operator which are simply complex numbers. So, I have to introduce. So, just like there I have to introduce the eigenstates of position and momentum to resolve the identity and insert a sequence of states in between.

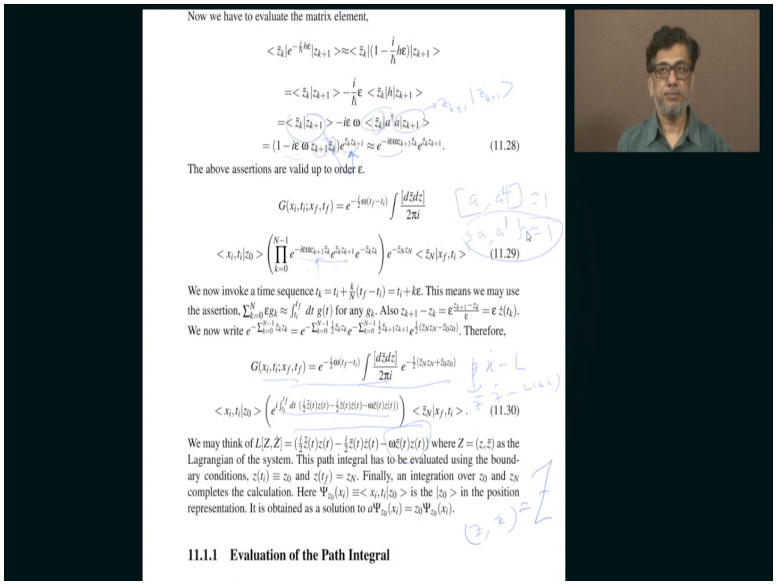
So, I divide up the time interval into small pieces and then, insert identities by resolving identities and so on. I have to do the same here, but now I have to insert the resolution of identities in terms of the coherent states which is why I require this over complete resolution of the identity; I mean resolution of the identity using coherent states.

So, as usual, I start off with this question, if I want to calculate this Green's function of the system this is what it is. So, now as usual I split this up into smaller pieces and I end up getting yeah. So, what is this h? So, remember h is basically h bar omega into a dagger a plus half. So, this h is this small letter h is h bar omega into a dagger a.

So, this plus half omega I have put it outside. So, that is anyway constant factor. So, now you see I write t f minus t i as t f minus t i by n and etcetera. So, I split this up into many pieces; sigma n pieces. So, that is what that is. So, and this will appear in the exponent. So, sigma exponential sigma is the product. So, I will get product here.

So, bottom line is that you see now I go ahead and insert my resolution of the identity; that means, this one and what is this one? This one. So, I keep inserting the resolution of the identity here and then, I will give this get this over complete weighting factor from here.

(Refer Slide Time: 37:15)


Now we have to evaluate the matrix element,

$$< \bar{z}_k|e^{-\frac{i}{\hbar}\mathcal{H}\varepsilon}|z_{k+1}> \approx < \bar{z}_k|(1-\frac{i}{\hbar}\mathcal{H}\varepsilon)|z_{k+1}>$$

$$= < \bar{z}_k|z_{k+1}> - \frac{i}{\hbar}\varepsilon < \bar{z}_k|H|z_{k+1}>$$

$$= < \bar{z}_k|z_{k+1}> - i\varepsilon\,\omega < \bar{z}_k|a^\dagger a|z_{k+1}>$$

$$= (1 - i\varepsilon\,\omega\,\bar{z}_{k+1}\bar{z}_k)e^{\bar{z}_k z_{k+1}} \approx e^{-i\varepsilon\omega\bar{z}_{k+1}z_k}e^{\bar{z}_k z_{k+1}}. \qquad (11.28)$$

The above assertions are valid up to order ε.

$$G(x_i,t_i;x_f,t_f) = e^{-\frac{i}{2}\omega(t_f-t_i)}\int\frac{[d\bar{z}dz]}{2\pi i}$$

$$< x_i,t_i|z_0> \left(\prod_{k=0}^{N-1} e^{-i\varepsilon\omega\bar{z}_{k+1}z_k}e^{\bar{z}_k z_{k+1}}e^{-\bar{z}_k z_k}\right) e^{-\bar{z}_N z_N} < \bar{z}_N|x_f,t_f > \qquad (11.29)$$

We now invoke a time sequence $t_k = t_i + \frac{k}{N}(t_f - t_i) = t_i + k\varepsilon$. This means we may use the assertion, $\sum_{k=0}^{N}\varepsilon g_k \approx \int_{t_i}^{t_f} dt\; g(t)$ for any g_k. Also $z_{k+1} - z_k = \varepsilon\frac{z_{k+1}-z_k}{\varepsilon} = \varepsilon\,\dot{z}(t_k)$. We now write $e^{-\sum_{k=0}^{N-1}\bar{z}_k z_k} = e^{-\sum_{k=0}^{N-1}\frac{1}{2}\bar{z}_k z_k}e^{-\sum_{k=0}^{N-1}\frac{1}{2}\bar{z}_{k+1}z_{k+1}}e^{\frac{1}{2}(\bar{z}_N z_N - \bar{z}_0 z_0)}$. Therefore,

$$G(x_i,t_i;x_f,t_f) = e^{-\frac{i}{2}\omega(t_f-t_i)}\int\frac{[d\bar{z}dz]}{2\pi i}\,e^{-\frac{1}{2}(\bar{z}_N z_N + \bar{z}_0 z_0)}$$

$$< x_i,t_i|z_0> \left(e^{\int_{t_i}^{t_f} dt\,(\frac{i}{2}\dot{\bar{z}}(t)z(t) - \frac{i}{2}\bar{z}(t)\dot{z}(t) - i\omega\bar{z}(t)z(t))}\right) < \bar{z}_N|x_f,t_f > . \qquad (11.30)$$

We may think of $L[Z,\dot{Z}] = (\frac{i}{2}\dot{\bar{z}}(t)z(t) - \frac{i}{2}\bar{z}(t)\dot{z}(t) - i\omega\bar{z}(t)z(t))$ where $Z = (z,\bar{z})$ as the Lagrangian of the system. This path integral has to be evaluated using the boundary conditions, $z(t_i) \equiv z_0$ and $z(t_f) = z_N$. Finally, an integration over z_0 and z_N completes the calculation. Here $\Psi_{z_0}(x_i) \equiv < x_i,t_i|z_0 >$ is the $|z_0>$ in the position representation. It is obtained as a solution to $a\Psi_{z_0}(x_i) = z_0\Psi_{z_0}(x_i)$.

11.1.1 Evaluation of the Path Integral

And then this is what this is ok. So, we are not done yet because I have to show that this is in some sense still it is related to the Lagrangian of the system. So, this gets multiplied by the rest of it ok. So, it is a it gets multiplied by this successive terms. So, this is the over complete terms times the remaining; the remaining is this. So, this is similar to what we did earlier. So, here you see the idea is the following because epsilon is small because what is epsilon t f minus t i by n and n is very large. So, epsilon is small. So, this is approximately this.

So, now if I go ahead and write it this way, then you can clearly show that since h is h bar omega into a dagger a this is clearly given by this. So, this is the eigenstate of a. So, therefore, it is just z z k plus 1 right. So, and this is the eigen eigenstate of the other one.

So, it is basically it gives you this ok; yeah. So, it just gives you back this times this. So, this is the eigen value. So, this is just z this is nothing but z k plus 1 into z k. It just gives

you the eigen value; similarly, this also just gives you the eigen value. So, clearly that it is equal to and what is this? This is equal to this. These two are the same things.

So, this overlap is basically remember I told you this is the overlap. So, the overlap of z and z bar is basically this one and that appears here also is after all the eigen values come outside these two are the same things. So, that comes out. So, this is again, you can re exponentiate because epsilon is small; this is equal to this.

So, now, you can go ahead and write it this way ok. So, it is going to be like this ok. So, then you identify this, this discrete sum with basically a kind of a discretized version of an integral. So, to cut a long story short if you discretize this integral, it gives you back this expression.

So, I have done the reverse; you can start from here and get there that is easier. So, discretize this integral, you will end up with this because the product is exponential of this sum. So, the thing is that now we may think of this as a Lagrangian of the system ok. Because this has this is something like you know p x dot minus L types; I mean instead of p, you have z dash; instead of x dot, you have z dot.

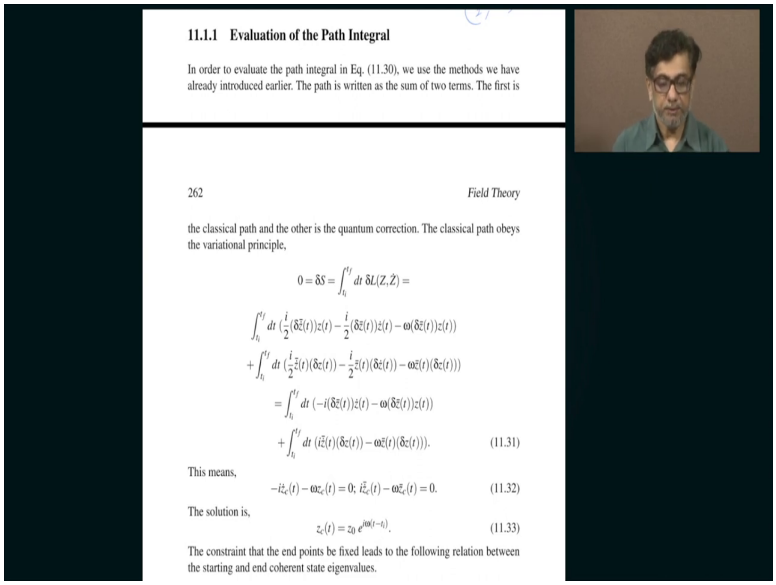
This is L z z dash. So, it is like that. So, its z z bar minus L and this is L right. So, its omega into a dagger a and a is a eigenvalue is z a dagger eigenvalue is z with a bar there. So, basically this is the Lagrangian generalized coordinate is now z z bar which is capital Z. So, that is your generalized coordinate. So, now, you have to simply.

So, the Green's function can also be written as a coherent state path integral. So, this is the coherent state path integral. So, it is a path integral in terms of coherent states ok yeah. So, you might be thinking that why am I doing this. Because I can solve harmonic oscillator quite nicely using just Hamiltonian approach; why do I want a Lagrangian approach in that too in terms of creation and annihilation de quantized versions of the creation and annihilation operators. See the reason for that is because in see all the modern relativistic quantum field theories are actually thought of as the coherent state path integral.

So, they are always phrased in terms of a coherent state path integral. So, they are written in terms of; so, all your quantum electrodynamics in. So, you write them in terms of so, but then, for that I have to introduce the coherent state path integral for fermions. So, till now I have only see this a a dagger as this commutator is 1; I have to do coherent state path integral when the anti commutator is 1. So, that I will do next.

But bottom line is that if once you are successful in doing this, then you can put in a spatial dependence to make it fields and once you make it up field, then you can describe you see electron field that is the field of charged particles whose excitations of are electrons and then, you can express the dynamics of such a field; the quantum dynamics of such a field as a coherent state path integral over these fields now.

(Refer Slide Time: 42:27)



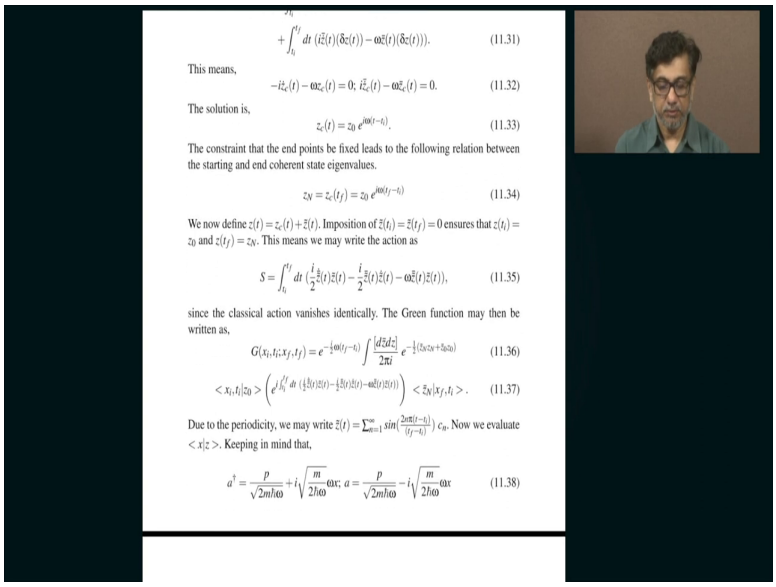
So, that is nice because it allows for a kind of a simpler description of see the matter and forces can be treated on an equal footing in a more elegant way, when you do that because. So, I would not get into the actual deeper motivations for why people do that, but you know once you start using it you will see its utility to some extent.

But in the end, it is true that many of these theories are you know I mean these kinds of changes in perspective are somewhat beneficial; but in a very deep sense they are still very cosmetic. Because it is not as if you can solve for the properties of interacting

systems simply by transforming your perspective from a Hamiltonian to a Lagrangian framework.

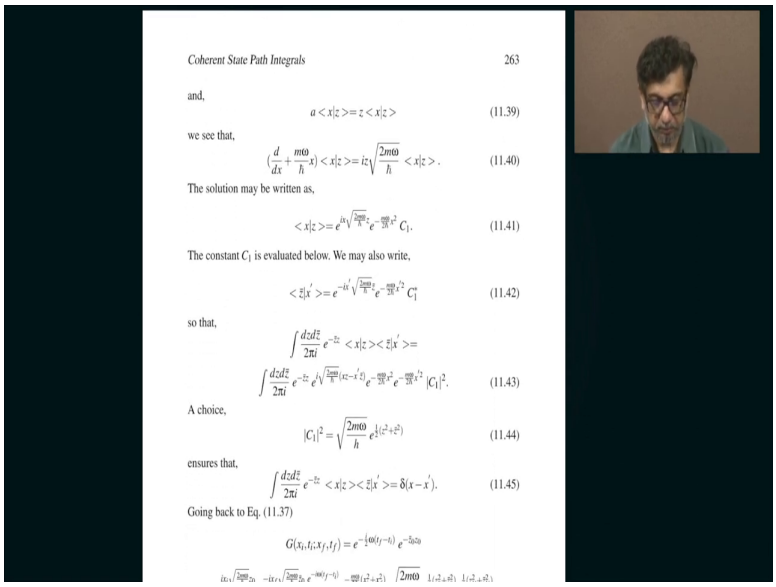
So, the fundamental problems namely that you are dealing with a strongly coupled system and so on that will still remain and there is nothing much you can do about it. So, ok. So, in an. So, basically what I have done next is I have shown you how to evaluate this path integral. So, remember in the case of quantum harmonic oscillator in terms of these actual physical paths x as a function of time, I had shown how to evaluate the path integral from using some saddle point method.

(Refer Slide Time: 44:04)



So, here also I should be able to show you that it can be evaluated and you get what you expect from traditional means in terms of the Green's function should agree with that not only you should agree with the x and p path integral, should agree with the Hamiltonian version of the Green's function also.

(Refer Slide Time: 44:26)



Coherent State Path Integrals 263

and,

$$a<x|z>=z<x|z> \tag{11.39}$$

we see that,

$$(\frac{d}{dx}+\frac{m\omega}{\hbar}x)<x|z>=iz\sqrt{\frac{2m\omega}{\hbar}}<x|z>. \tag{11.40}$$

The solution may be written as,

$$<x|z>=e^{ix\sqrt{\frac{2m\omega}{\hbar}}z}e^{-\frac{m\omega}{2\hbar}z^2}C_1. \tag{11.41}$$

The constant C_1 is evaluated below. We may also write,

$$<\bar{z}|x'>=e^{-ix'\sqrt{\frac{2m\omega}{\hbar}}\bar{z}}e^{-\frac{m\omega}{2\hbar}\bar{z}^2}C_1^* \tag{11.42}$$

so that,

$$\int\frac{dzd\bar{z}}{2\pi i}e^{-\bar{z}z}<x|z><\bar{z}|x'>=$$

$$\int\frac{dzd\bar{z}}{2\pi i}e^{-\bar{z}z}e^{i\sqrt{\frac{2m\omega}{\hbar}}(xz-x'\bar{z})}e^{-\frac{m\omega}{2\hbar}x^2}e^{-\frac{m\omega}{2\hbar}x'^2}|C_1|^2. \tag{11.43}$$

A choice,

$$|C_1|^2=\sqrt{\frac{2m\omega}{\hbar}}e^{\frac{1}{2}(z^2+\bar{z}^2)} \tag{11.44}$$

ensures that,

$$\int\frac{dzd\bar{z}}{2\pi i}e^{-\bar{z}z}<x|z><\bar{z}|x'>=\delta(x-x'). \tag{11.45}$$

Going back to Eq. (11.37)

$$G(x_i,t_i;x_f,t_f)=e^{-\frac{i}{2}\omega(t_f-t_i)}e^{-\frac{i}{\hbar}2\omega_0}$$

$$e^{ix_i\sqrt{\frac{2m\omega}{\hbar}}z_0}e^{-ix_f\sqrt{\frac{2m\omega}{\hbar}}z_0}e^{-i\omega(t_f-t_i)}e^{-\frac{m\omega}{2\hbar}(x_i^2+x_f^2)}\sqrt{\frac{2m\omega}{h}}e^{\frac{1}{4}(z_0^2+\bar{z}_0^2)}e^{\frac{1}{2}(z_0\bar{z}_0+\bar{z}_0z_0)}$$

$$\int\frac{dzd\bar{z}}{2\pi i}\left(e^{\int_{t_i}^{t_f}dt\,(\frac{1}{2}\dot{\bar{z}}(t)\bar{z}(t)-\frac{1}{2}\dot{z}(t)\bar{z}(t)-\omega\bar{z}(t)\bar{z}(t))}\right). \tag{11.46}$$

We may now evaluate the same quantity using conventional Hamiltonian methods. For this we invoke the occupation number basis.

$$G(x_i,t_i;x_f,t_f)\equiv<x_i,t_i|e^{-\frac{i}{\hbar}H(t_f-t_i)}|x_f,t_i>$$

$$=\sum_{n=0}^{\infty}<x_i,t_i|n><n|x_f,t_i>e^{-\frac{i}{\hbar}(n+\frac{1}{2})\hbar\omega(t_f-t_i)}$$

264 Field Theory

$$<x_i,t_i|n>=\frac{(-\frac{i\hbar}{\sqrt{2m\hbar\omega}}\frac{\partial}{\partial x_i}+i\sqrt{\frac{m}{2m\omega}}\omega x_i)^n}{\sqrt{n!}}<x_i|0>$$

$$<n|x_f,t_i>=\frac{(\frac{i\hbar}{\sqrt{2m\hbar\omega}}\frac{\partial}{\partial x_f}-i\sqrt{\frac{m}{2m\omega}}\omega x_f)^n}{\sqrt{n!}}<0|x_f> \tag{11.47}$$

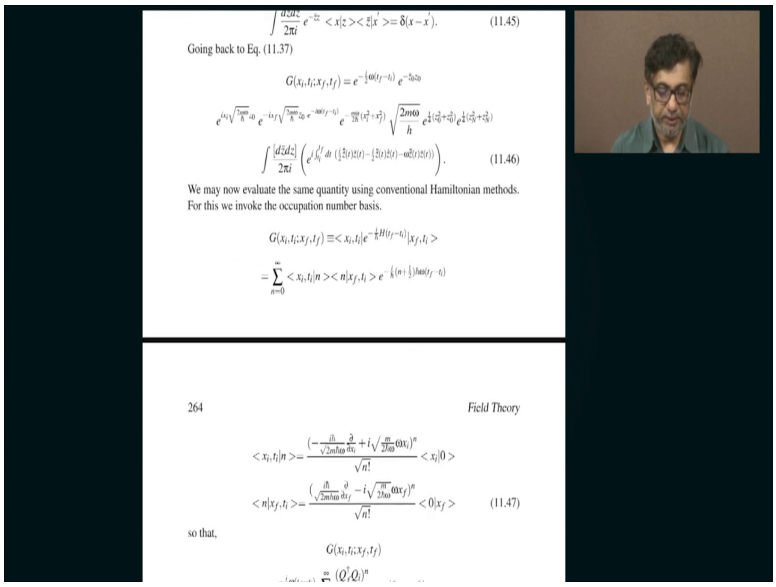
so that,

$$G(x_i,t_i;x_f,t_f)$$

$$\sum^{\infty}(Q_i^{\dagger}Q_i)^n$$

So, I am going to spend the next lecture probably explaining to you how that comes about.

(Refer Slide Time: 44:32)



$$\int\frac{dzd\bar{z}}{2\pi i}e^{-\bar{z}z}<x|z><\bar{z}|x'>=\delta(x-x'). \tag{11.45}$$

Going back to Eq. (11.37)

$$G(x_i,t_i;x_f,t_f)=e^{-\frac{i}{2}\omega(t_f-t_i)}e^{-\frac{i}{\hbar}2\omega_0}$$

$$e^{ix_i\sqrt{\frac{2m\omega}{\hbar}}z_0}e^{-ix_f\sqrt{\frac{2m\omega}{\hbar}}z_0}e^{-i\omega(t_f-t_i)}e^{-\frac{m\omega}{2\hbar}(x_i^2+x_f^2)}\sqrt{\frac{2m\omega}{h}}e^{\frac{1}{4}(z_0^2+\bar{z}_0^2)}e^{\frac{1}{2}(z_0\bar{z}_0+\bar{z}_0z_0)}$$

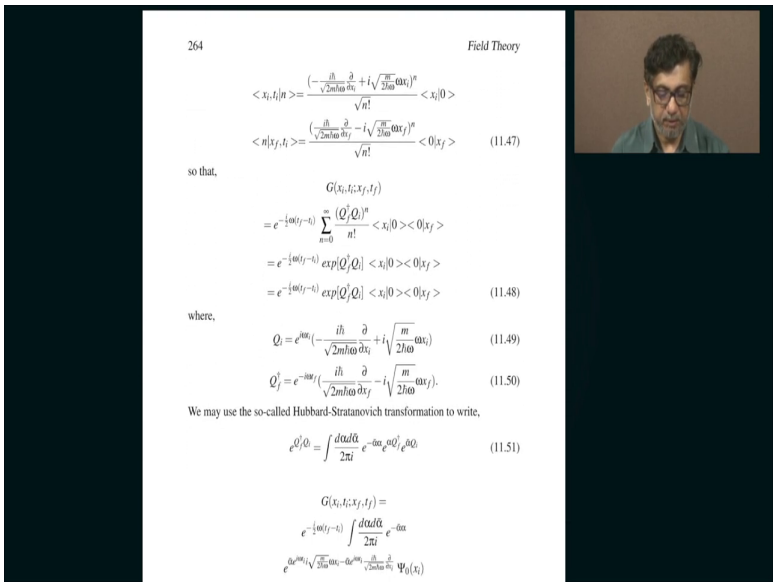
$$\int\frac{dzd\bar{z}}{2\pi i}\left(e^{\int_{t_i}^{t_f}dt\,(\frac{1}{2}\dot{\bar{z}}(t)\bar{z}(t)-\frac{1}{2}\dot{z}(t)\bar{z}(t)-\omega\bar{z}(t)\bar{z}(t))}\right). \tag{11.46}$$

We may now evaluate the same quantity using conventional Hamiltonian methods. For this we invoke the occupation number basis.

$$G(x_i,t_i;x_f,t_f)\equiv<x_i,t_i|e^{-\frac{i}{\hbar}H(t_f-t_i)}|x_f,t_i>$$

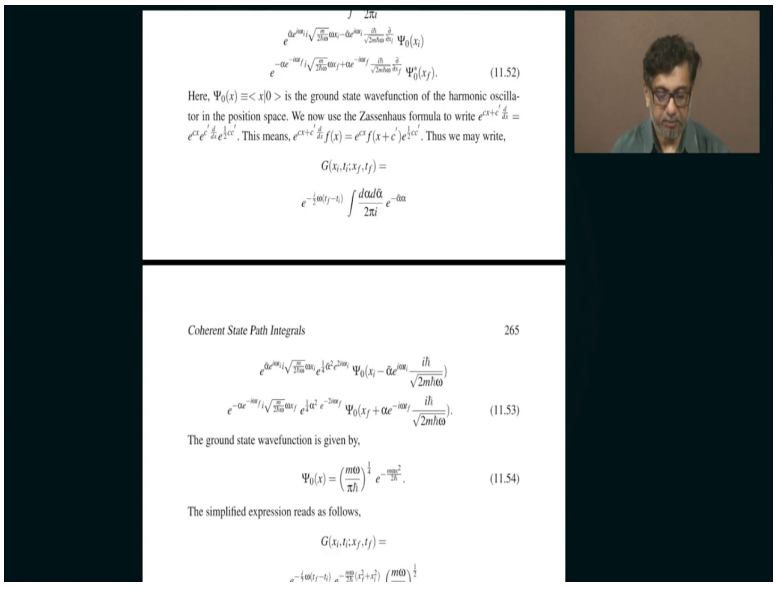
$$=\sum_{n=0}^{\infty}<x_i,t_i|n><n|x_f,t_i>e^{-\frac{i}{\hbar}(n+\frac{1}{2})\hbar\omega(t_f-t_i)}$$

(Refer Slide Time: 44:34)



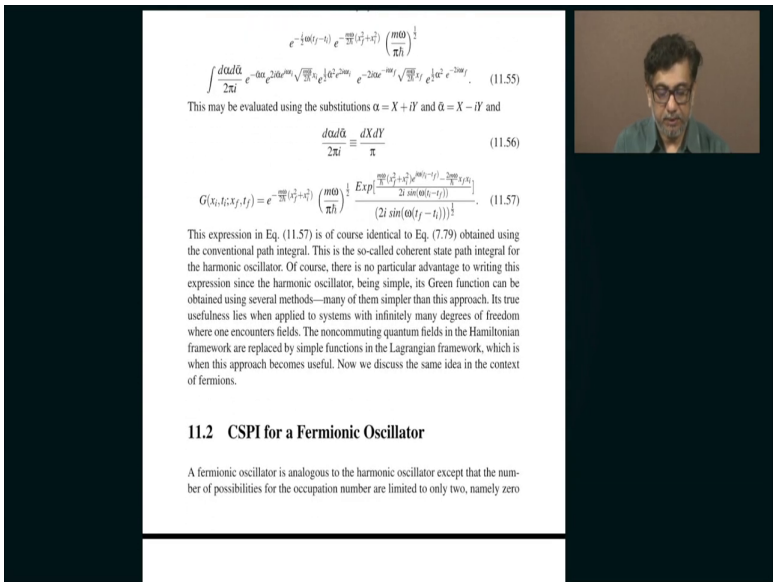
But in the meanwhile, you should go ahead and read all this quite I mean the technical details can be somewhat overwhelming.

(Refer Slide Time: 44:38)



Because I do not know how much I can explain to you.

(Refer Slide Time: 44:43)



$$e^{-\frac{i}{2}\omega(t_f-t_i)}\, e^{-\frac{m\omega}{2\hbar}(x_f^2+x_i^2)} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}$$

$$\int \frac{d\alpha d\bar{\alpha}}{2\pi i}\, e^{-\bar{\alpha}\alpha} e^{\sqrt{\frac{m\omega}{2\hbar}}\bar{\alpha}x_i} e^{\sqrt{\frac{m\omega}{2\hbar}}x_i\alpha} e^{\frac{1}{2}\bar{\alpha}^2 e^{i\omega t_i}} e^{-2\bar{\alpha}\alpha e^{-i\omega t_f}} \sqrt{\frac{m\omega}{2\hbar}}x_f\, e^{\frac{1}{2}\alpha^2 e^{-2i\omega t_f}}. \quad (11.55)$$

This may be evaluated using the substitutions $\alpha = X + iY$ and $\bar{\alpha} = X - iY$ and

$$\frac{d\alpha d\bar{\alpha}}{2\pi i} \equiv \frac{dXdY}{\pi} \quad (11.56)$$

$$G(x_i,t_i;x_f,t_f) = e^{-\frac{m\omega}{2\hbar}(x_f^2+x_i^2)} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{Exp\left[\frac{\frac{m\omega}{\hbar}(x_f^2+x_i^2)e^{i\omega(t_f-t_i)} - \frac{2m\omega}{\hbar}x_f x_i}{2i\sin(\omega(t_f-t_i))}\right]}{(2i\sin(\omega(t_f-t_i)))^{\frac{1}{2}}}. \quad (11.57)$$

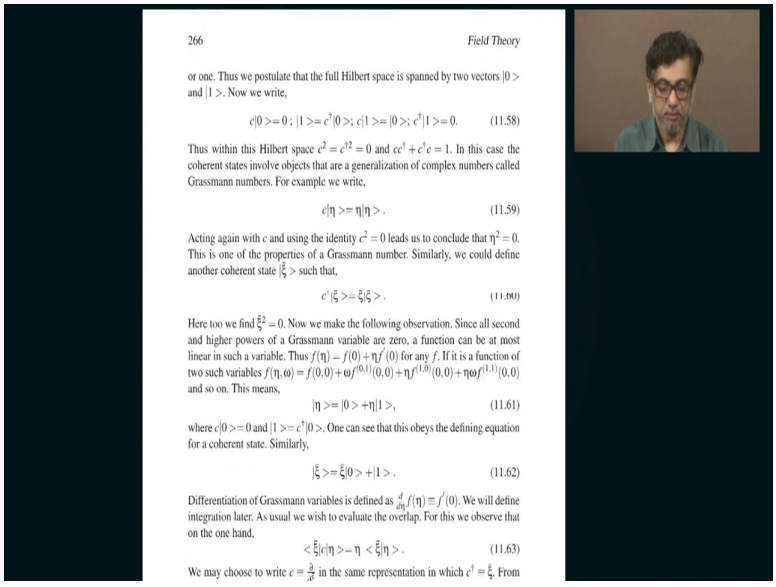
This expression in Eq. (11.57) is of course identical to Eq. (7.79) obtained using the conventional path integral. This is the so-called coherent state path integral for the harmonic oscillator. Of course, there is no particular advantage to writing this expression since the harmonic oscillator, being simple, its Green function can be obtained using several methods—many of them simpler than this approach. Its true usefulness lies when applied to systems with infinitely many degrees of freedom where one encounters fields. The noncommuting quantum fields in the Hamiltonian framework are replaced by simple functions in the Lagrangian framework, which is when this approach becomes useful. Now we discuss the same idea in the context of fermions.

11.2 CSPI for a Fermionic Oscillator

A fermionic oscillator is analogous to the harmonic oscillator except that the number of possibilities for the occupation number are limited to only two, namely zero

Because all the steps are here. So, you should not complain that I have not explained because I have explicitly derived everything nothing is missing. So, you just have to follow all the steps. So, if I even if I verbally describe what is going on, it will pretty much be just saying the same thing that you are seeing here. So, you just have to go ahead and work this out.

(Refer Slide Time: 45:08)



or one. Thus we postulate that the full Hilbert space is spanned by two vectors $|0>$ and $|1>$. Now we write,

$$c|0>=0\,;\ |1>=c^\dagger|0>;\ c|1>=|0>;\ c^\dagger|1>=0. \quad (11.58)$$

Thus within this Hilbert space $c^2 = c^{\dagger 2} = 0$ and $cc^\dagger + c^\dagger c = 1$. In this case the coherent states involve objects that are a generalization of complex numbers called Grassmann numbers. For example we write,

$$c|\eta>=\eta|\eta>. \quad (11.59)$$

Acting again with c and using the identity $c^2 = 0$ leads us to conclude that $\eta^2 = 0$. This is one of the properties of a Grassmann number. Similarly, we could define another coherent state $|\xi>$ such that,

$$c^\dagger|\xi>=\xi|\xi>. \quad (11.60)$$

Here too we find $\xi^2 = 0$. Now we make the following observation. Since all second and higher powers of a Grassmann variable are zero, a function can be at most linear in such a variable. Thus $f(\eta) = f(0) + \eta f'(0)$ for any f. If it is a function of two such variables $f(\eta,\omega) = f(0,0) + \omega f^{(0,1)}(0,0) + \eta f^{(1,0)}(0,0) + \eta\omega f^{(1,1)}(0,0)$ and so on. This means,

$$|\eta>=|0>+\eta|1>, \quad (11.61)$$

where $c|0>=0$ and $|1>=c^\dagger|0>$. One can see that this obeys the defining equation for a coherent state. Similarly,

$$|\xi>=\xi|0>+|1>. \quad (11.62)$$

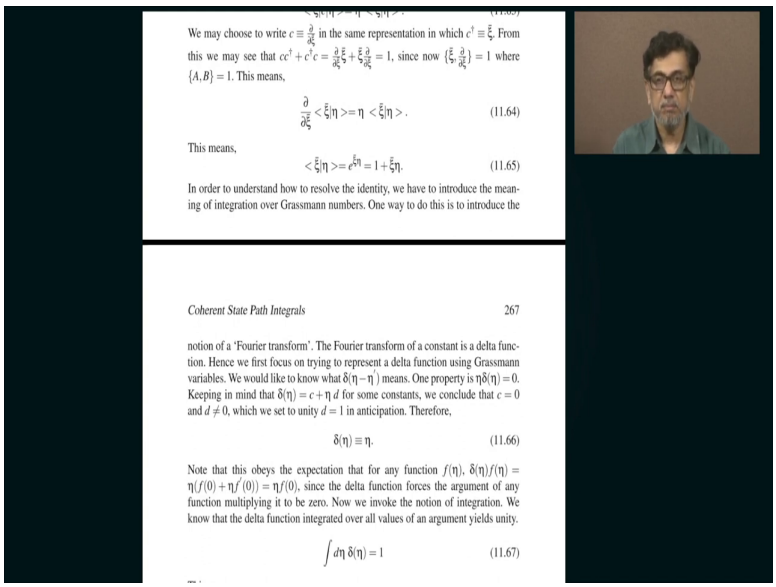
Differentiation of Grassmann variables is defined as $\frac{d}{d\eta}f(\eta) \equiv f'(0)$. We will define integration later. As usual we wish to evaluate the overlap. For this we observe that on the one hand,

$$<\xi|c|\eta>=\eta<\xi|\eta>. \quad (11.63)$$

We may choose to write $c = \frac{\partial}{\partial\xi}$ in the same representation in which $c^\dagger = \xi$. From

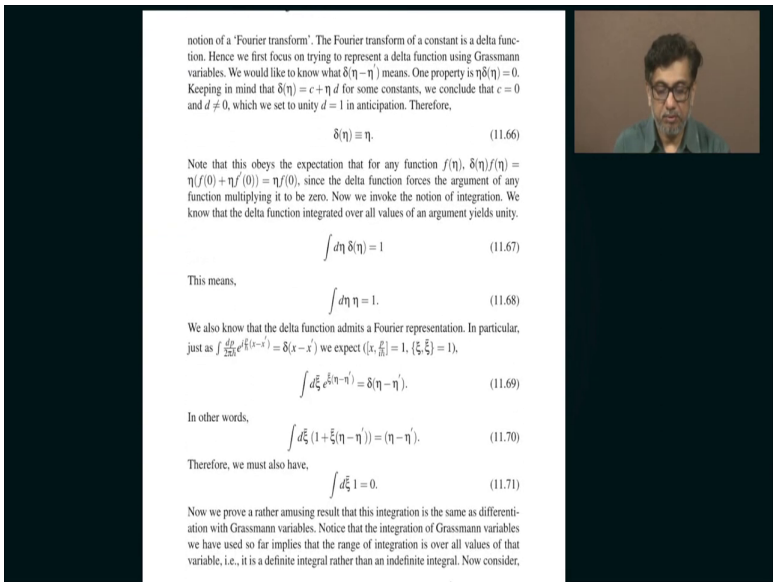
But the more interesting thing will be when I generalize all these two fermions which is really the reason why people do this because you can study matter fields.

(Refer Slide Time: 45:20)



And the fermionic coherent states are very peculiar. So, they involve the eigenvalues of the annihilation operator are not complex numbers they are what are called Grassmann numbers. Grassmann numbers are you know some anti-commuting versions of complex numbers. So, they are complex numbers of a very peculiar kinds. So, there is a 0 complex number whose square is 0. So, that seems like impossible, but actually that is what a Grassmann number is.

(Refer Slide Time: 45:58)



notion of a 'Fourier transform'. The Fourier transform of a constant is a delta function. Hence we first focus on trying to represent a delta function using Grassmann variables. We would like to know what $\delta(\eta-\eta')$ means. One property is $\eta\delta(\eta)=0$. Keeping in mind that $\delta(\eta)=c+\eta\,d$ for some constants, we conclude that $c=0$ and $d\neq 0$, which we set to unity $d=1$ in anticipation. Therefore,

$$\delta(\eta)\equiv\eta. \tag{11.66}$$

Note that this obeys the expectation that for any function $f(\eta)$, $\delta(\eta)f(\eta)=\eta(f(0)+\eta f'(0))=\eta f(0)$, since the delta function forces the argument of any function multiplying it to be zero. Now we invoke the notion of integration. We know that the delta function integrated over all values of an argument yields unity.

$$\int d\eta\,\delta(\eta)=1 \tag{11.67}$$

This means,

$$\int d\eta\,\eta=1. \tag{11.68}$$

We also know that the delta function admits a Fourier representation. In particular, just as $\int\frac{dp}{2\pi\hbar}e^{\frac{i}{\hbar}p(x-x')}=\delta(x-x')$ we expect ($[x,\frac{p}{|p|}]=1$, $\{\xi,\frac{\bar{\xi}}{\xi}\}=1$),

$$\int d\bar{\xi}\,e^{\bar{\xi}(\eta-\eta')}=\delta(\eta-\eta'). \tag{11.69}$$

In other words,

$$\int d\bar{\xi}\,(1+\bar{\xi}(\eta-\eta'))=(\eta-\eta'). \tag{11.70}$$

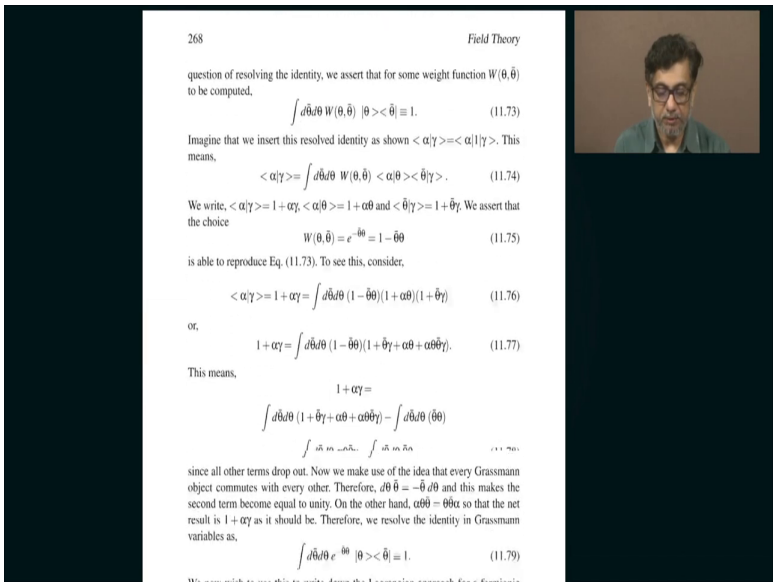
Therefore, we must also have,

$$\int d\bar{\xi}\,1=0. \tag{11.71}$$

Now we prove a rather amusing result that this integration is the same as differentiation with Grassmann variables. Notice that the integration of Grassmann variables we have used so far implies that the range of integration is over all values of that variable, i.e., it is a definite integral rather than an indefinite integral. Now consider,

So, you will be forced to introduce all these bizarre kinds of objects which are called Grassmann numbers.

(Refer Slide Time: 46:01)



268 *Field Theory*

question of resolving the identity, we assert that for some weight function $W(\theta,\bar{\theta})$ to be computed,

$$\int d\bar{\theta}d\theta\,W(\theta,\bar{\theta})\;|\theta><\bar{\theta}|\equiv 1. \tag{11.73}$$

Imagine that we insert this resolved identity as shown $<\alpha|\gamma>=<\alpha|1|\gamma>$. This means,

$$<\alpha|\gamma>=\int d\bar{\theta}d\theta\;W(\theta,\bar{\theta})\;<\alpha|\theta><\bar{\theta}|\gamma>. \tag{11.74}$$

We write, $<\alpha|\gamma>=1+\alpha\gamma$, $<\alpha|\theta>=1+\alpha\theta$ and $<\bar{\theta}|\gamma>=1+\bar{\theta}\gamma$. We assert that the choice

$$W(\theta,\bar{\theta})=e^{-\bar{\theta}\theta}=1-\bar{\theta}\theta \tag{11.75}$$

is able to reproduce Eq. (11.73). To see this, consider,

$$<\alpha|\gamma>=1+\alpha\gamma=\int d\bar{\theta}d\theta\,(1-\bar{\theta}\theta)(1+\alpha\theta)(1+\bar{\theta}\gamma) \tag{11.76}$$

or,

$$1+\alpha\gamma=\int d\bar{\theta}d\theta\,(1-\bar{\theta}\theta)(1+\bar{\theta}\gamma+\alpha\theta+\alpha\theta\bar{\theta}\gamma). \tag{11.77}$$

This means,

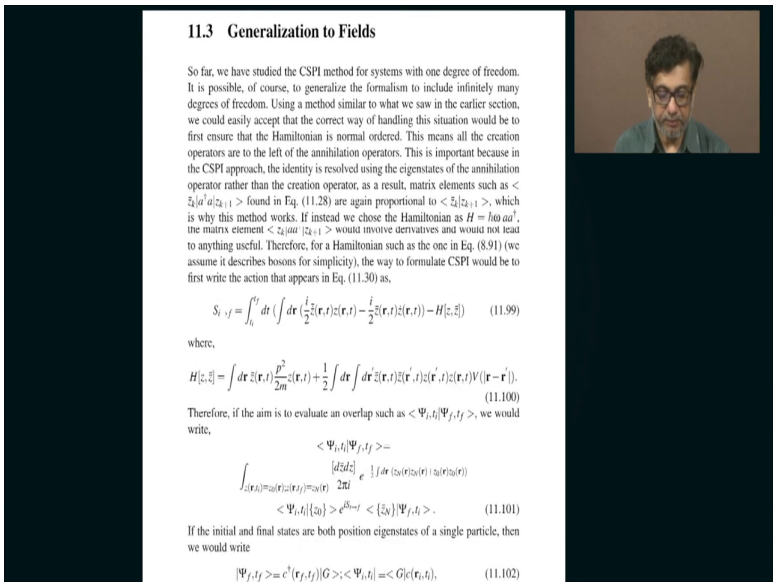
$$1+\alpha\gamma=$$
$$\int d\bar{\theta}d\theta\,(1+\bar{\theta}\gamma+\alpha\theta+\alpha\theta\bar{\theta}\gamma)-\int d\bar{\theta}d\theta\,(\bar{\theta}\theta)$$

since all other terms drop out. Now we make use of the idea that every Grassmann object commutes with every other. Therefore, $d\theta\,\bar{\theta}=-\bar{\theta}\,d\theta$ and this makes the second term become equal to unity. On the other hand, $\alpha\theta\bar{\theta}=\theta\bar{\theta}\alpha$ so that the net result is $1+\alpha\gamma$ as it should be. Therefore, we resolve the identity in Grassmann variables as,

$$\int d\bar{\theta}d\theta\,e^{-\bar{\theta}\theta}\;|\theta><\bar{\theta}|\equiv 1. \tag{11.79}$$

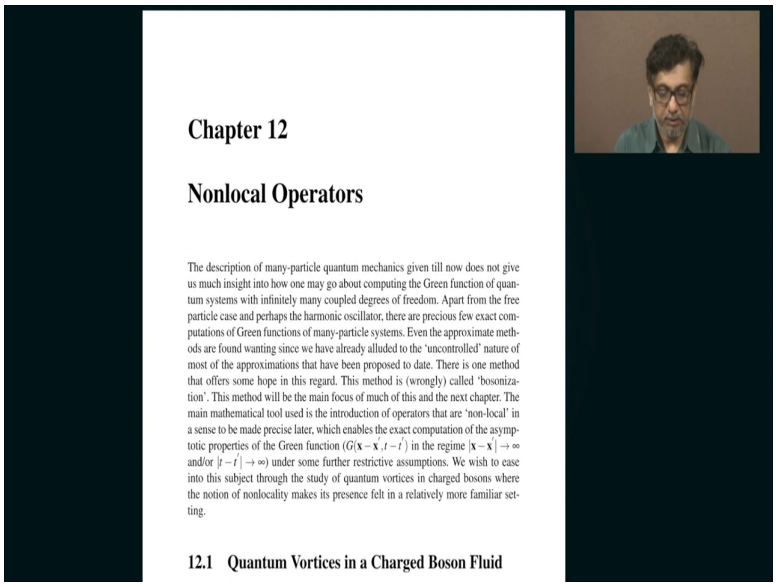
And so, all your path integrals for fermions will involve that.

(Refer Slide Time: 46:07)



And then finally, we will generalize to fields. So, when you generalize to fields, you will understand why I am doing all this or perhaps you will not; but certainly, you will get some inkling as to why I am doing this. So, once we are done with that, we will move on to my favourite topic which is also my research area.

(Refer Slide Time: 46:37)



So, you see most of the top ideas in the later chapters are pretty much subjects, I mean ideas from my own research works. So, I want to spend some time explaining all that

because there are some very important questions that have to be answered which have not yet been answered; but that is something I want people listening to these lectures to be able to contribute and answer. Because some of it, we are already making progress in some of those questions, but there are many questions which are still largely unanswered. So, I need time to describe those issues and that will conclude this course ok.

I am going to stop now. In the next class, I try to wind up this path integrals using coherent states ok.

Thank you.