Dynamics of Classical and Quantum Fields: An Introduction
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Nonequilibrium Green functions
Lecture - 44
Keldysh Contour

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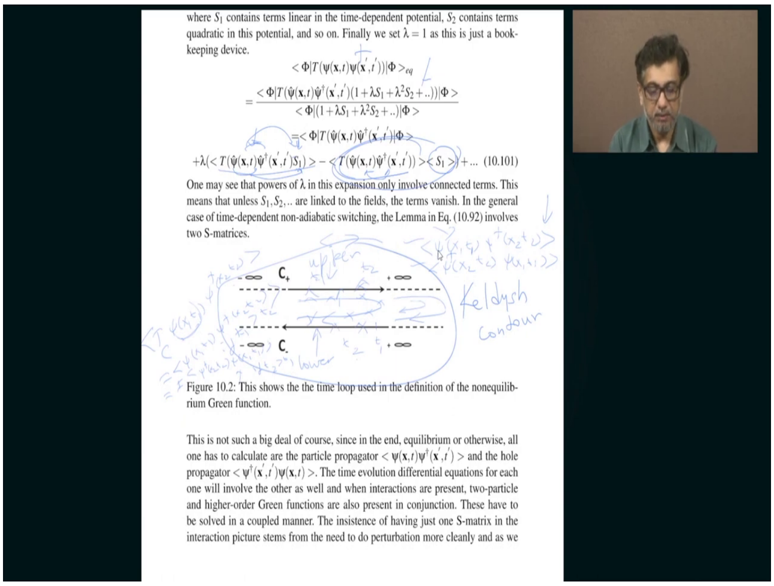

where S_1 contains terms linear in the time-dependent potential, S_2 contains terms quadratic in this potential, and so on. Finally we set $\lambda = 1$ as this is just a book-keeping device.

$$< \Phi|T(\psi(\mathbf{x},t)\psi(\mathbf{x}^{'},t^{'}))|\Phi >_{eq}$$
$$= \frac{< \Phi|T(\hat{\psi}(\mathbf{x},t)\hat{\psi}^{\dagger}(\mathbf{x}^{'},t^{'})(1+\lambda S_1+\lambda^2 S_2+..))|\Phi >}{< \Phi|(1+\lambda S_1+\lambda^2 S_2+..)|\Phi >}$$
$$=< \Phi|T(\hat{\psi}(\mathbf{x},t)\hat{\psi}^{\dagger}(\mathbf{x}^{'},t^{'})|\Phi >$$
$$+\lambda(< T(\hat{\psi}(\mathbf{x},t)\hat{\psi}^{\dagger}(\mathbf{x}^{'},t^{'})S_1) > - < T(\hat{\psi}(\mathbf{x},t)\hat{\psi}^{\dagger}(\mathbf{x}^{'},t^{'})) >< S_1 >) + ... \quad (10.101)$$

One may see that powers of λ in this expansion only involve connected terms. This means that unless $S_1, S_2, ..$ are linked to the fields, the terms vanish. In the general case of time-dependent non-adiabatic switching, the Lemma in Eq. (10.92) involves two S-matrices.

Figure 10.2: This shows the the time loop used in the definition of the nonequilibrium Green function.

This is not such a big deal of course, since in the end, equilibrium or otherwise, all one has to calculate are the particle propagator $< \psi(\mathbf{x},t)\psi^{\dagger}(\mathbf{x}^{'},t^{'}) >$ and the hole propagator $< \psi^{\dagger}(\mathbf{x}^{'},t^{'})\psi(\mathbf{x},t) >$. The time evolution differential equations for each one will involve the other as well and when interactions are present, two-particle and higher-order Green functions are also present in conjunction. These have to be solved in a coupled manner. The insistence of having just one S-matrix in the interaction picture stems from the need to do perturbation more cleanly and as we

Ok. So let us try to see if we can wind up our discussion of non equilibrium Green's functions. So, remember that, I told you that you see the point is that regardless of whether the system is in equilibrium by that I mean you know the system has a Hamiltonian that is time independent. And it could possibly be coupled to some reservoir at some finite temperature or it can be coupled to as reservoir which is not only at a finite temperature, but also at some finite chemical potential.

So, implying therefore, that the system can exchange part both particles and energy with the surroundings. So, if that is the case and even if that is not the case even for an truly isolated system, the quantities of interest especially with respect to the one particle Green's function would necessarily be the particle and whole Green's functions.

Because after all that is all there is to it is. You see you either calculate this or you calculate the other thing. So, this corresponds to the particle Green's function this would correspond to whole Green's function. So, the only difference is that you see the this bracket has different meanings depending upon the situation.

So, if the system is truly isolated this bracket; obviously, means expectation value with respect to ground state, but if the system is not isolated; for example, if the system is a grand canonical a part of a grand canonical ensemble; that means, the system is not only exchanging energy with surroundings, it is also exchanging particles with surroundings.

So, that it you know comes to an equilibrium with a chemical potential and temperature that is common to both the system and the surroundings, then you see the average the brackets here I am talking about for the particle and whole Green's function. They have a different meaning, namely it is not merely the expectation value. It is actually the trace over all possible states of the system with an appropriate weight which is similar to the Boltzmann weight, if it were merely canonical.

So, it is basically e raised to minus beta h minus mu n that is what it would be. So, it is basically it is a weighted average over all the states of the system not merely the ground state. So, that is all you have to keep in mind that basically that is the only difference between a truly isolated system versus a system that is in contact with some surroundings. So, the meaning of the brackets differ, but otherwise it in both situations you are simply called upon to calculate the either the particle Green's function or the whole Green's function that is all there is to it.

See if you are allowed to remove a particle or insert one particle or remove one particle, then after all that is the only thing you can calculate either you calculate the particle Green's function where you insert a particle first or you calculate the whole Green's function where you remove a particle first. But the point is that you see that the fact that there are two different such Green's function that you have to calculate becomes inconvenient when you are trying to solve.

See if your Hamiltonian typically will be not exactly diagonalizable for various reasons and the most serious reason being that particles interact amongst themselves not only do they interact with possible external fields, but they also interact among themselves. So, when they interact among themselves the implication is that you would not be able to handle it. So, you will have to expand in powers of that interaction between particles.

So, when you do that you will necessarily be called upon to perform a perturbation series expansion. Now, when you do that the problem is that it is not that much of a problem. It is just an annoyance the fact that you have two different Green's function that you have to perturb and find the series expansion for is somewhat irritating, but there is a more I mean there is a deeper reason for why it is desirable to have a kind of a single definition for a Green's function which results in these two particle and whole Green's function as special cases.

So, the reason for seeking more compact definition of the one particle Green's function is because in that compact definition will actually then obey an exact equation even when interactions are present so and it will only involve that Green's function and nothing else see right. Now, if you try to write down the equation of motion for the particle Green's function for example.

You will soon find that because the particles interact amongst themselves the equation of motion for the particle. Green's function will start to in involve a Green's function where you are going to be creating two particles and annihilating two particles. So, the one particle Green's function the equation of motion for the one particle Green's function will involve two particle Green's function.

And the equation of motion for the two particle Green's function will involve three particle Green's function and so on and so forth and the endlessly; and that is of course, not convenient. So, it is better to see if there is some way in which you can have a situation where the equation of motion for the one particle Green's function will involve only the one particle Green's function and not anything else.

So, if you want to achieve that goal it is not going to be possible if you do the name thing or the obvious thing which is deal with this particle in whole Green's functions separately. So, it is important in that case to be clever about it. So; that means, we are now going to introduce. Well, I have already introduced that definition its basically called the time ordered Green's function.

So, the time ordered Green's function specifically if especially if you have non equilibrium external fields then what happens is that you are called upon to define them in this peculiar way. So, the idea is that if so, the time ordered Green's function is. So, this C is called the Keldysh contour. So, what you are staring at here is called the Keldysh contour. So, the in the Keldysh contour the times that I am referring to here t 1 and t 2 they basically live on one of these branches.

So, in the implication is that if suppose t 2 is here on this branch this is called the lower branch that is called the upper branch for obvious reasons. So, this is upper, and this is lower. So, if t 2 is on the lower branch regardless of where t 1 is on the. So, long as it is on the upper branch it does not matter where t 1 is t 2 is always greater than t 1.

So, if t 2 is on the lower branch and t 1 is on the upper branch, it does not matter what the relative magnitudes of t 1 and t 2 are ok. So, t 2 will always be greater than t 1. So, whereas, if both t 1 and t 2 are on the same branch then at t 2 is said to be greater than t 1 if it is numerically greater than t 1 and they are both on the upper branch, but t 2 is greater than t 1 if it is to the left of t 1 if it is on the lower branch. So, basically on the upper branch t 2 is greater than t 1 if it is to the right of t 1.

So, the implication is that you are going like this. So, the path of increasing times is like this. So, this time is greater than this time is greater than this. So, this is always greater this is greater, even greater, even greater, even greater, even greater like that. So, you are going like this. So, your this is increasing times time is increasing like this.

So, this is called the Keldysh contour. So, if you decide that your times are in this defined in this peculiar way and then having done that you can always freeze the times to some numerical values so either. So, then what happens is that you see you will end up getting all these particle in whole Green's functions.

So, suppose you want to you do not care about this contour ordered Green's function you just want to extract say the particle Green's function all you do is simply put t two on the lower branch t 1 on the upper branch and you got your particle Green's function. So, if you want the reverse you want a whole Green's function.

So, you see if I put t 1 on the upper branch it is going to be to the left of. So, time ordering means if t 1 is greater than t 2 in the sense of this contour. So, if t 1 is greater than t 2 on the contour, it this is basically psi of x 1 t 1 psi dagger x 2 t 2 if t 1 is greater than t 2 on the contour.

And it is plus or minus the other way side dagger x 2 t 2 psi x 1 t 1, if t 2 is greater than t 1 plus minus depending upon whether its boson or fermion. So, you see this is particle Green's function this is whole Green's function. So, this is on the contour. So, if you really so all you have to do is if you want to get particle Green's function all you have to do is make sure that t 1 is on the lower branch and our t 2 is on the upper branch.

And regardless of the numerical values of t 1 and t 2 you will always be studying the particle Green's function. And suppose you want the whole Green's function you make sure that t 2 is on the lower branch and t 1 is on the upper branch and regardless of now the numerical values. So, t 2 can be as negative or as positive as it wants and independently t 1 can be anything. So, the point is once t 2 is on the lower branch you will always be studying the whole Green's function.

So, long as t 1 is on the upper branch. So, now, if both are on the same branch then you will be getting your usual time ordering. So, if both are on the upper branch it you will get time ordering in the sense in which you normally expect if numerically t 2 is larger then you will get a whole Green's function, and if numerically t 1 is larger you will get the particle Green's function.

Yeah, Am I saying that right? So, if numerically t one is larger then this will be on the left and this will be on the right yeah. So, then you will get the particle Green's function and numerically t 2 is larger, t 2 will come to the left and you will make it the whole Green's function. So, if t 1 and t 2 are on the upper branch the ordering is same as the numerical ordering whichever is numerically larger is also larger in the sense of time ordering.

So, in time ordering means that it time ordering of a bracket t and b bracket t dash is same as a into b if t is greater than t dash its b into a if t dash is greater than t, but that greater and lesser have different meanings depending upon the situation. So, if the times are on two different branches I told you what it is. If they are on the same branch it is the same as numerical ordering if both are on the upper branch.

And it is the anti numerical ordering if they are on the lower branch that by that I mean if t 1 is greater than t 2 numerically, it will be regarded as this t 1 will be regarded as the smaller quantity if both are on the lower branch. See so, the numerical smallness will be the contour largeness if they are on both are on the lower branch.

But if both are on the upper branch then the numerical largeness is same as contour largerness. So, if they are on different branches then the lower branch is always larger than the upper branch ok. So, that is all there is to it. So, that is the Keldysh contour. So, you might think that why are we making a simple situation complicated because after all we had a particle Green's function a whole Green's function. So, we could have simply lived with two different Green's functions.

So, we are struggling to you know somehow combine these two Green's function into one contour ordered Green's function. It is not at all clear that that is worth the effort because it seems like a lot of effort.

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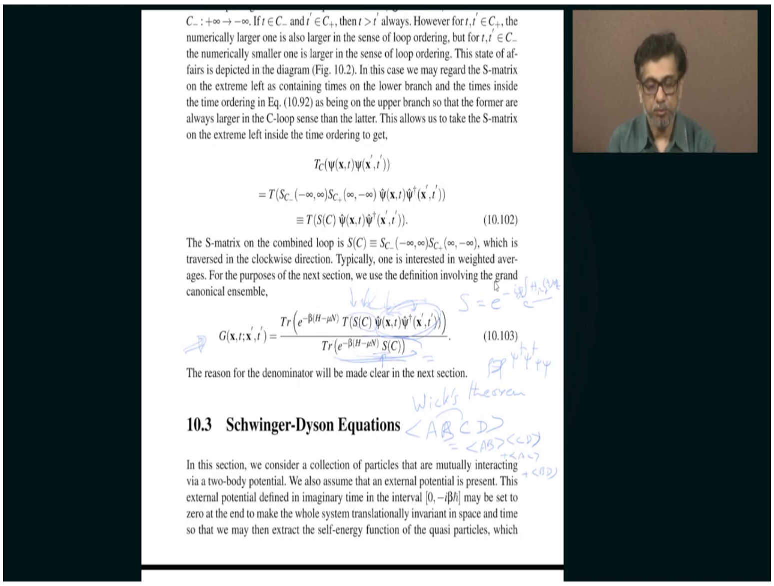

$C_- : +\infty \to -\infty$. If $t \in C_-$ and $t' \in C_+$, then $t > t'$ always. However for $t, t' \in C_+$, the numerically larger one is also larger in the sense of loop ordering, but for $t, t' \in C_-$ the numerically smaller one is larger in the sense of loop ordering. This state of affairs is depicted in the diagram (Fig. 10.2). In this case we may regard the S-matrix on the extreme left as containing times on the lower branch and the times inside the time ordering in Eq. (10.92) as being on the upper branch so that the former are always larger in the C-loop sense than the latter. This allows us to take the S-matrix on the extreme left inside the time ordering to get,

$$\begin{aligned} &T_C(\psi(\mathbf{x},t)\psi(\mathbf{x}',t')) \\ &= T(S_{C_-}(-\infty,\infty)S_{C_+}(\infty,-\infty)\, \hat{\psi}(\mathbf{x},t)\hat{\psi}^\dagger(\mathbf{x}',t')) \\ &\equiv T(S(C)\, \hat{\psi}(\mathbf{x},t)\hat{\psi}^\dagger(\mathbf{x}',t')). \qquad (10.102) \end{aligned}$$

The S-matrix on the combined loop is $S(C) \equiv S_{C_-}(-\infty,\infty)S_{C_+}(\infty,-\infty)$, which is traversed in the clockwise direction. Typically, one is interested in weighted averages. For the purposes of the next section, we use the definition involving the grand canonical ensemble,

$$S = e^{-\frac{i}{\hbar}\int H_{int}(t)dt}$$

$$G(\mathbf{x},t;\mathbf{x}',t') = \frac{Tr\left(e^{-\beta(H-\mu N)}\, T(S(C)\, \hat{\psi}(\mathbf{x},t)\hat{\psi}^\dagger(\mathbf{x}',t'))\right)}{Tr\left(e^{-\beta(H-\mu N)}\, S(C)\right)}. \qquad (10.103)$$

$\psi^\dagger\psi^\dagger\psi\psi$

The reason for the denominator will be made clear in the next section.

Wick's theorem

10.3 Schwinger-Dyson Equations

$\langle ABCD\rangle = \langle AB\rangle\langle CD\rangle + \langle AC\rangle + \langle BD\rangle$

In this section, we consider a collection of particles that are mutually interacting via a two-body potential. We also assume that an external potential is present. This external potential defined in imaginary time in the interval $[0, -i\beta\hbar]$ may be set to zero at the end to make the whole system translationally invariant in space and time so that we may then extract the self-energy function of the quasi particles, which

So, you will see that it is useful because you see in the end what will happen is that if you decide to define contour ordering in this way then you can write down your contour order Green's function in the grand canonical ensemble in this way and then the this is your s matrix with respect to. So, if your fields evolve according to the part of the Hamiltonian that is independent of time the part of the Hamiltonian that depends on time can be absorbed into this S matrix.

And this s matrix is nothing but that part of the Hamiltonian which depends on time and it is again along the contour. So, now, see when you do perturbation the series in this interaction term. So, what this is going to do is that it is basically gives you a whole bunch of terms. And so, there will be a whole bunch of terms in the denominator as well, but the point is that they will systematically cancel out the terms in the numerator where the fields actually couple.

So, in other words that this is same as just doing the numerator expansion, but keep remembering to throw away terms where these two get coupled. So, yeah I know that I have not spent enough time explaining to you how the perturbation series is supposed to be carried out. So, the idea is that you see the point is that when the when you turn off this external fields.

Then typically what will happen is that something called Wick's theorem which is applicable. So, Wick's theorem simply says that if you want to calculate the average of A, B, C, D. So, this is same as calculating the average of A into B times C into D or calculating average of A into C plus B into D like that.

So, it is like all sorts of pairings. So, it basically says that is what it says. So, if mutual interactions between particles are absent then Wick's theorem is applicable. So; that means, so, whether it is 4 or 6 or 8 or whatever it is because that is what will happen here. So, if this is for example, the interaction between particles it will involve density, density.

So, it will be a 4 for beyond interaction we will have sciatica psi and you will get these types of terms and they will all pair like this. So, and one of those pairings will involve pairing these two, but; and so, once you pair up these two and take the average it becomes some number and goes outside and then you will end up having to deal with only the operators that are sitting inside S.

By the whole point is that that will exactly cancel the terms coming from the denominator. So, even though superficially it seems like we have not really achieved the sort of goal that we set out to achieve in the beginning namely that we wanted the S matrix to appear exactly in one place, but it looks like there it still appears in two places.

But now you see that is only superficially. So, because what this means is this is just a shorthand of for saying that even though it superficially appears in two places, the implication is that when you do a perturbation series the terms in the numerator that correspond to coupling of the psi and psi dagger are to be consciously omitted while performing the expansion.

So, once you remember to consciously omit the terms that correspond to coupling of psi and psi dagger, then you simply can afford to ignore this denominator entirely. So, then my original claim is then valid so; that means, the original claim was that this way of doing this way of defining Green's function will only involve s matrix in one place namely in the numerator.

So, but that is a troop. So, long as you remember to omit terms which in this expansion in the perturbation expansion you omit terms which involve pairing psi and psi dagger. So, if you omit those terms then even though superficially it appears that the S matrix is there in both places numerator and denominator it is practically it is there only in the numerator ok.

So, yeah. So, if you are having trouble understanding these issues. Firstly, I do not blame you because it is a lot of there is a lot of details that have either omitted or implied and not explicitly worked out everything. So, that is something that as a student you have to try on your own. And if you are simply unable to understand what I am talking about you really should be reaching out to me by email or asking me questions and you know saying that look I do not know what you are talking about you know explain it to me a little better.

But try and understand whatever I have explained till now as much as possible. So, whatever is remaining I can explain to you separately through email and that sort of thing or your live sessions, but you know I yesterday I had a live session I was somewhat disappointed because I expected very specific technical questions, but most of you did not come prepared to ask me specific technical questions.

So, please come prepare to ask me specific technical questions next time ok. So, the whole point is that this. So, one goal has been achieved namely that this way of defining the Green’s function through this contour ordering in this grand canonical sort of ensemble picture achieves this goal of ensuring that the perturbation series is done in a clean economical way namely.

That you simply only expand one of the s matrices in the numerator perturbatively and then you start using your Wick’s theorem and pair up all the terms, but there is a deeper more a better reason for introducing this definition this peculiar way of defining Green’s functions. So, through contour ordering and that deeper more convincing reason is because you can now derive a kind of an equation it is called a functional differential equation.

So, it is not the sort of differential equation that you are accustomed to, but it is a it is a kind of equation that I think even mathematicians seldom encounter in their own research, but it is one of those peculiar equations which involves both conventional part partial derivatives as well as functional derivatives the sort of functional differentiation and integration have been discussing in the last few lectures.

So, it involves both those equations, but the important benefit to thinking this way is that the equation such a functional differential equation for this contour ordered Green's function of this 10.103 will be such that it will only involve this green function back again, it would not involve anything else.

So, in other words see if you if you do not do it this way if you if you insist on using your conventional the original nave and even well it is physically well motivated, but technically naive definition of particle and Green's function as simply the average of psi and side dagger or average of psi dagger and psi. Even though that is physically well motivated it is technically naive because the equation of motion for those will involve now four point functions.

That means those are two point functions available four point functions and then if you then go ahead and try to see what equations those four point functions will obey they will now involve six point function etcetera etcetera endlessly; and that is not a desirable situation. So, you see this. So, we have to be technically clever also.

So, we have to define the Green's function in a technically clever way so, that the equation that you write down for that Green's function will not involve anything other than versions of itself. So, that is precisely what the Schwinger Dyson equation is basically it is an equation it is a functional differential equation of the Green's function which involves other versions of itself ok.

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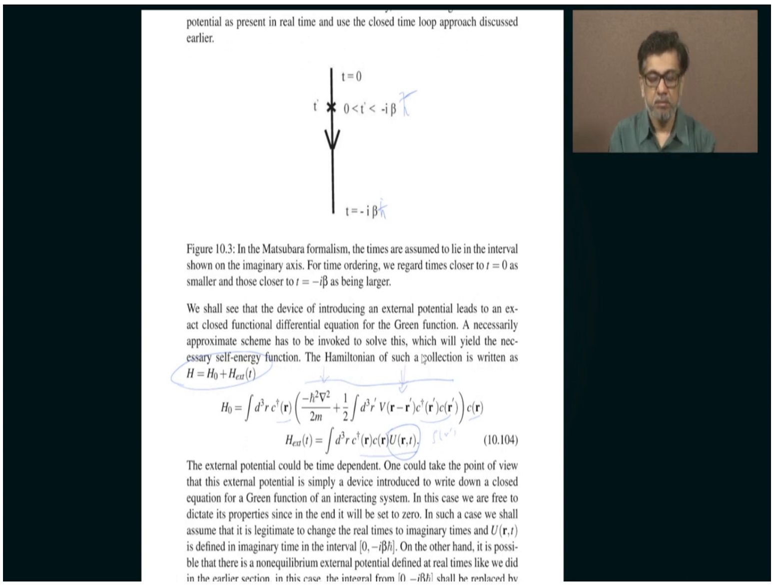

potential as present in real time and use the closed time loop approach discussed earlier.

Figure 10.3: In the Matsubara formalism, the times are assumed to lie in the interval shown on the imaginary axis. For time ordering, we regard times closer to $t = 0$ as smaller and those closer to $t = -i\beta$ as being larger.

We shall see that the device of introducing an external potential leads to an exact closed functional differential equation for the Green function. A necessarily approximate scheme has to be invoked to solve this, which will yield the necessary self-energy function. The Hamiltonian of such a collection is written as $H = H_0 + H_{ext}(t)$

$$H_0 = \int d^3r\, c^{\dagger}(\mathbf{r}) \left(\frac{-\hbar^2\nabla^2}{2m} + \frac{1}{2}\int d^3r'\, V(\mathbf{r}-\mathbf{r}')c^{\dagger}(\mathbf{r}')c(\mathbf{r}') \right) c(\mathbf{r})$$

$$H_{ext}(t) = \int d^3r\, c^{\dagger}(\mathbf{r})c(\mathbf{r})U(\mathbf{r},t). \quad (10.104)$$

The external potential could be time dependent. One could take the point of view that this external potential is simply a device introduced to write down a closed equation for a Green function of an interacting system. In this case we are free to dictate its properties since in the end it will be set to zero. In such a case we shall assume that it is legitimate to change the real times to imaginary times and $U(\mathbf{r},t)$ is defined in imaginary time in the interval $[0, -i\beta\hbar]$. On the other hand, it is possible that there is a nonequilibrium external potential defined at real times like we did in the earlier section, in this case, the integral from $[0, -i\beta\hbar]$ shall be replaced by

So, how do you do that? So, you do that by first let us see what is the system we are trying to study. So, the system we are trying to study is the following. I want to study this system, what is this system? This is the usual kinetic energy, but this is this is this will involve four fermions you see this is usually C dagger, C is just kinetic energy.

But this involves this is the density of particles sitting at r dash and this is the density of particle r and. So, in some sense it is basically the interaction of particles sitting at r n and r dash. And this is the mutual interaction potential energy, and this half is there because you are counting it twice and you want to count only once.

So, bottom line is that this is really what I want to study a collection of quantum particles that are mutually interacting by a two body pairwise potential called V of r minus r dash. So, but. So, I want to find the Green’s function of such a system as particle and whole Green’s function. So, the way I am going to do this is in order for me to derive an equation for that Green’s function which does not involve anything.

But other versions of that Green’s function I will be forced to introduce an external time dependent potential. So, now, I am see because this is merely a device to generate the I mean basically it is a device to write down those Green’s functions I mean. So, write down those functional differential equations.

Then it is clear that I am at liberty to choose this external time dependent field according to my convenience, because after all in the end I intend to set it equal to 0 because really speaking it is not there. The system is in equilibrium and so I am introducing it because I want to derive a functional differential equation for the Green's function which will involve this external time dependent field which will finally, go away.

But because it will finally, go away and it is not really there I am putting it in for my own technical convenience, I am completely at liberty to choose its properties. So, specifically I am going to postulate that this time dependent external potential U of r comma t is such that the times involved are also along the imaginary axis.

Now, I am no longer going to be studying the this Keldysh contour because you see the Keldysh contour is useful when you have actual time dependent fields which are you know time varying in real time, but this is some ex mathematical device which I am introducing. So, because it is a mathematical device I can pretend that this time dependent external source also is meaningful only on the imaginary axis.

And it is defined for times between t equal to 0 and t equals minus beta h bar ok. So, physicists have this instinctive habit of setting h bar equals one I occasionally slip up and forget the h bar because in other places I have written h bar ok. So, the point is that. So, that is how I choose to introduce this external device this external field which is actually not there, but I introduce it for my convenience and I will finally, get rid of it.

So, now the question is if I decide that now this this is my combined Hamiltonian which now involves both the external field as well as the field which involves I mean as well as the kinetic energy of the particles and the mutual interaction potential energy now I can go ahead and ask myself what is the equation of motion obeyed by the time dependent annihilation operator.

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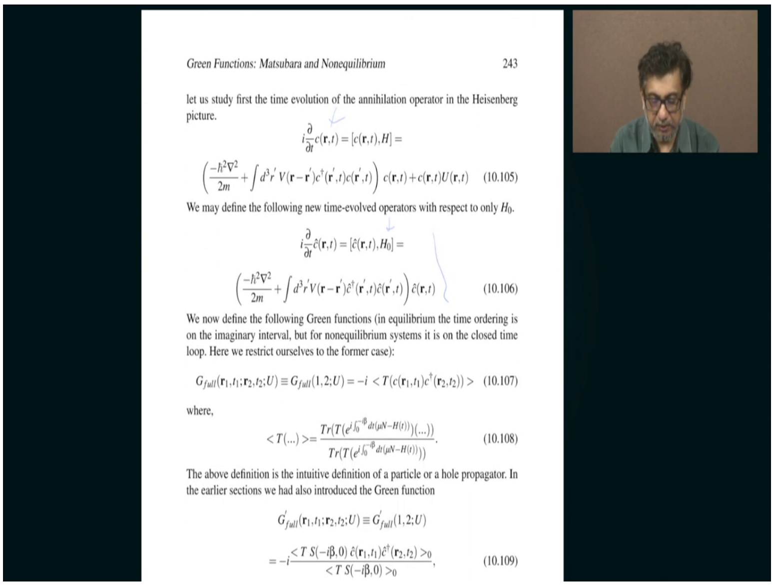

Green Functions: Matsubara and Nonequilibrium 243

let us study first the time evolution of the annihilation operator in the Heisenberg picture.

$$i\frac{\partial}{\partial t}c(\mathbf{r},t) = [c(\mathbf{r},t),H] =$$

$$\left(\frac{-\hbar^2\nabla^2}{2m} + \int d^3r'\, V(\mathbf{r}-\mathbf{r}')c^{\dagger}(\mathbf{r}',t)c(\mathbf{r}',t)\right) c(\mathbf{r},t) + c(\mathbf{r},t)U(\mathbf{r},t) \quad (10.105)$$

We may define the following new time-evolved operators with respect to only H_0.

$$i\frac{\partial}{\partial t}\hat{c}(\mathbf{r},t) = [\hat{c}(\mathbf{r},t),H_0] =$$

$$\left(\frac{-\hbar^2\nabla^2}{2m} + \int d^3r' V(\mathbf{r}-\mathbf{r}')\hat{c}^{\dagger}(\mathbf{r}',t)\hat{c}(\mathbf{r}',t)\right) \hat{c}(\mathbf{r},t) \quad (10.106)$$

We now define the following Green functions (in equilibrium the time ordering is on the imaginary interval, but for nonequilibrium systems it is on the closed time loop. Here we restrict ourselves to the former case):

$$G_{full}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2;U) \equiv G_{full}(1,2;U) = -i < T(c(\mathbf{r}_1,t_1)c^{\dagger}(\mathbf{r}_2,t_2)) > \quad (10.107)$$

where,

$$< T(...) >= \frac{Tr(T(e^{i\int_0^{-i\beta}dt(\mu N - H(t))})(...))}{Tr(T(e^{i\int_0^{-i\beta}dt(\mu N - H(t))}))}. \quad (10.108)$$

The above definition is the intuitive definition of a particle or a hole propagator. In the earlier sections we had also introduced the Green function

$$G'_{full}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2;U) \equiv G'_{full}(1,2;U)$$

$$= -i\frac{< T\, S(-i\beta,0)\, \hat{c}(\mathbf{r}_1,t_1)\hat{c}^{\dagger}(\mathbf{r}_2,t_2) >_0}{< T\, S(-i\beta,0) >_0}, \quad (10.109)$$

So, clearly it is going to be this. See whereas, if the part of the Hamiltonian that evolves only according to the time independent part of the Hamiltonian will clearly obey this equation right.

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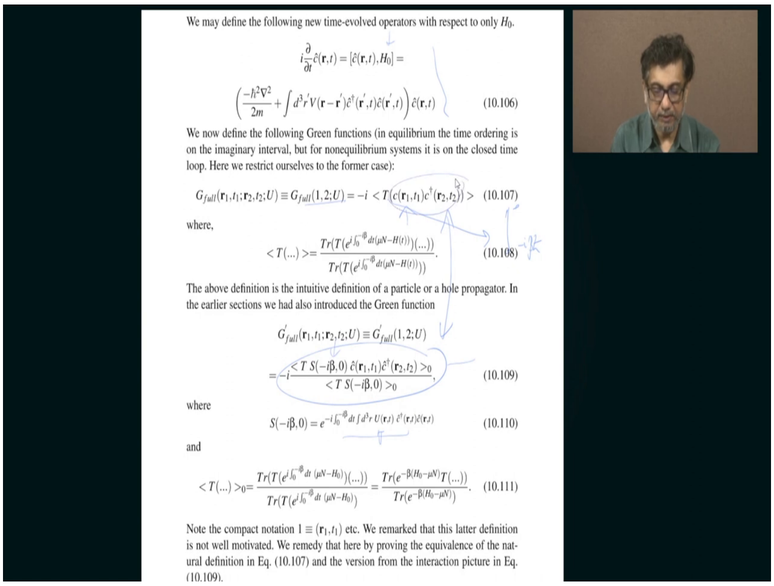

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$$G_{full}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2;U) \equiv G_{full}(1,2;U) = -i < T(c(\mathbf{r}_1,t_1)c^{\dagger}(\mathbf{r}_2,t_2)) > \quad (10.107)$$

where,

$$< T(...) >= \frac{Tr(T(e^{i\int_0^{-i\beta}dt(\mu N - H(t))})(...))}{Tr(T(e^{i\int_0^{-i\beta}dt(\mu N - H(t))}))}. \quad (10.108)$$

The above definition is the intuitive definition of a particle or a hole propagator. In the earlier sections we had also introduced the Green function

$$G'_{full}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2;U) \equiv G'_{full}(1,2;U)$$

$$= -i\frac{< T\, S(-i\beta,0)\, \hat{c}(\mathbf{r}_1,t_1)\hat{c}^{\dagger}(\mathbf{r}_2,t_2) >_0}{< T\, S(-i\beta,0) >_0}, \quad (10.109)$$

where

$$S(-i\beta,0) = e^{-i\int_0^{-i\beta}dt \int d^3r\, U(\mathbf{r},t)\, \hat{c}^{\dagger}(\mathbf{r},t)\hat{c}(\mathbf{r},t)} \quad (10.110)$$

and

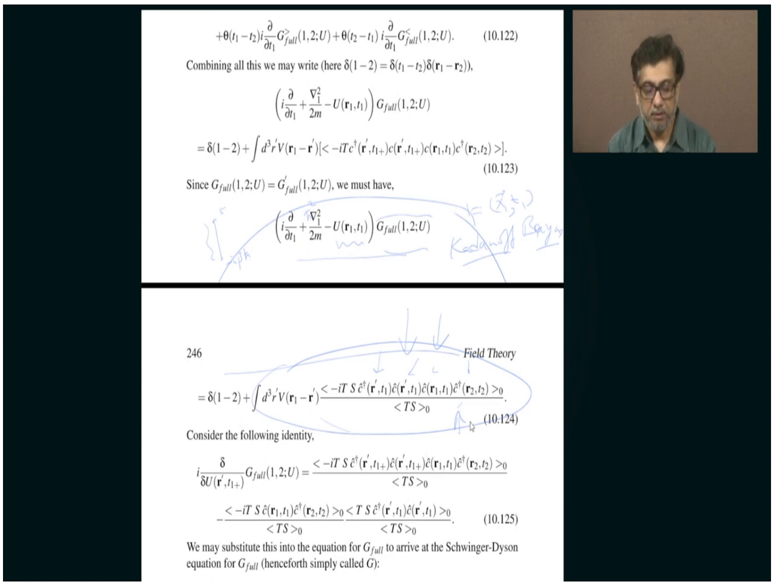
$$< T(...) >_0 = \frac{Tr(T(e^{i\int_0^{-i\beta}dt\, (\mu N - H_0)})(...))}{Tr(T(e^{i\int_0^{-i\beta}dt\, (\mu N - H_0)}))} = \frac{Tr(e^{-\beta(H_0 - \mu N)}T(...))}{Tr(e^{-\beta(H_0 - \mu N)})}. \quad (10.111)$$

Note the compact notation $1 \equiv (\mathbf{r}_1,t_1)$ etc. We remarked that this latter definition is not well motivated. We remedy that here by proving the equivalence of the natural definition in Eq. (10.107) and the version from the interaction picture in Eq. (10.109).

So, the implication is that these two are related through this sort of S matrix ok. So, if your external field was zero it these two would coincide because if U was 0 that these two are the same things.

So, the hat refers to the time evolution purely with respect to the time independent part of the Hamiltonian. So, I have spent some effort trying to convince you that things work out I mean that they are mutually consistent and so on. So, I am going to skip that those are for some I mean for those of you who are not entirely convinced by some of these steps, but they are not terribly important. So, let us proceed. So, the point is that this is the Green's function that I am going to be dealing with ok. So, let me go all the way here yeah.

(Refer Slide Time: 31:27)



$$+\theta(t_1-t_2)i\frac{\partial}{\partial t_1}G^{>}_{full}(1,2;U)+\theta(t_2-t_1)\,i\frac{\partial}{\partial t_1}G^{<}_{full}(1,2;U). \qquad (10.122)$$

Combining all this we may write (here $\delta(1-2)=\delta(t_1-t_2)\delta(\mathbf{r}_1-\mathbf{r}_2)$),

$$\left(i\frac{\partial}{\partial t_1}+\frac{\nabla_1^2}{2m}-U(\mathbf{r}_1,t_1)\right)G_{full}(1,2;U)$$

$$=\delta(1-2)+\int d^3r'V(\mathbf{r}_1-\mathbf{r}')[<-iTc^{\dagger}(\mathbf{r}',t_{1+})c(\mathbf{r}',t_{1+})c(\mathbf{r}_1,t_1)c^{\dagger}(\mathbf{r}_2,t_2)>]. \qquad (10.123)$$

Since $G_{full}(1,2;U)=G'_{full}(1,2;U)$, we must have,

$$\left(i\frac{\partial}{\partial t_1}+\frac{\nabla_1^2}{2m}-U(\mathbf{r}_1,t_1)\right)G_{full}(1,2;U)$$

246 *Field Theory*

$$=\delta(1-2)+\int d^3r'V(\mathbf{r}_1-\mathbf{r}')\frac{<-iT\,S\,\hat{c}^{\dagger}(\mathbf{r}',t_1)\hat{c}(\mathbf{r}',t_1)\hat{c}(\mathbf{r}_1,t_1)\hat{c}^{\dagger}(\mathbf{r}_2,t_2)>_0}{<TS>_0}. \qquad (10.124)$$

Consider the following identity,

$$i\frac{\delta}{\delta U(\mathbf{r}',t_{1+})}G_{full}(1,2;U)=\frac{<-iT\,S\,\hat{c}^{\dagger}(\mathbf{r}',t_{1+})\hat{c}(\mathbf{r}',t_{1+})\hat{c}(\mathbf{r}_1,t_1)\hat{c}^{\dagger}(\mathbf{r}_2,t_2)>_0}{<TS>_0}$$

$$-\frac{<-iT\,S\,\hat{c}(\mathbf{r}_1,t_1)\hat{c}^{\dagger}(\mathbf{r}_2,t_2)>_0}{<TS>_0}\frac{<T\,S\,\hat{c}^{\dagger}(\mathbf{r}',t_1)\hat{c}(\mathbf{r}',t_1)>_0}{<TS>_0}. \qquad (10.125)$$

We may substitute this into the equation for G_{full} to arrive at the Schwinger-Dyson equation for G_{full} (henceforth simply called G):

So, this is a Green. So, this is the Green's function I am going to be dealing with. So, this see this one notation one is a compact way of writing X 1 comma t 1 ok. So, this is called Kadanoff Baym notation.

So, this was in mind this notation was invented by the well known physicist Kadanoff Baym in the early sixties when they wrote this textbook called quantum statistical mechanics. So, it that is where I learned all this from ok point is that there are these. So, again I instinctively forgot my h bar. So, the bottom line is that this Green's function which is now defined it remember that the time ordering is now still on the imaginary axis.

So, all these times at t 1, t 2 they all live on the imaginary axis between t equal to 0 and t equals minus b i beta h bar. So, this is clearly the equation of motion; that means the equation the time evolution equation obeyed by this Green's function. See the thing is that the reason why we have chosen to introduce this external source is I mean initially it seems like unnecessary complication.

Because after all we are only interested in studying the system at equilibrium, but a system where the particles perhaps are in contact with surroundings at some temperature, but more importantly they interact amongst themselves, but they are still in equilibrium. So, it means that there is the Green's functions look the same if you decide to shift the origin of your time to somewhere else.

And they are also spatially homogeneous; that means, that if you shift the origin of your coordinate system especially also the Green's function look the same. So, there is special and time homogeneity as well so; that means, the system is in equilibrium. So, if it is in equilibrium it is kind of annoying to introduce some external potential now that spoils all that nice equilibrium property.

But you will see that it is important to do this because the term that involves interactions between particles can now be written you see this what is this term that inverse the interaction between particles it involves four formulas that two creation and two annihilations whereas, this Green's function involves one creation and one annihilation.

Because that is how we defined it somewhere here. So, it was defined like this. So, it involves one creation and one annihilation ok. So, these two are the same this is, but I mean this is the S matrix perturbation series friendly version of this. So, this is the original average that I am looking. So, this is along the time ordering is along the imaginary axis remember.

So, this time ordering is along this imaginary axis. So, this is the original definition now I replace the C s by C hats and this is kind of the same as I mean this is mathematically the same that is what I have done in all these steps this theorem basically shows that this is same as this and this is basically involves the part of the Hamiltonian this is the interaction picture.

It only involves part of the Hamiltonian that does not change with time the part that does change with time is lumped into the size matrix. So, bottom line is that whatever the Green's function is it involves one creation one annihilation operator; however, its equation of motion now unfortunately seems to involve two creation and two annihilation.

Now, if you go ahead and write down the equation of motion for this it will involve three creation three annihilation and so on and so forth endlessly, and that is where you are stuck because then you do not have a convenient equation that you can solve. So, the whole point of introducing this device of an external time dependent source is to see if you can go ahead. And now express this term which involves two annihilation two creation purely in terms of a Green's function that has only one annihilation one creation.

So in fact, I will show you in the next class that it will it is possible to do that. So, if I am successful and showing you that then what this equation becomes is basically it becomes an equation involving a version of the Green's function which creates only one particle and annihilates only one particle and nothing else it does not involve Green's functions where you are annihilating two particles or creating two particles.

So, I am going to stop here. And in the next class, I am going to continue and prove this important claim that the terms which I have written down here which involves creating and annihilating two particles that correspond two interactions between particles can now be expressed purely in terms of the Green's function involving one particle so long as I involve the sources that I have artificially introduced into the problem ok. I am going to stop here and in the next class let us continue.

Thank you.