Dynamics of Classical and Quantum Fields: An Introduction
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Green Functions in Grand Canonical Ensemble
Lecture - 42
Self-Energy and Spectral Functions

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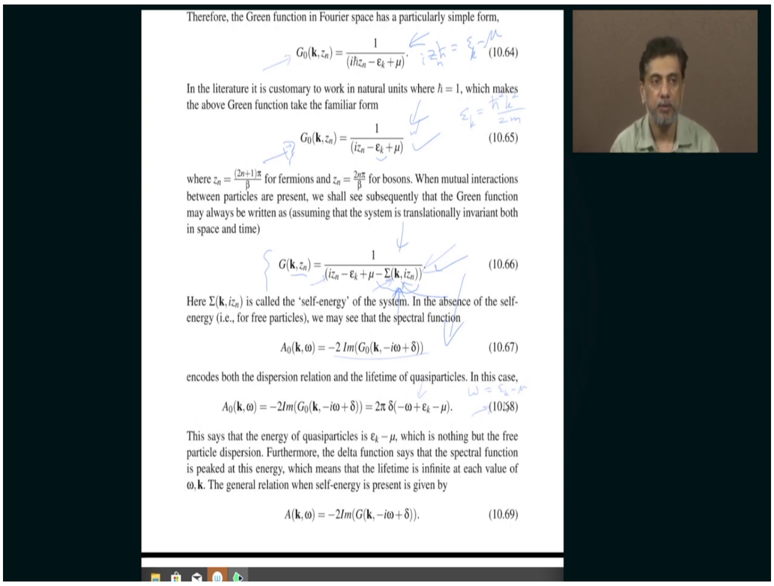

Therefore, the Green function in Fourier space has a particularly simple form,

$$G_0(\mathbf{k}, z_n) = \frac{1}{(i\hbar z_n - \varepsilon_k + \mu)}. \qquad (10.64)$$

$i z_n \hbar = \xi_k - \mu$

In the literature it is customary to work in natural units where $\hbar = 1$, which makes the above Green function take the familiar form

$\varepsilon_k = \frac{\hbar^2 k^2}{2m}$

$$G_0(\mathbf{k}, z_n) = \frac{1}{(i z_n - \varepsilon_k + \mu)} \qquad (10.65)$$

where $z_n = \frac{(2n+1)\pi}{\beta}$ for fermions and $z_n = \frac{2n\pi}{\beta}$ for bosons. When mutual interactions between particles are present, we shall see subsequently that the Green function may always be written as (assuming that the system is translationally invariant both in space and time)

$$G(\mathbf{k}, z_n) = \frac{1}{(i z_n - \varepsilon_k + \mu - \Sigma(\mathbf{k}, i z_n))}. \qquad (10.66)$$

Here $\Sigma(\mathbf{k}, i z_n)$ is called the 'self-energy' of the system. In the absence of the self-energy (i.e., for free particles), we may see that the spectral function

$$A_0(\mathbf{k}, \omega) = -2\, Im(G_0(\mathbf{k}, -i\omega + \delta)) \qquad (10.67)$$

encodes both the dispersion relation and the lifetime of quasiparticles. In this case,

$\omega = \varepsilon_k - \mu$

$$A_0(\mathbf{k}, \omega) = -2 Im(G_0(\mathbf{k}, -i\omega + \delta)) = 2\pi\, \delta(-\omega + \varepsilon_k - \mu). \qquad (10.68)$$

This says that the energy of quasiparticles is $\varepsilon_k - \mu$, which is nothing but the free particle dispersion. Furthermore, the delta function says that the spectral function is peaked at this energy, which means that the lifetime is infinite at each value of $\omega, \mathbf{k}$. The general relation when self-energy is present is given by

$$A(\mathbf{k}, \omega) = -2 Im(G(\mathbf{k}, -i\omega + \delta)). \qquad (10.69)$$

So today's class let us continue our discussion of Green's functions of many body systems at finite temperature. So, these are called Matsubara Green's functions. So, I told you that in the last class that the Green's functions basically what we had described in real space that is you remove or insert a particle at some position r at time t and then you insert or remove the same particle at some other position at some other time.

So, that would correspond to Green's functions and then you find the overlap between the two states starting and ending. So, that would correspond to the Green's function in real space real time.

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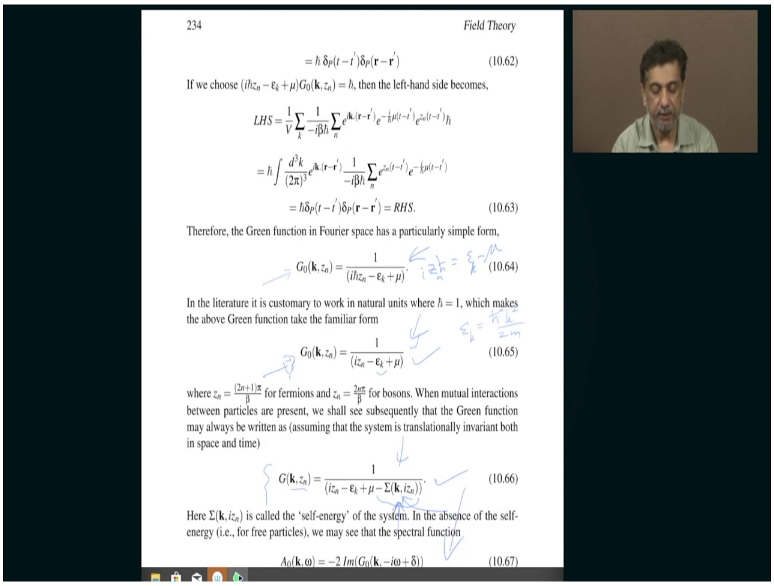

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$$= \hbar\, \delta_P(t-t')\delta_P(\mathbf{r}-\mathbf{r}') \qquad (10.62)$$

If we choose $(i\hbar z_n - \varepsilon_k + \mu)G_0(\mathbf{k}, z_n) = \hbar$, then the left-hand side becomes,

$$LHS = \frac{1}{V}\sum_k \frac{1}{-i\beta\hbar}\sum_n e^{i\mathbf{k}.(\mathbf{r}-\mathbf{r}')} e^{-\frac{i}{\hbar}\mu(t-t')} e^{z_n(t-t')}\hbar$$

$$= \hbar\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}.(\mathbf{r}-\mathbf{r}')} \frac{1}{-i\beta\hbar}\sum_n e^{z_n(t-t')} e^{-\frac{i}{\hbar}\mu(t-t')}$$

$$= \hbar\delta_P(t-t')\delta_P(\mathbf{r}-\mathbf{r}') = RHS. \qquad (10.63)$$

Therefore, the Green function in Fourier space has a particularly simple form,

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Here $\Sigma(\mathbf{k}, i z_n)$ is called the 'self-energy' of the system. In the absence of the self-energy (i.e., for free particles), we may see that the spectral function

$$A_0(\mathbf{k}, \omega) = -2\, Im(G_0(\mathbf{k}, -i\omega + \delta)) \qquad (10.67)$$

So, but; however, I told you that there is an advantage to thinking of the times as being purely imaginary in the case of a system at finite temperature. So, if you have a system which is exchanging not only energy with its surroundings but also particles are flowing in and out. So, if you are uncomfortable with that type of situation, you can just think of you know an imaginary boundary of a subsystem within a larger system.

So, if you think of it that way then clearly both energy and particles can freely flow in and out and then you have a kind of an equilibrium that is established with and then you will have a chemical potential; that means, these in that instead of the number of particles being specified what is specified is.

The chemical potential, but I told you again and again that for systems that are large; that means, in the thermodynamic limit there is no distinction at all between the grand canonical ensemble and the canonical ensemble. So, if you do have a system with a genuinely fixed number of particles it is not incorrect to first study the system using grand canonical ensemble because it is mathematically simpler and then you find a relation between the chemical potential which you are not interested in.

And you relate that to the density of particles which is well and truly fixed in the case of canonical ensemble. So, then you get back your canonical answers very easily because you have started with the grand canonical ensemble. So, bottom line is that you do what is needed to get the answer as quickly as possible.

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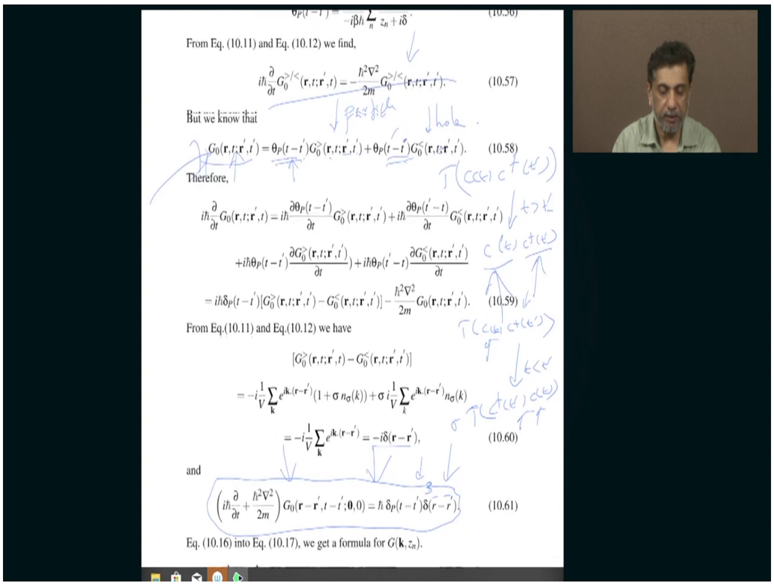

From Eq. (10.11) and Eq. (10.12) we find,

$$i\hbar\frac{\partial}{\partial t}G_0^{>/<}(\mathbf{r},t;\mathbf{r}',t) = -\frac{\hbar^2\nabla^2}{2m}G_0^{>/<}(\mathbf{r},t;\mathbf{r}',t'). \quad (10.57)$$

But we know that

particle hole

$$G_0(\mathbf{r},t;\mathbf{r}',t') = \theta_P(t-t')G_0^{>}(\mathbf{r},t;\mathbf{r}',t') + \theta_P(t'-t)G_0^{<}(\mathbf{r},t;\mathbf{r}',t'). \quad (10.58)$$

Therefore,

$$i\hbar\frac{\partial}{\partial t}G_0(\mathbf{r},t;\mathbf{r}',t) = i\hbar\frac{\partial\theta_P(t-t')}{\partial t}G_0^{>}(\mathbf{r},t;\mathbf{r}',t') + i\hbar\frac{\partial\theta_P(t'-t)}{\partial t}G_0^{<}(\mathbf{r},t;\mathbf{r}',t')$$
$$+ i\hbar\theta_P(t-t')\frac{\partial G_0^{>}(\mathbf{r},t;\mathbf{r}',t')}{\partial t}) + i\hbar\theta_P(t'-t)\frac{\partial G_0^{<}(\mathbf{r},t;\mathbf{r}',t')}{\partial t}$$
$$= i\hbar\delta_P(t-t')[G_0^{>}(\mathbf{r},t;\mathbf{r}',t') - G_0^{<}(\mathbf{r},t;\mathbf{r}',t')] - \frac{\hbar^2\nabla^2}{2m}G_0(\mathbf{r},t;\mathbf{r}',t'). \quad (10.59)$$

$T(c(t)\, c^{\dagger}(t'))$

$t > t'$

$c(t)\, c^{\dagger}(t')$

From Eq.(10.11) and Eq.(10.12) we have

$t < t'$

$\sigma\, T(c^{\dagger}(t')\, c(t))$

$$[G_0^{>}(\mathbf{r},t;\mathbf{r}',t) - G_0^{<}(\mathbf{r},t;\mathbf{r}',t')]$$
$$= -i\frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}.(\mathbf{r}-\mathbf{r}')}(1+\sigma\, n_\sigma(k)) + \sigma\, i\frac{1}{V}\sum_{k} e^{i\mathbf{k}.(\mathbf{r}-\mathbf{r}')}n_\sigma(k)$$
$$= -i\frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}.(\mathbf{r}-\mathbf{r}')} = -i\delta(\mathbf{r}-\mathbf{r}'), \quad (10.60)$$

and

$$\boxed{\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\nabla^2}{2m}\right)G_0(\mathbf{r}-\mathbf{r}',t-t';\mathbf{0},0) = \hbar\,\delta_P(t-t')\overset{3}{\delta}(\vec{r}-\vec{r}').} \quad (10.61)$$

Eq. (10.16) into Eq. (10.17), we get a formula for $G(\mathbf{k},z_n)$.

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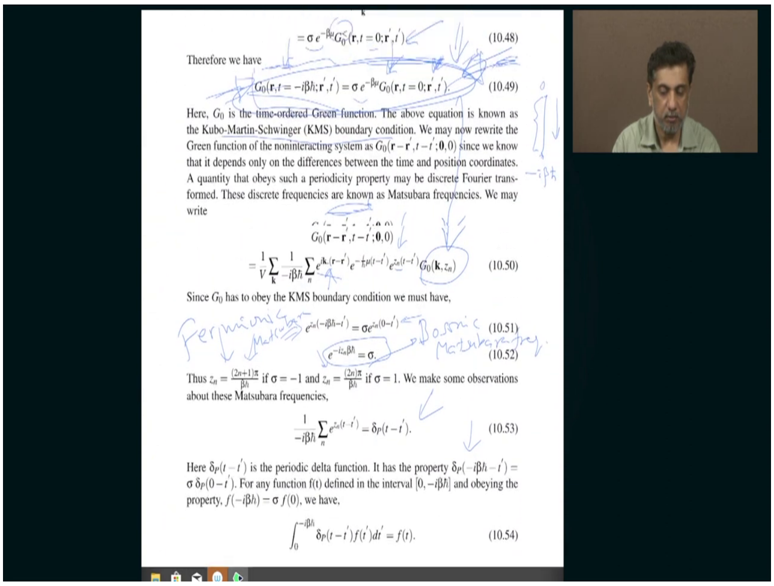

$$= \sigma\, e^{-\beta\mu}G_0^{<}(\mathbf{r},t=0;\mathbf{r}',t') \quad (10.48)$$

Therefore we have

$$G_0(\mathbf{r},t=-i\beta\hbar;\mathbf{r}',t') = \sigma\, e^{-\beta\mu}G_0(\mathbf{r},t=0;\mathbf{r}',t'). \quad (10.49)$$

Here, G_0 is the time-ordered Green function. The above equation is known as the Kubo-Martin-Schwinger (KMS) boundary condition. We may now rewrite the Green function of the noninteracting system as $G_0(\mathbf{r}-\mathbf{r}',t-t';\mathbf{0},0)$ since we know that it depends only on the differences between the time and position coordinates. A quantity that obeys such a periodicity property may be discrete Fourier transformed. These discrete frequencies are known as Matsubara frequencies. We may write

0

$-i\beta\hbar$

$$G_0(\mathbf{r}-\mathbf{r}',t-t';\mathbf{0},0)$$
$$= \frac{1}{V}\sum_{\mathbf{k}}\frac{1}{-i\beta\hbar}\sum_n e^{i\mathbf{k}.(\mathbf{r}-\mathbf{r}')}e^{-\frac{1}{\hbar}\mu(t-t')}e^{z_n(t-t')}G_0(\mathbf{k},z_n) \quad (10.50)$$

Since G_0 has to obey the KMS boundary condition we must have,

Fermionic Matsubara

$$e^{z_n(-i\beta\hbar-t')} = \sigma e^{z_n(0-t')} \quad (10.51)$$
$$\boxed{e^{-iz_n\beta\hbar} = \sigma.} \quad (10.52)$$

Bosonic Matsubara freq.

Thus $z_n = \frac{(2n+1)\pi}{\beta\hbar}$ if $\sigma = -1$ and $z_n = \frac{(2n)\pi}{\beta\hbar}$ if $\sigma = 1$. We make some observations about these Matsubara frequencies,

$$\frac{1}{-i\beta\hbar}\sum_n e^{z_n(t-t')} = \delta_P(t-t'). \quad (10.53)$$

Here $\delta_P(t-t')$ is the periodic delta function. It has the property $\delta_P(-i\beta\hbar-t') = \sigma\,\delta_P(0-t')$. For any function f(t) defined in the interval $[0,-i\beta\hbar]$ and obeying the property, $f(-i\beta\hbar) = \sigma\, f(0)$, we have,

$$\int_0^{-i\beta\hbar}\delta_P(t-t')f(t')dt' = f(t). \quad (10.54)$$

But having done that you see I told you that it is possible to rewrite these Green's functions in momentum and frequency domain.

And specifically we exploit what are called the KMS boundary conditions. So, in order to exploit this we have to first define a kind of time ordering. So, I told you that you imagine the times to be between 0 and minus i beta h bar. So, if you do that then you can define the particle Green's function as the with a subscript greater than sine. So, that way you can define what is called a time ordered Green's function on the imaginary axis.

So, having done that you can show that basically the time ordered Green's function can be written in Fourier space and the Fourier components of the time ordered Green's functions are nothing, but these quantities. So, that is if you are talking about free particles then it is going to be this and Z n is basically the Matsubara frequency. So, for fermions the Matsubara frequencies odd multiple of pi divided by beta remember that beta is inverse temperature and for bosons it is even multiple of pi divided by beta.

And mu is the chemical potential and epsilon k is the dispersion relation between of the particles. So, it is basically h bar square k square by 2 m. So, this is for genuinely free particles; that means, particles that do not interact with anything. So, if that is the case then this is the result, but I told you that in situations where particles interact with each other it is clear that you know especially if the interaction is between particles the translational invariance of the system is still respected.

So, momentum continues to be a good quantum number and on top of that the system is still in equilibrium because there is no external driving force the interactions are not due to some external source. So, it is some internally generated interaction. So, the system continues to be in equilibrium. So, the Matsubara frequencies are still valid and I told you the KMS boundary conditions are very general.

They just have to do with the fact that you know the cyclic it just utilizes the cyclic property of trace which is pretty much very general. So, because of that you see the Green's function is just a function of k and the appropriate Matsubara frequency. So, bottom line is that you can always write the Green's function with interactions in this way so; that means, you introduce a new function which accounts for the fact that the system interacts with each other the particles interact with each other.

So, that is called the self-energy. So, basically self-energy corresponds to the it has a very intuitively appealing name it is called self-energy; that means, that is the energy generated by interaction of one component one part of the system with another. So, basically the mutual interaction between particles. So, this would be. So, calculating the self-energy for a system of mutually interacting particles is one of the central goals of many body theory.

It is not easy and usually many people use many approximations like perturbation theory loop approximation so on and so forth. So, physically you see I told you that one can also start studying some specific quantities related to this Green's function in momentum and frequency domain and specifically if you calculate what is called a spectral function. So, this is called a spectral function.

So, if you calculate this sort of thing. So, where omega. So, you replace formally Z n by minus i omega plus delta. So, remember z n is supposed to be some discrete quantity, but formally you can always replace it by some continuous quantity called minus i omega plus delta where delta is a small real number. So, if you do that and you think of omega as real and as small and real you will see that as delta tends to 0 this spectral function for free particles becomes a Dirac delta function.

Evaluated at this quantity so; that means, the spectral function peaks at omega equals E k minus mu. So, that corresponds to E k is the energy of the particles and mu is the chemical potential. So, the difference corresponds to the excess energy that the particle has over and above the chemical potential.

So, this we identify with the quasi particles. So, basically quasi particles are particles that appear to move independent of each other in the system. So, you see this 10.68 is assuming that the there is no mutual interaction between particles. So, there is no distinction between the original particle and the quasi particle they are basically one and the same because there is no mutual interaction between particles.

So, in that case clearly the quasi particle energy should be just trivially related to the energy of the original particles which is epsilon k. So, that is the reason why omega equals epsilon k minus mu is just shifted by the chemical potential, but; however, more interesting things are possible when you have the self-energy; that means, when you have mutual interaction between particles.

Then the spectral function firstly, will have two features one is that it will not be a delta function, but it will be it will resemble a delta function. It will be it will not be strictly a delta function, but it will have a very pronounced peak usually at some value of omega and the basically the width of that peak the fact that it is not delta function, but its broadened to some extent reflects the fact that the quasi particles in the system have a finite lifetime.

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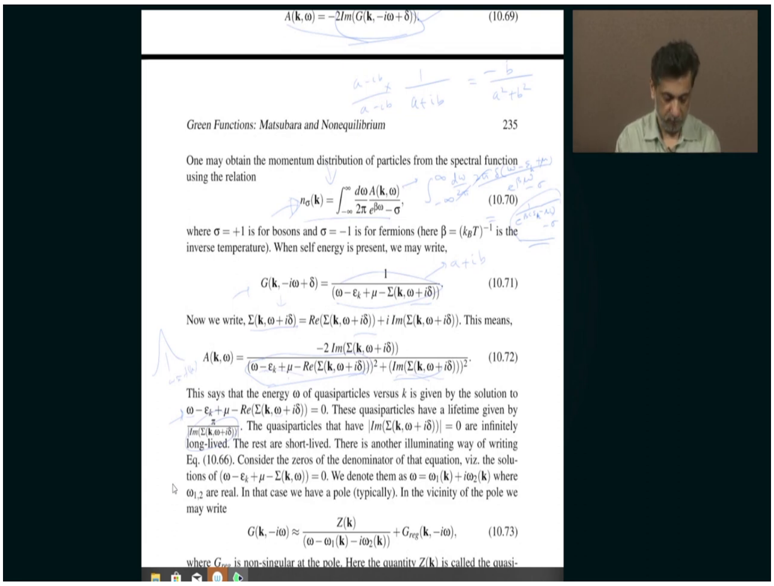

$$A(\mathbf{k},\omega) = -2Im(G(\mathbf{k},-i\omega+\delta)). \qquad (10.69)$$

$$\frac{a-ib}{a-ib} \times \frac{1}{a+ib} = \frac{-b}{a^2+b^2}$$

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One may obtain the momentum distribution of particles from the spectral function using the relation

$$n_\sigma(\mathbf{k}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{A(\mathbf{k},\omega)}{e^{\beta\omega} - \sigma}, \qquad (10.70)$$

where $\sigma = +1$ is for bosons and $\sigma = -1$ is for fermions (here $\beta = (k_B T)^{-1}$ is the inverse temperature). When self energy is present, we may write,

$$G(\mathbf{k},-i\omega+\delta) = \frac{1}{(\omega - \varepsilon_k + \mu - \Sigma(\mathbf{k},\omega+i\delta))}. \qquad (10.71)$$

Now we write, $\Sigma(\mathbf{k},\omega+i\delta) = Re(\Sigma(\mathbf{k},\omega+i\delta)) + i\,Im(\Sigma(\mathbf{k},\omega+i\delta))$. This means,

$$A(\mathbf{k},\omega) = \frac{-2\,Im(\Sigma(\mathbf{k},\omega+i\delta))}{(\omega - \varepsilon_k + \mu - Re(\Sigma(\mathbf{k},\omega+i\delta)))^2 + (Im(\Sigma(\mathbf{k},\omega+i\delta)))^2}. \qquad (10.72)$$

This says that the energy ω of quasiparticles versus k is given by the solution to $\omega - \varepsilon_k + \mu - Re(\Sigma(\mathbf{k},\omega+i\delta)) = 0$. These quasiparticles have a lifetime given by $\frac{\pi}{|Im(\Sigma(\mathbf{k},\omega+i\delta))|}$. The quasiparticles that have $|Im(\Sigma(\mathbf{k},\omega+i\delta))| = 0$ are infinitely long-lived. The rest are short-lived. There is another illuminating way of writing Eq. (10.66). Consider the zeros of the denominator of that equation, viz. the solutions of $(\omega - \varepsilon_k + \mu - \Sigma(\mathbf{k},\omega)) = 0$. We denote them as $\omega = \omega_1(\mathbf{k}) + i\omega_2(\mathbf{k})$ where $\omega_{1,2}$ are real. In that case we have a pole (typically). In the vicinity of the pole we may write

$$G(\mathbf{k},-i\omega) \approx \frac{Z(\mathbf{k})}{(\omega - \omega_1(\mathbf{k}) - i\omega_2(\mathbf{k}))} + G_{reg}(\mathbf{k},-i\omega), \qquad (10.73)$$

where G_{reg} is non-singular at the pole. Here the quantity $Z(\mathbf{k})$ is called the quasi-

So, that is the physical meaning of the width of that and the exact location of the peak also tells you the energy dispersion of the quasi particles which may be somewhat different from what you see for free particles ok. So, that we will come to a little later because that is the interpretation we are going to it is not some guess or anything we can actually deduce that interpretation very soon in just a few minutes.

So, but before I do that let me point out to you that it is possible to also deduce the distribution the Fermi Dirac and Bose Einstein distribution through this approach. So; that means, if it is genuinely free particles. So, suppose this is for genuinely free particles I told you it is just this much. So, if I substitute for genuinely free particles you see you will get this result it is going to be 2 pi delta of epsilon k plus mu I mean I have just flipped a sign you know that delta function is even function.

So, I am allowed to flip sign. So, it is e raised to sigma. So, now, you see, omega is if I integrate over omega I will just get 2 pi 2 pi cancels and this becomes nothing, but e raised to beta epsilon k minus mu then minus sigma. So, you see if sigma is minus 1 it is fermi on. So, it is fermi Dirac distribution if sigma is plus 1 it is boson.

So, it is Bose Einstein distribution, but you see the implication here is that this is not valid only for free particles. So, if A is I have just verified that if a is given very simply by this Dirac delta then it is reduced to this familiar form of fermi Dirac Bose Einstein, but the interest situation is when this is not given by Dirac delta when you have quasi particles when you have lifetimes and all that.

So, when you have those types of things then this is still valid, but A is complicated because A is complicated n sigma is also complicated. So, that is the distribution of quasi particles will not be exactly fermi Dirac or Bose Einstein because the particles interact with each other ok. So, how do you calculate the lifetime and the new dispersion? So, the way to do that is you first re rewrite first formally insert z n as minus i omega plus delta you get this.

So, now what you do is you formally decompose this self-energy into real and imaginary parts. So, clearly because this is as i delta it will have some formally a real part formally an imaginary part and I write it this way. So, if I do this then I take the imaginary part of. So, you see imaginary part of a plus i b is nothing, but b over. So, because i can multiply and divide both sides by a minus i b.

So, if I do that I will get a squared plus b squared into a minus i b and the imaginary part of that is minus b ok. So, basically that is what I have done here. So, you see it was this is something like a plus i b here this is something like a plus i b because you see sigma is something plus i something. So, I can always write this as a plus i b and if I take imaginary part. So, it is like taking of 1 by a plus i b minus 2 imaginary part of that this one.

So, it is like taking imaginary part of this quantity on both sides. So, imaginary part of one over a plus i b. So, that is basically b over a square plus. So, that is what this a squared plus b squared and this is b there are two there.

So, bottom line is that you see if you are successful somehow with finding this difficult quantity or self-energy, then you can always find many interesting things one is that you can convince yourself that the quasi particle energies are not really trivially related to epsilon k which is the kinetic energy of free particles, but it can be very complicated because you will have to solve this non-linear equation is the is basically this Dirac delta is peaked at value where omega means when this becomes 0.

So, that is where that. So, it is basically something it will look something like a Gaussian with this peak at omega equals that appropriate value of so; that means, some appropriate function of k.

So, it involves solving this non-linear equation because there is omega outside omega is unknown. So, you have to solve for omega. So, omega is outside it is also inside this sigma. So, somehow if you solve this you will get omega versus k. So, that is basically that it tells you the location of the means, it will tell you the dispersion relation of the quasi particle which may not be as simple as k squared h bar squared k squared by two m it can be more complicated.

But not only that is not only the dispersion relation of the quasi particle different from free particles, it also could have a lifetime so; that means, you can have a situation where the imaginary part of self-energy is not 0 so; that means, that these quasi particles not only have a different dispersion from free particles they are also not infinitely long lived; that means, they have a finite lifetime.

So, in general they will have a finite lifetime because in general imaginary part of sigma k comma omega plus i delta may not be 0, but there will be some situations where. So, the lifetime can be read off like this. So, it is just pi by imaginary remember sigma is the units of energy. So, this has units of one by energy. So, and h bar is one in our calculation. So, basically this has dimensions of time.

So, but of course, you can have a situation where some parts of you know you can simultaneously solve for this and you can ask yourself when this is 0 is this also 0. So, if this is also 0 then it means that for those particular values of omega and k that particular quasi particle can be infinitely long lived and that is likely to happen very rarely if at all ok because it involves some specific value of k and some specific value of omega.

But in general most of the quasi particles will have a finite lifetime when there is mutual interaction between particles ok.

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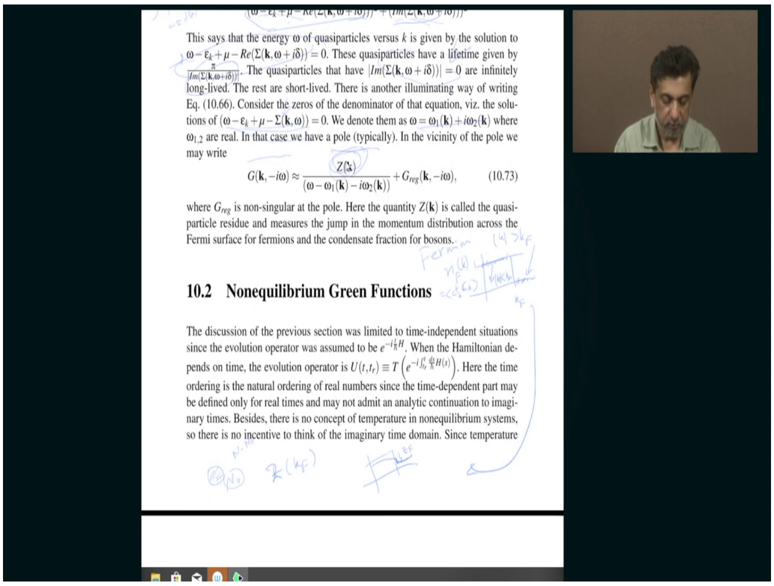

This says that the energy ω of quasiparticles versus k is given by the solution to $\omega - \varepsilon_k + \mu - Re(\Sigma(\mathbf{k}, \omega + i\delta)) = 0$. These quasiparticles have a lifetime given by $\frac{\pi}{|Im(\Sigma(\mathbf{k},\omega+i\delta))|}$. The quasiparticles that have $|Im(\Sigma(\mathbf{k}, \omega + i\delta))| = 0$ are infinitely long-lived. The rest are short-lived. There is another illuminating way of writing Eq. (10.66). Consider the zeros of the denominator of that equation, viz. the solutions of $(\omega - \varepsilon_k + \mu - \Sigma(\mathbf{k}, \omega)) = 0$. We denote them as $\omega = \omega_1(\mathbf{k}) + i\omega_2(\mathbf{k})$ where $\omega_{1,2}$ are real. In that case we have a pole (typically). In the vicinity of the pole we may write

$$G(\mathbf{k}, -i\omega) \approx \frac{Z(\mathbf{k})}{(\omega - \omega_1(\mathbf{k}) - i\omega_2(\mathbf{k}))} + G_{reg}(\mathbf{k}, -i\omega), \qquad (10.73)$$

where G_{reg} is non-singular at the pole. Here the quantity $Z(\mathbf{k})$ is called the quasi-particle residue and measures the jump in the momentum distribution across the Fermi surface for fermions and the condensate fraction for bosons.

10.2 Nonequilibrium Green Functions

The discussion of the previous section was limited to time-independent situations since the evolution operator was assumed to be $e^{-i\frac{t}{\hbar}H}$. When the Hamiltonian depends on time, the evolution operator is $U(t, t_r) \equiv T\left(e^{-i\int_{t_r}^{t}\frac{ds}{\hbar}H(s)}\right)$. Here the time ordering is the natural ordering of real numbers since the time-dependent part may be defined only for real times and may not admit an analytic continuation to imaginary times. Besides, there is no concept of temperature in nonequilibrium systems, so there is no incentive to think of the imaginary time domain. Since temperature

So, the other perspective you can utilize is that you can instead of decomposing the fermi energy this way you can say that look the real part of this. So, rather than decomposing into real part imaginary part you just look at the complex sigma itself right.

So, you just look at the complex sigma itself this just the complex sigma and then you ask yourself what should be. So, if you solve this for complex omega. So, because sigma is complex and omega if the solution will in general be complex. So, now, you imagine that the solution is formally written as omega 1 which is real and plus i omega two where omega two is the imaginary part of omega.

So, omega 1 is real part of omega, omega 2 is imaginary part of omega which is the solution of this complex equation because sigma is complex omega is a complex solution of this equation. So, if you decide to do it that way then you see it is very easy to insert the appropriate omega there and you can write down the Green's function in this way also and this quantity basically is called so, you can usually think of this as a pole.

So, usually if this has a if this has a 0 then clearly the 0 of that if this is the if this is the root of this equation this is a root of this equation, you can always think of it as omega minus the root because it will just be you know clearly is going to be like that, but then it will be proportional to that and that proportionality constant or will be a function of k and that Z k is basically called the quasi particle residue.

So, it measures the jump in the momentum distribution across the fermi surface for fermions and it measures the condensate fraction for bosons. So, basically it. So, you might be wondering what all that is and what am I talking about. So, basically let me tell you what that is. So, if I am talking about fermions, you see you can always plot this n F k which is the average number of particles with momentum k.

So, if I am talking about 0 temperature and I am talking about particles that do not interact with each other clearly the answer is this it is one if k is less than k F and 0 if k is greater than k F. So, if this is and this is for k less than k F. So, it is just a step function for non-interacting particles at 0 temperature, but the interesting situation is that when you have interactions between particles this becomes like this.

So, this is called the quasi particle residue. So, usually this will. So, you can have that quasi particle residue at some other value of k also, but usually it is evaluated at k equal to k F. So, that is the interesting situation which tells you the jump in the fermi means is the jump in the value of the discontinuity in the fermi Dirac distribution. So, that is what that is. So, it just tells you.

So, whereas, if it is bosons you see that there are N number of particle N 0 number of particles in the condensate and N minus N 0 number outside. So, basically this will tell you N 0 value this this will tell you the value of N 0 if k is 0 this condensate corresponds to k equal 0 and Z of 0 will correspond to N 0 so, that in that in some sense like a sudden jump. So, this is also sudden that is also sudden jump.

So, basically that is how you describe Green’s functions for interacting fermions and interacting bosons ok. So, now, I am just going to introduce to you in the remaining few minutes I have I will simply introduce to you the idea of non-equilibrium Green’s function. So, till now whatever I have described is basically systems in thermodynamic equilibrium; that means, I have a system which is exchanging energy with surroundings and possibly also particles with surroundings and coming to an equilibrium.

But many interesting most realistic systems or many realistic systems in found in nature are not in equilibrium the most interesting ones are living systems. You see you know phenomena that take place inside a living organisms like inside a cell, they are clearly not in equilibrium because energy is constantly being supplied to the cell and many processes are taking place dynamically.

So, if you want to study a living system using you know the tools of physics which is of course, incredibly ambitious and possibly quite foolish, but still there are people who try to do that in some crude way and they are forced to reckon with the fact that many of these systems are not in equilibrium. So, now, you will have to encounter our you know take into account the fact that you will.

So, in other words, you have to be prepared to generalize the idea of Green's functions to encompass systems which are not in equilibrium. So, of course, you know the systems that we are studying are quantum mechanical in nature. So, what I have in mind is some subatomic particles that are possibly driven by some external fields like electromagnetic fields which are out of equilibrium.

So, whatever it is whatever the source of that non equilibrium nature of the system is, that I am still forced to you know define and study thesis appropriate generalization of the Green's function to systems out of equilibrium.

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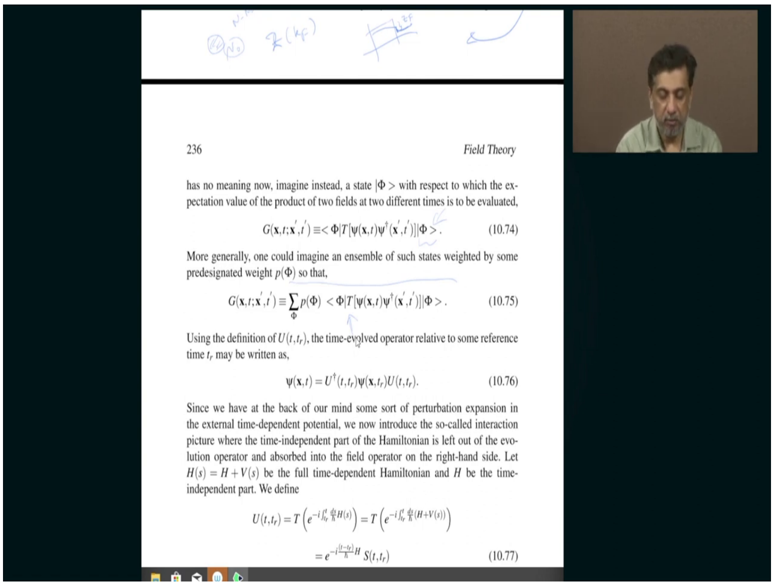

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has no meaning now, imagine instead, a state $|\Phi>$ with respect to which the expectation value of the product of two fields at two different times is to be evaluated,

$$G(\mathbf{x},t;\mathbf{x}',t') \equiv <\Phi|T[\psi(\mathbf{x},t)\psi^{\dagger}(\mathbf{x}',t')]|\Phi>. \tag{10.74}$$

More generally, one could imagine an ensemble of such states weighted by some predesignated weight $p(\Phi)$ so that,

$$G(\mathbf{x},t;\mathbf{x}',t') \equiv \sum_{\Phi} p(\Phi) \; <\Phi|T[\psi(\mathbf{x},t)\psi^{\dagger}(\mathbf{x}',t')]|\Phi>. \tag{10.75}$$

Using the definition of $U(t,t_r)$, the time-evolved operator relative to some reference time t_r may be written as,

$$\psi(\mathbf{x},t) = U^{\dagger}(t,t_r)\psi(\mathbf{x},t_r)U(t,t_r). \tag{10.76}$$

Since we have at the back of our mind some sort of perturbation expansion in the external time-dependent potential, we now introduce the so-called interaction picture where the time-independent part of the Hamiltonian is left out of the evolution operator and absorbed into the field operator on the right-hand side. Let $H(s) = H + V(s)$ be the full time-dependent Hamiltonian and H be the time-independent part. We define

$$U(t,t_r) = T\left(e^{-i\int_{t_r}^{t}\frac{ds}{\hbar}H(s)}\right) = T\left(e^{-i\int_{t_r}^{t}\frac{ds}{\hbar}(H+V(s))}\right)$$
$$= e^{-i\frac{(t-t_r)}{\hbar}H}\,S(t,t_r) \tag{10.77}$$

So, the question is how do I do that. So, to do that I as usual introduce my usual you know evolution operators s matrix and all that which I discussed earlier.

But it is very similar except that now I have to take into account the fact that the systems that I am studying are not in imaginary time. So, the times are going to be purely real, but because there is no concept of temperature. So, there is no particular benefit in moving to imaginary time. So, the times are going to be purely real. So, I have to make sense out of this concept of time ordering if at all required I have to redefine what it means for time.

See remember that I defined time order ordering in this peculiar way along the imaginary axis because it was very useful in ensuring that the Green’s function defined in that peculiar way obeys periodic boundary conditions. So, I am entitled to define any kind of time ordering I want so, long as I also list out all the properties that kind of a definition has so, that I can utilize it later on.

So, I mean I should be introducing such notions only if it benefits me in some way. So, it is going to benefit me if I can show that type of time ordering leads to say periodic boundary conditions in the Green’s function and that is what I was successful in showing for the case of equilibrium systems, but for non-equilibrium system is not at all clear what I should do because there is no question of a temperature in non-equilibrium system so, there is no question of minus i beta h bar and all that.

So, beta is inverse temperature there is no concept of temperature in non-equilibrium system. So, I have to deal with that aspect. So, the idea is that you first make a mental note that in general the Green's function that you will be dealing with even if you are successful in defining time ordering in some appropriate way you will have to not just you know find the expectation value of these two operators. But rather perhaps you will be compelled to form a kind of an ensemble average over some probability distribution.

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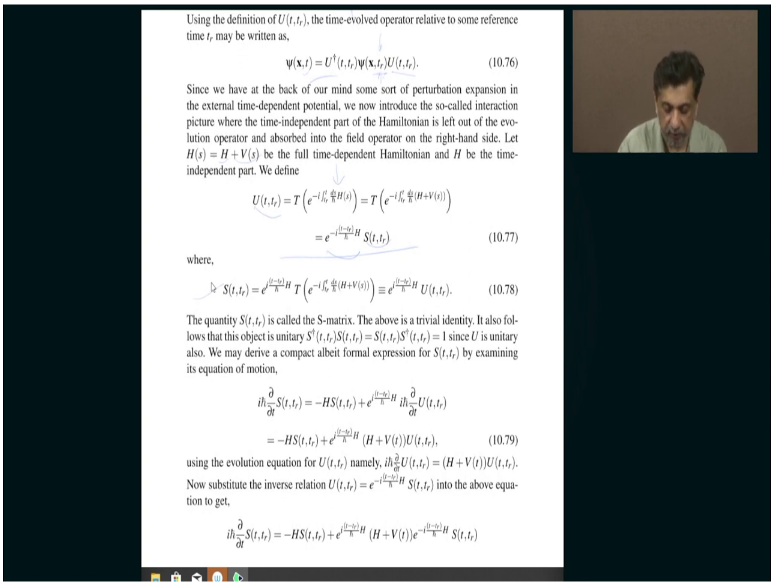

Using the definition of $U(t,t_r)$, the time-evolved operator relative to some reference time t_r may be written as,

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$$U(t,t_r) = T\left(e^{-i\int_{t_r}^{t}\frac{ds}{\hbar}H(s)}\right) = T\left(e^{-i\int_{t_r}^{t}\frac{ds}{\hbar}(H+V(s))}\right)$$

$$= e^{-i\frac{(t-t_r)}{\hbar}H}\, S(t,t_r) \quad (10.77)$$

where,

$$S(t,t_r) = e^{i\frac{(t-t_r)}{\hbar}H}\, T\left(e^{-i\int_{t_r}^{t}\frac{ds}{\hbar}(H+V(s))}\right) \equiv e^{i\frac{(t-t_r)}{\hbar}H}\, U(t,t_r). \quad (10.78)$$

The quantity $S(t,t_r)$ is called the S-matrix. The above is a trivial identity. It also follows that this object is unitary $S^{\dagger}(t,t_r)S(t,t_r) = S(t,t_r)S^{\dagger}(t,t_r) = 1$ since U is unitary also. We may derive a compact albeit formal expression for $S(t,t_r)$ by examining its equation of motion,

$$i\hbar\frac{\partial}{\partial t}S(t,t_r) = -HS(t,t_r) + e^{i\frac{(t-t_r)}{\hbar}H}\, i\hbar\frac{\partial}{\partial t}U(t,t_r)$$

$$= -HS(t,t_r) + e^{i\frac{(t-t_r)}{\hbar}H}\,(H+V(t))U(t,t_r), \quad (10.79)$$

using the evolution equation for $U(t,t_r)$ namely, $i\hbar\frac{\partial}{\partial t}U(t,t_r) = (H+V(t))U(t,t_r)$. Now substitute the inverse relation $U(t,t_r) = e^{-i\frac{(t-t_r)}{\hbar}H}\, S(t,t_r)$ into the above equation to get,

$$i\hbar\frac{\partial}{\partial t}S(t,t_r) = -HS(t,t_r) + e^{i\frac{(t-t_r)}{\hbar}H}\,(H+V(t))e^{-i\frac{(t-t_r)}{\hbar}H}\, S(t,t_r)$$

So, having kept that in mind now let us go ahead and so, I think this I am sort of repeating myself. So, remember that I was had defined something called the evolution operator where I am evolving. So, if I start as the Heisenberg picture, I am just unnecessarily repeating myself is probably from your elementary quantum mechanics.

So, you have this time operator at some fixed time reference time t r and then I want to evolve it to some other later time t. So, how do I do that? I simply introduce this Heisenberg evolution operators and the point is that if this system Hamiltonian is time dependent I have to necessarily define evolution in this peculiar way which will involve time ordering over times between t r and t.

So, now the thing is that because I can split up this Hamiltonian into two part parts 1 is time independent part, which is h and the other is time dependent part which is V of s I can always rewrite the evolution as a standard unitary evolution with respect to the time independent part and a part which is basically depends on the interaction.

So, this would correspond to the interaction picture ok. I think now is a good time to stop because in the next class I am going to tell you that this s of a certain well known equations which I can then use to exploit to rewrite it in this in this very standard way.

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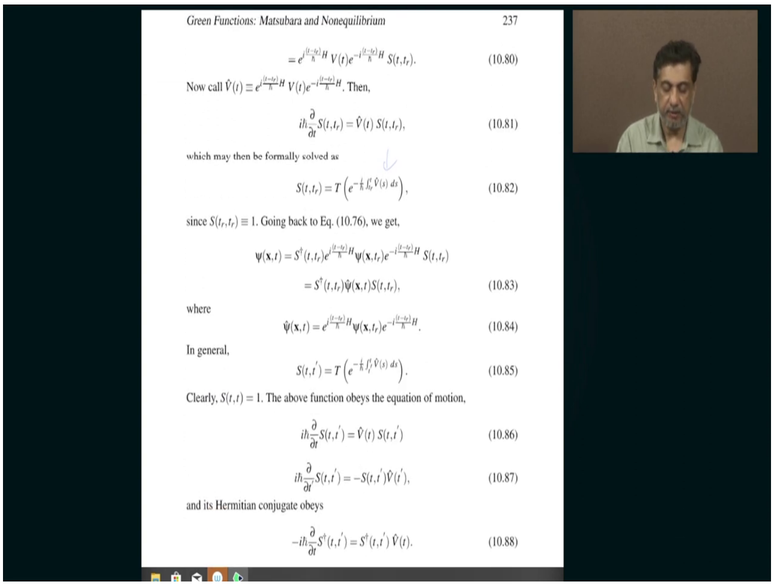

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$$= e^{i\frac{(t-t_r)}{\hbar}H}\, V(t) e^{-i\frac{(t-t_r)}{\hbar}H}\, S(t,t_r). \qquad (10.80)$$

Now call $\hat{V}(t) \equiv e^{i\frac{(t-t_r)}{\hbar}H}\, V(t) e^{-i\frac{(t-t_r)}{\hbar}H}$. Then,

$$i\hbar\frac{\partial}{\partial t}S(t,t_r) = \hat{V}(t)\, S(t,t_r), \qquad (10.81)$$

which may then be formally solved as

$$S(t,t_r) = T\left(e^{-\frac{i}{\hbar}\int_{t_r}^{t}\hat{V}(s)\,ds}\right), \qquad (10.82)$$

since $S(t_r,t_r) \equiv 1$. Going back to Eq. (10.76), we get,

$$\begin{aligned}\psi(\mathbf{x},t) &= S^{\dagger}(t,t_r) e^{i\frac{(t-t_r)}{\hbar}H}\psi(\mathbf{x},t_r) e^{-i\frac{(t-t_r)}{\hbar}H}\, S(t,t_r)\\ &= S^{\dagger}(t,t_r)\hat{\psi}(\mathbf{x},t)S(t,t_r), \qquad (10.83)\end{aligned}$$

where

$$\hat{\psi}(\mathbf{x},t) = e^{i\frac{(t-t_r)}{\hbar}H}\psi(\mathbf{x},t_r) e^{-i\frac{(t-t_r)}{\hbar}H}. \qquad (10.84)$$

In general,

$$S(t,t') = T\left(e^{-\frac{i}{\hbar}\int_{t'}^{t}\hat{V}(s)\,ds}\right). \qquad (10.85)$$

Clearly, $S(t,t) = 1$. The above function obeys the equation of motion,

$$i\hbar\frac{\partial}{\partial t}S(t,t') = \hat{V}(t)\, S(t,t') \qquad (10.86)$$

$$i\hbar\frac{\partial}{\partial t'}S(t,t') = -S(t,t')\hat{V}(t'), \qquad (10.87)$$

and its Hermitian conjugate obeys

$$-i\hbar\frac{\partial}{\partial t}S^{\dagger}(t,t') = S^{\dagger}(t,t')\, \hat{V}(t). \qquad (10.88)$$

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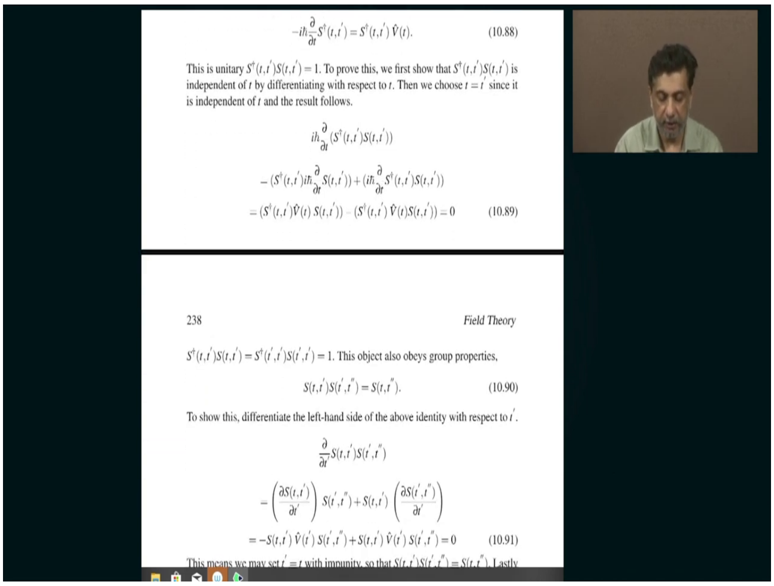

$$-i\hbar\frac{\partial}{\partial t}S^{\dagger}(t,t') = S^{\dagger}(t,t')\,\hat{V}(t). \qquad (10.88)$$

This is unitary $S^{\dagger}(t,t')S(t,t') = 1$. To prove this, we first show that $S^{\dagger}(t,t')S(t,t')$ is independent of t by differentiating with respect to t. Then we choose $t = t'$ since it is independent of t and the result follows.

$$i\hbar\frac{\partial}{\partial t}(S^{\dagger}(t,t')S(t,t'))$$

$$= (S^{\dagger}(t,t')i\hbar\frac{\partial}{\partial t}S(t,t')) + (i\hbar\frac{\partial}{\partial t}S^{\dagger}(t,t')S(t,t'))$$

$$= (S^{\dagger}(t,t')\hat{V}(t)\,S(t,t')) - (S^{\dagger}(t,t')\,\hat{V}(t)S(t,t')) = 0 \qquad (10.89)$$

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$S^{\dagger}(t,t')S(t,t') = S^{\dagger}(t',t')S(t',t') = 1$. This object also obeys group properties,

$$S(t,t')S(t',t'') = S(t,t''). \qquad (10.90)$$

To show this, differentiate the left-hand side of the above identity with respect to t'.

$$\frac{\partial}{\partial t'}S(t,t')S(t',t'')$$

$$= \left(\frac{\partial S(t,t')}{\partial t'}\right) S(t',t'') + S(t,t')\left(\frac{\partial S(t',t'')}{\partial t'}\right)$$

$$= -S(t,t')\,\hat{V}(t')\,S(t',t'') + S(t,t')\,\hat{V}(t')\,S(t',t'') = 0 \qquad (10.91)$$

This means we may set $t' = t$ with impunity, so that $S(t,t')S(t',t'') = S(t,t'')$. Lastly

And then I can rewrite my fields in the interaction picture and then I can go ahead and recast the whole Green's function in the interaction picture and we can proceed that way. So, I can understand that many of these lectures can be quite technical and they may seem kind of very mechanical to many of you listeners, but that is something that cannot be helped.

So, you should if you feel bored you can always you know pause the video you know think about these issues on your own and then resume the video or you can consult the text book relevant chapter and go through all the steps because you see that text book has been written with the specific purpose of filling in all the missing steps usually most of the text books follow a lot of steps whereas, in my book I have made sure that many of the steps are explicitly given so, that you know a student will not feel taxed and harassed by these concepts. So, they can easily follow many of the steps that are being explained because they are explicitly written down.

So, I think you should spend some time reading the textbook at leisure and try to follow all the steps because in the lectures I will just be breezing past all those steps by just repeating all the steps verbally which can be quite boring and also quite repetitive. So, you should necessarily go through the book and follow the steps on your own and then ask questions during the live sessions or by email or whatever ok.

I am going to stop here and in the next class let us discuss how to make peace with this concept of non-equilibrium Green’s function how to make sense out of them and how to utilize them in practice.

Thank you.