Dynamics of Classical and Quantum Fields: An Introduction

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Creation and Annihilation

Lecture - 33

Creation and annihilation operators - Many-body physics

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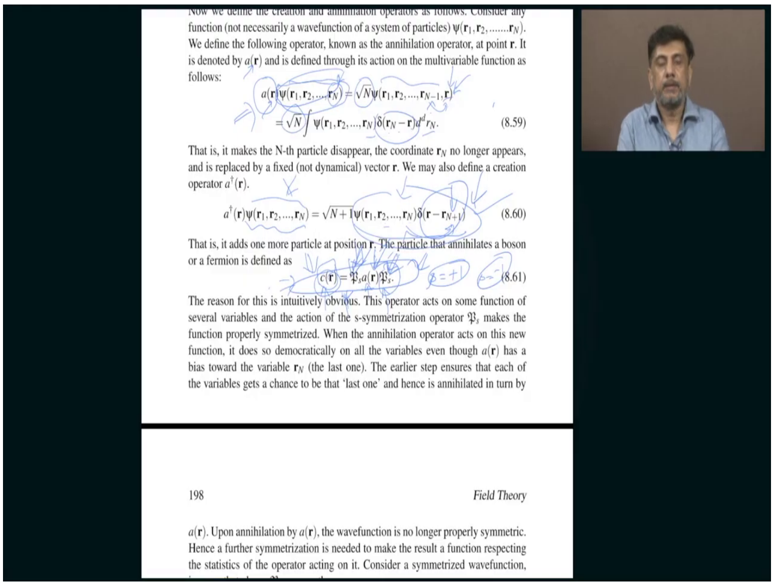

So, in today's class, let us continue our discussion of Creation and annihilation operators in Many particle condensed matter systems. So, that means, you see in the earlier classes we had seen how creation and annihilation operators are very useful in studying excitations of systems with finite number of degrees of freedom or even in the case of say a crystalline solid like in one dimension, you had an infinitely many degrees of freedom.

But then, the point is that the masses and the masses involved in that system are fixed in number; but it is the excitations that are varying in number. So, we could study excitations of such systems through creation and annihilation operators very conveniently and a very similar approach was useful also in the study of the quantum nature of the electromagnetic field.

Because the excitations of the electromagnetic field when studied quantum mechanically result in this very important notion of photons and photons are very critical in understanding, a very important phenomenon which perplexed most of the physics community in the early part of the 20th century namely the photoelectric effect.

But however, the utility of the creation and annihilation operator method is not limited to the study of excitations. It is also very useful in studying a more general class of systems, where not only the excitations are can be created and annihilated; but the particles themselves are now viewed as excitations of some vacuum.

So, that is typical in relativistic field theory says especially the Dirac theory of the electron, where the vacuum is not just empty in the sense of being absolutely devoid of any dynamics; but rather it is an infinite reservoir of negative energy particle. So, if you pump enough energy into vacuum, the energy is swallowed by the vacuum and it generates particles and anti particles.

So, in that sense, it is very important to learn how to study how to create and annihilate material particles, not only excitations. Because material particles also in this modern way of thinking are merely excitations of some other field ok. With that preamble, I had started off explaining to you the mathematical definitions of the annihilation operator specifically and also, the creation operator.

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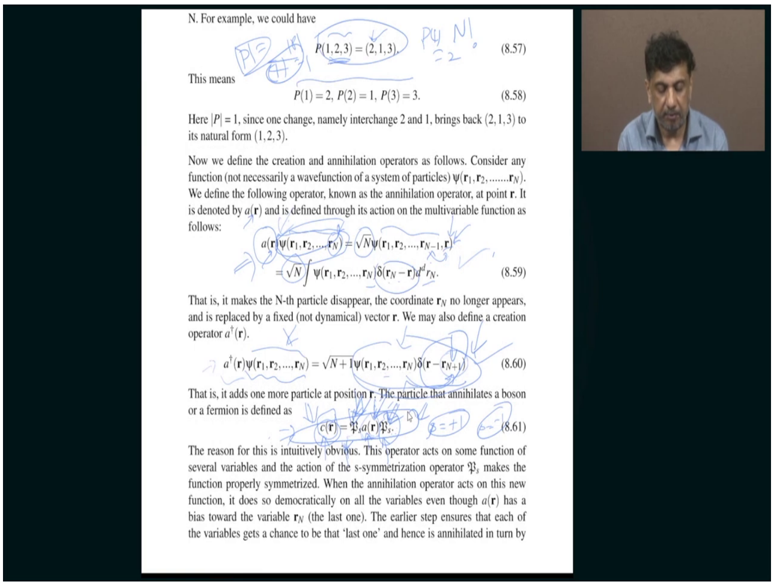
N. For example, we could have

$$P(1,2,3) = (2,1,3). \quad (8.57)$$

This means

$$P(1) = 2,\ P(2) = 1,\ P(3) = 3. \quad (8.58)$$

Here $|P| = 1$, since one change, namely interchange 2 and 1, brings back $(2,1,3)$ to its natural form $(1,2,3)$.

Now we define the creation and annihilation operators as follows. Consider any function (not necessarily a wavefunction of a system of particles) $\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots\ldots \mathbf{r}_N)$. We define the following operator, known as the annihilation operator, at point $\mathbf{r}$. It is denoted by $a(\mathbf{r})$ and is defined through its action on the multivariable function as follows:

$$a(\mathbf{r})\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \sqrt{N}\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N-1}, \mathbf{r})$$
$$= \sqrt{N}\int \psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\delta(\mathbf{r}_N - \mathbf{r})d^d r_N. \quad (8.59)$$

That is, it makes the N-th particle disappear, the coordinate $\mathbf{r}_N$ no longer appears, and is replaced by a fixed (not dynamical) vector $\mathbf{r}$. We may also define a creation operator $a^\dagger(\mathbf{r})$.

$$a^\dagger(\mathbf{r})\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \sqrt{N+1}\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\delta(\mathbf{r} - \mathbf{r}_{N+1}) \quad (8.60)$$

That is, it adds one more particle at position $\mathbf{r}$. The particle that annihilates a boson or a fermion is defined as

$$c(\mathbf{r}) = \mathfrak{P}_s a(\mathbf{r})\mathfrak{P}_s. \quad (8.61)$$

The reason for this is intuitively obvious. This operator acts on some function of several variables and the action of the s-symmetrization operator $\mathfrak{P}_s$ makes the function properly symmetrized. When the annihilation operator acts on this new function, it does so democratically on all the variables even though $a(\mathbf{r})$ has a bias toward the variable $\mathbf{r}_N$ (the last one). The earlier step ensures that each of the variables gets a chance to be that 'last one' and hence is annihilated in turn by

So, I had reached up to this point, where I had told you that in order to define the annihilation operator, you first introduce a concept called a symmetrization operator which ensures that the wave function of many particles which may not necessarily be properly symmetrized is first properly symmetrized and then, it is then acted upon by an annihilation operator which annihilates a coordinate rather than a particle.

So, specifically this operator called a of r is preferentially annihilates the last coordinate, it finds in the wave function. And because it is biased towards the last coordinate, it is very important to first properly symmetries the wave function so that every coordinate, then gets an opportunity to be that last coordinate.

Because the symmetrization ensures that there is a linear combination of terms where that last coordinate is successively replaced by a different one in each term. So, point is that you could go ahead and define the annihilation operator in this way. But then, this annihilates the last coordinate. So, similarly, you have a creation operator which creates an additional coordinate which was not there earlier.

So, imagine you have a system of N particles. So, it depends upon N coordinates r 1, r 2 up to r N; but then, you also create one more coordinate called r N plus 1. So, that r N plus 1, this coordinate was not there earlier. So, but then you are creating it through this.

So, that amounts to creating a new particle at position r N plus 1. But then, remember that we have it is still not a bona fide particle; in the sense that it right now, it is merely creating a coordinate.

In order to create a particle, we have to ensure that the wave function that is finally, obtained is again properly symmetrized, properly symmetrized means it could be fully symmetric under pairwise exchange which would correspond to bosons or it could be fully anti symmetric under pair wise exchange which would correspond to fermions.

So, the way to accomplish that I already explained to you in the last class is to sandwich this operator which annihilates a coordinate between the symmetrization between two symmetrization operators. So, symmetrization means symmetrization or anti symmetrization as the case may be.

So, by doing this, you see you are ensuring that before you annihilate, you first properly symmetrize the wave function so that every coordinate gets an opportunity to be the last one and then finally, this annihilation operator annihilates the last coordinate and having annihilated the last coordinate, again the wave function is not guaranteed to be properly symmetric.

So, you again anti symmetrize or symmetrize as the case may be and then, you get back a wave function which has one fewer particle; but still represents the same class of particles whether they are bosons or fermions as they were to start with.

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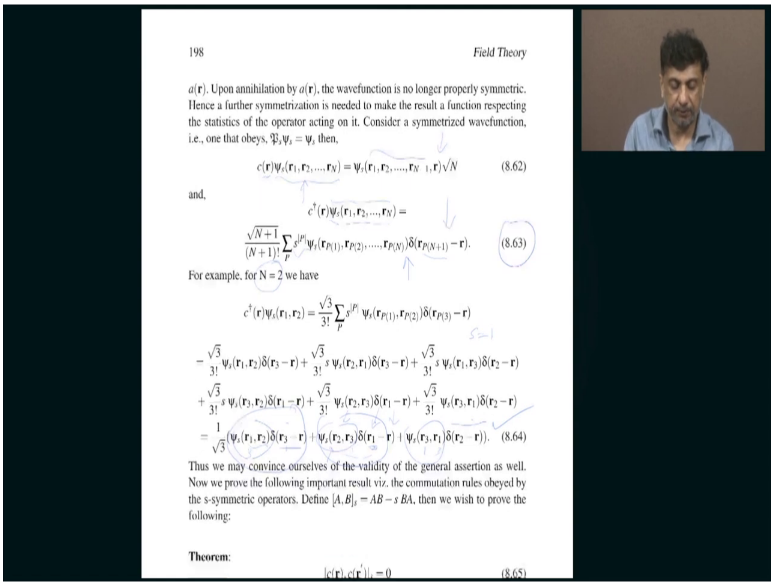

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$a(\mathbf{r})$. Upon annihilation by $a(\mathbf{r})$, the wavefunction is no longer properly symmetric. Hence a further symmetrization is needed to make the result a function respecting the statistics of the operator acting on it. Consider a symmetrized wavefunction, i.e., one that obeys, $\mathfrak{P}_s\psi_s = \psi_s$ then,

$$c(\mathbf{r})\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N) = \psi_s(\mathbf{r}_1,\mathbf{r}_2,....,\mathbf{r}_{N-1},\mathbf{r})\sqrt{N} \quad (8.62)$$

and,

$$c^\dagger(\mathbf{r})\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N) = \frac{\sqrt{N+1}}{(N+1)!}\sum_P s^{|P|}\psi_s(\mathbf{r}_{P(1)},\mathbf{r}_{P(2)},....,\mathbf{r}_{P(N)})\delta(\mathbf{r}_{P(N+1)}-\mathbf{r}). \quad (8.63)$$

For example, for N = 2 we have

$$c^\dagger(\mathbf{r})\psi_s(\mathbf{r}_1,\mathbf{r}_2) = \frac{\sqrt{3}}{3!}\sum_P s^{|P|}\psi_s(\mathbf{r}_{P(1)},\mathbf{r}_{P(2)})\delta(\mathbf{r}_{P(3)}-\mathbf{r})$$

$$= \frac{\sqrt{3}}{3!}\psi_s(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}_3-\mathbf{r}) + \frac{\sqrt{3}}{3!}s\,\psi_s(\mathbf{r}_2,\mathbf{r}_1)\delta(\mathbf{r}_3-\mathbf{r}) + \frac{\sqrt{3}}{3!}s\,\psi_s(\mathbf{r}_1,\mathbf{r}_3)\delta(\mathbf{r}_2-\mathbf{r})$$

$$+ \frac{\sqrt{3}}{3!}s\,\psi_s(\mathbf{r}_3,\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}) + \frac{\sqrt{3}}{3!}\psi_s(\mathbf{r}_2,\mathbf{r}_3)\delta(\mathbf{r}_1-\mathbf{r}) + \frac{\sqrt{3}}{3!}\psi_s(\mathbf{r}_3,\mathbf{r}_1)\delta(\mathbf{r}_2-\mathbf{r})$$

$$= \frac{1}{\sqrt{3}}(\psi_s(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}_3-\mathbf{r}) + \psi_s(\mathbf{r}_2,\mathbf{r}_3)\delta(\mathbf{r}_1-\mathbf{r}) + \psi_s(\mathbf{r}_3,\mathbf{r}_1)\delta(\mathbf{r}_2-\mathbf{r})). \quad (8.64)$$

Thus we may convince ourselves of the validity of the general assertion as well. Now we prove the following important result viz. the commutation rules obeyed by the s-symmetric operators. Define $[A,B]_s = AB - s\,BA$, then we wish to prove the following:

Theorem:

$$[c(\mathbf{r}),c(\mathbf{r}')]_s = 0 \quad (8.65)$$

So, that is the whole point of this method of defining annihilation operators. So, similarly, with creation operators also we have to you know first. So, imagine suppose have you already have a symmetrized wave function. So, if you already have a symmetrized wave function things are much simpler. So, if it is already properly symmetrized, you do not have to again you could again symmetrize; but you will get back the same result. So, you see the this is an operator that annihilates a particle.

So, the relation between an operator that annihilates a particle which is called c of r and an operator that annihilates a coordinate is basically that the two are related through this formula 8.61, where you sandwich the operator that annihilates a coordinate between two operators that symmetrize the wave function. But then, if it is already symmetrized properly to begin with, then you do not need to further symmetrize, you simply annihilate the last one.

Because then, the last one, annihilating the last one does not mean that you are you know preferentially you are giving a bias towards the last one. Because then, you see the all the wave functions have you know the properly symmetrized wave function has already taken care of the fact that the remaining one. So, the remaining coordinates will be properly symmetric or anti symmetric as it was earlier ok.

So, that is the reason why if you start off with a properly symmetrized wave function, you are in luck because you do not have to struggle too hard while defining the annihilation operator. So, similarly, while defining the creation operator you end up doing this. So, the creation operator is a little bit harder because you see once you create a particle at position r N plus 1.

Then of course, you are maximally spoiling the symmetry because you see the wave function is symmetric under the exchange of any two coordinates. So, long as those two coordinates are in that you know starting set from r 1, r 2 all the way up to r N only. So, but then on the other hand, you have now created a new coordinate called r N plus 1; now that is no longer guaranteed to be properly symmetric or anti symmetric with respect to the remaining ones.

So, that means, if you exchange r 1 with r 2 or r N, you are going to get the right answer. But if you interchange r 1 with r N plus 1 which has now been added that is no longer guaranteed to be properly symmetric, unless you do what 8.63 suggests to you. Namely that it democratically interchanges r N plus 1 through permutations with all the remaining coordinates. So, it is kind of it permutes over all the permutations of 1, 2, 3 up to r N plus 1 and it makes sure that the proper signs are being counted.

So, if you are talking about fermions, you have to make sure that you put a minus sign every time the permutation is odd and a plus sign every time the permutation is even. So, when you do that, you are guaranteed to; so, if you start off with a system of N fermions for example. And then you create a fermion at position r, you are guaranteed to end up with a wave function that corresponds to you know a particle that is I mean you are guaranteed to find yourself with a system with one more particle, where one of the fermions is at r which is what you want.

So, you know just to be concrete. So, this may look little formidable, but it will look easier to understand, if you specialize to a specific value of the number of particles. So, if you select capital N to be 2 for example, which means your starting number of particles had two particles in it and then, you create one more particle right. So, you end up with you know a wave function that looks like this. So, this is perfectly symmetric or

I mean I am talking about say Bosons here. Let us see yeah I think I am specializing to s equal to plus 1 here.

So, if it is Bosons, then you can clearly see that you know if you interchange 1 and 2, you get back what you are looking for. But if you interchange say 1 and 3 right. So, this becomes 3 and this becomes 1 and this becomes 1 and this becomes 3 and this becomes 1, this becomes 3 ok. So, yeah this is in general, not necessarily for a sequence 1 because you see if you interchange 1 and 3, what is going to happen is that this will become psi s r 3, r 2 right.

So, r 3, r 2 is basically s times r 2, r 3. So, it will become. So, this term will become s times this term and this delta function will become this delta function. So, that this entire term will become s times this function right and this will become s times. So, r 3, r 1 will go to r 1, r 3; but then psi of r 1, r 3 is same as s times psi of r 3, r 1. So, it again becomes s of itself. So, finally, if you interchange say 1 and 3, you get the same wave function multiplied by s.

So, the interchange of any two of them, any two coordinates is basically s times itself. I mean s times the original wave function. So, therefore, this is the correct way of creating a particle new particle at r; it is been created at r, obviously because you see it is democratically, so r is the position at which a fermion or Boson has been created, but then you see every coordinate here has an opportunity to sit at r.

So, in this term r 1 has that opportunity to sit at r here; r 2 has the opportunity to sit at r and here, it is r 3 has that opportunity

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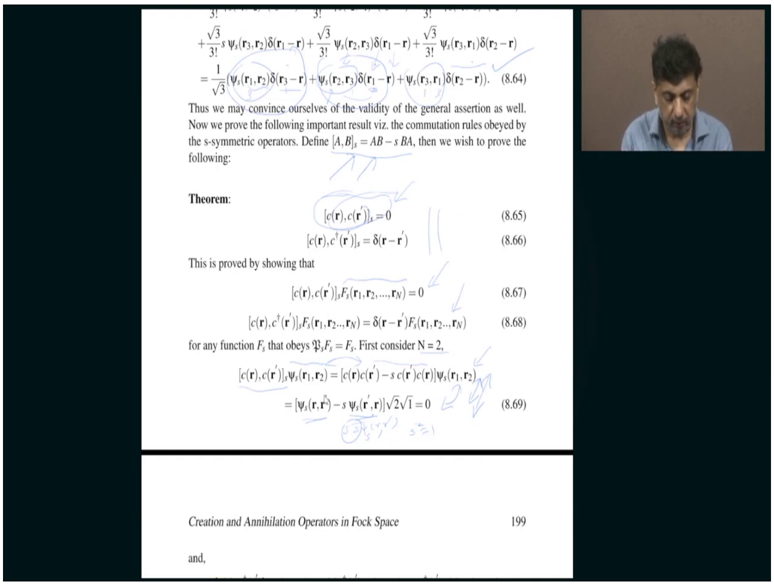

So, this way of doing things you know democratises how the creation operator acts on the wave functions ok. So, now, with this sort of a machinery, we can go ahead and prove certain important theorems that are going to be useful later on and that is for example, the first theorem that I am going to prove is that the first, I am going to define this s commutator.

So, that means, that if s is plus 1 is the usual commutator describing Boson, so that means, is plus 1, then A commutator B subscript plus 1 means AB minus BA. See, however, if s is minus 1 A commutator B with subscript minus 1 is AB plus BA. So, that is called the anti commutator. So, that is of importance when you are studying fermions. So, when s is plus 1, this definition corresponds to the usual commutator, but if s is minus one it corresponds to an anti commutator.

So, the anti commutator of A and B is AB plus BA. So, the point is that I am going to be I am going to now prove that the commutator or anti commutator as the case may be of two c s, two different positions is always 0. However, c and it is adjoints so that means, that so the two annihilation operators will always properly commute; properly commute means it will either the commutator is 0, if it is Bosons and anti commutator is 0, if it is fermions.

So, that is what that is what I mean by properly commute. So, the word commute actually means travel. See you commute from your home to work. So, commutator means it is a device that tells an operator how to travel across another operator. So, that is what commutator means. So, commutator is a device that or it is a prescription that pins down you know how an operator, what rules an operator has to obey in order to travel across another operator.

So, that is why it is called commutator. So, the point is that the commutator of c and c; I mean the commutator of two annihilation operators is going to be 0. So, I am going to prove this. So, I am going to prove that the s commutator of the two annihilation operators is 0; whereas, the s commutator of c and the annihilation and the creation operators is basically the Dirac delta function.

So, how do you prove this? So, this is very crucial. This is one of the central ideas in I mean this is one of the central relations that are going to be repeatedly used in all the calculations that you are going to do using creation and annihilation operators in many body physics. So, how do you prove this? Of course, you see these are these operators act on many body wave functions.

So, you should imagine that there is a many body wave function and as usual, I am going to assume that it is properly assymmetrize. So, that means, it is either properly symmetric or properly anti symmetric and then, now I am going to act this operator which acts supposed to act on N particle wave functions and clearly, there are two annihilations. So, I am going to be you know eliminating two coordinates from this wave function.

So, clearly, I should make sure that there are at least two particles in my system. Because otherwise proving this is meaningless. If there is less than two particles in the system, this is trivially always correct. But to prove this I need to show that regardless of how many particles there are, this is always true. So, in other words, I have to show this and this ok.

So, how do I show the first one? So, as usual I am going to. So, in this course, I am not going to be very careful about proving in the sense in which mathematicians prove things; I am going to prove by examples which is really not a proof at all. Because you see nobody is going to accept examples as substitutes for proofs. So, these are not proofs, they are plausibility arguments which I have wrongly characterized as proofs.

But then, the point is that you know as physicists, we have this intuitive feeling for when things are going to work out properly. So, through maybe inductive reasoning we work out a few examples and we do not expect any pathological exceptions. So, many times our intuition is correct and when we work out a small number of examples and if things are ok, we are completely justified in assuming that it is going to be ok in general.

There are some rare exceptions even in physics; but those are of interest only because they are exceptions rather than the rule. So, that is also true even in mathematics that many times your intuition works perfectly fine; but when it does not work, it actually becomes a research topic in itself that mathematicians make a big fuss about exceptions precisely because those are exceptions to.

So, in other words, that is where your intuition and guesswork fail. So, for the most part, I am not going to actually prove anything; I am going to rather use examples and then, use inductive reasoning and just claim that it is going to work. So, if those of you are dissatisfied with my approach can of course invited to go ahead and prove it rigorously using the tools that they are comfortable with.

In fact, that is a good idea to do it properly because you know it trains you in logical thinking. But the reason why I do not do it is because it is basically a lot of effort and as a physicist, I have better things to do in the sense of getting to the more difficult physics parts of the subject, which are anyway going to be very hard in themselves all right. So, now, I start off with a simple example, where there are only two particles in which case I start off with this wave function with two particles.

Then, I act it on this s commutator with respect to r and r dash and now, the question is this is what the s commutator is. Now, if I go ahead and annihilate r dash; so, clearly it is going to be this and it is going to be that ok. So, I mean, I will allow you to at least work this out. This is going from here to here is something you should do it yourself. If you want me to even explain how to go from here to here; that means, you are not following anything.

So, it is important for you to do this yourself. So, this to this is obvious; but this to this is less obvious, but you should figure that out. So, the point is that having reached here is now obvious that this is same as s times. So, it is s s times r r dash, but then s squared is always 1 because s is either plus 1 or minus 1. So, therefore, this is just psi minus psi which is 0.

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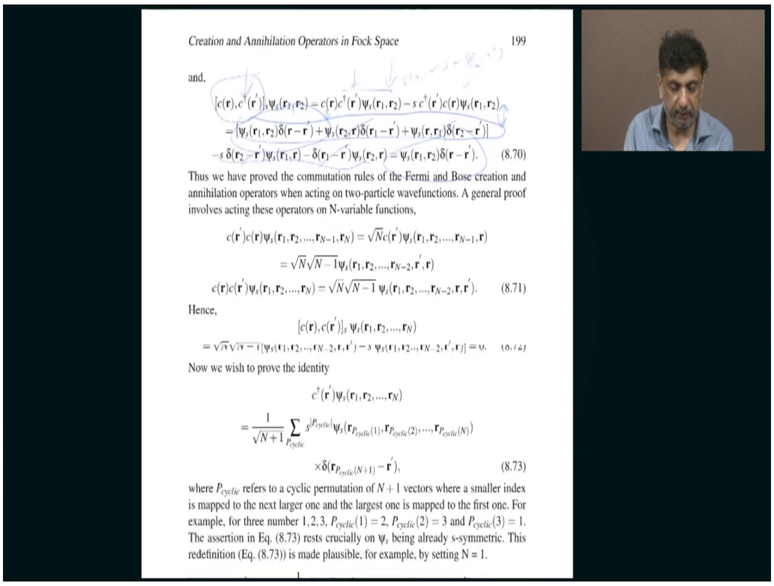

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and,

$$[c(\mathbf{r}),c^{\dagger}(\mathbf{r}')]_s\psi_s(\mathbf{r}_1,\mathbf{r}_2)=c(\mathbf{r})c^{\dagger}(\mathbf{r}')\psi_s(\mathbf{r}_1,\mathbf{r}_2)-s\,c^{\dagger}(\mathbf{r}')c(\mathbf{r})\psi_s(\mathbf{r}_1,\mathbf{r}_2)$$
$$=[\psi_s(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}-\mathbf{r}')+\psi_s(\mathbf{r}_2,\mathbf{r})\delta(\mathbf{r}_1-\mathbf{r}')+\psi_s(\mathbf{r},\mathbf{r}_1)\delta(\mathbf{r}_2-\mathbf{r}')]$$
$$-s\,\delta(\mathbf{r}_2-\mathbf{r}')\psi_s(\mathbf{r}_1,\mathbf{r})-\delta(\mathbf{r}_1-\mathbf{r}')\psi_s(\mathbf{r}_2,\mathbf{r})=\psi_s(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}-\mathbf{r}'). \quad (8.70)$$

Thus we have proved the commutation rules of the Fermi and Bose creation and annihilation operators when acting on two-particle wavefunctions. A general proof involves acting these operators on N-variable functions,

$$c(\mathbf{r}')c(\mathbf{r})\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_{N-1},\mathbf{r}_N)=\sqrt{N}c(\mathbf{r}')\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_{N-1},\mathbf{r})$$
$$=\sqrt{N}\sqrt{N-1}\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_{N-2},\mathbf{r}',\mathbf{r})$$
$$c(\mathbf{r})c(\mathbf{r}')\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N)=\sqrt{N}\sqrt{N-1}\,\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_{N-2},\mathbf{r},\mathbf{r}'). \quad (8.71)$$

Hence,

$$[c(\mathbf{r}),c(\mathbf{r}')]_s\,\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N)$$
$$=\sqrt{N}\sqrt{N-1}[\psi_s(\mathbf{r}_1,\mathbf{r}_2,..,\mathbf{r}_{N-2},\mathbf{r},\mathbf{r}')-s\,\psi_s(\mathbf{r}_1,\mathbf{r}_2..,\mathbf{r}_{N-2},\mathbf{r}',\mathbf{r})]=0. \quad (8.72)$$

Now we wish to prove the identity

$$c^{\dagger}(\mathbf{r}')\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N)$$
$$=\frac{1}{\sqrt{N+1}}\sum_{P_{cyclic}}s^{|P_{cyclic}|}\psi_s(\mathbf{r}_{P_{cyclic}(1)},\mathbf{r}_{P_{cyclic}(2)},...,\mathbf{r}_{P_{cyclic}(N)})$$
$$\times\delta(\mathbf{r}_{P_{cyclic}(N+1)}-\mathbf{r}'), \quad (8.73)$$

where P_{cyclic} refers to a cyclic permutation of $N+1$ vectors where a smaller index is mapped to the next larger one and the largest one is mapped to the first one. For example, for three number 1,2,3, $P_{cyclic}(1)=2$, $P_{cyclic}(2)=3$ and $P_{cyclic}(3)=1$. The assertion in Eq. (8.73) rests crucially on ψ_s being already s-symmetric. This redefinition (Eq. (8.73)) is made plausible, for example, by setting N = 1.

So, that concludes my plausibility argument and now, I am going to claim that it is valid for all N which is certainly no proof at all. But it is good enough for me you know my standards are very low ok. So, I am going to next prove that the s commutator of the annihilation and the creation operators acting on psi gives back the same wave function multiplied by Dirac delta function. So, how do I prove that?

Again, here I am going to start with two particles wave functions just for simplicity. Now, I create one more particle here. So, I create one more particle. So, how do I create a particle? So, first, I have to rewrite this as see first I have skipped many steps going from here to here. See first I have to create a particle here. So, what does that mean? So, I have to do all the things I have been doing there.

So, that means, I first multiply this by Dirac delta r 1 r dash, then I multiply by you know r 2 minus r dash; but with an s because now, I am doing the second one. So, r 2 minus r dash. So, I have to do all that. So, I have skipped many steps. So, here I first create, then I annihilate. So, here I first annihilate and then create. So, when I do all that I end up with this result ok.

So, this is the term I will get when I do this. So, that means, I first create and annihilate I get this; whereas, this one. So, this one I will get when I do this ok. So, this is basically I am annihilating. So, I am annihilating, then creating. So, I am annihilating r 1 and r 2 and then, creating one more particle. So, anyway bottom line is that put together, I am going to get this.

So, again, I have skipped step just like here I skipped the step from going from here to here which is not obvious; but not that difficult. But going from here to here is somewhat less obvious. So, you really should work this out using the definitions of c and c dagger, I have already explained.

This is how c behaves when it acts on an particle wave function, this is how c dagger acts. So, now, that you know how c and c dagger acts, you have to go ahead and evaluate all this on your own. So, when you do that, you will see that terms cancel out in pairs. So, for example, this cancels with this and this cancels with this because you see r 1 comma r is s times r comma r 1.

So, but s squared is one or alternatively, psi s of r comma r 1 is s times psi s of r 1 comma r and those two will cancel out and you get only end up with this and this is basically this. So, in other words, it is so the s commutator of c and c dagger acting on the wave function of two particles is the same wave function multiplied by delta of r minus r dash ok.

So, you can go ahead and prove that in general for many particles, I have just indicated how you might go about doing it for more than two particles.

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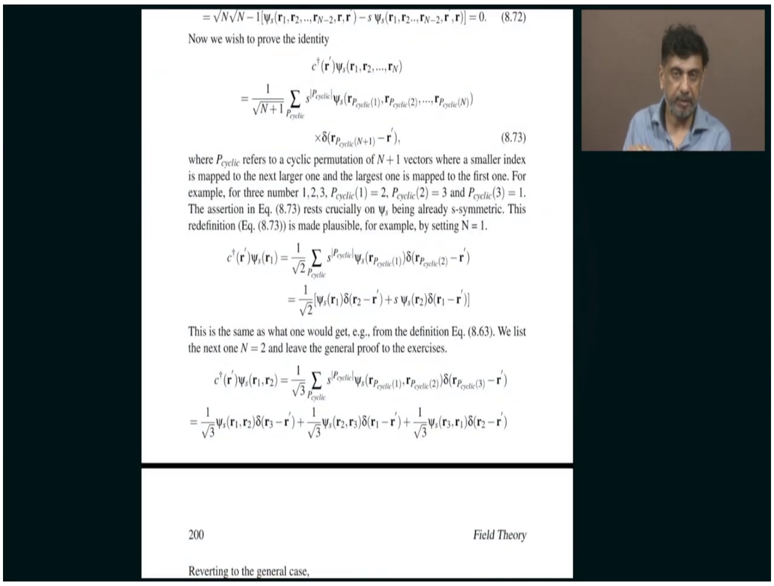

$$= \sqrt{N}\sqrt{N-1}[\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_{N-2},\mathbf{r},\mathbf{r}') - s\,\psi_s(\mathbf{r}_1,\mathbf{r}_2..,\mathbf{r}_{N-2},\mathbf{r}',\mathbf{r})] = 0. \quad (8.72)$$

Now we wish to prove the identity

$$c^\dagger(\mathbf{r}')\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N) = \frac{1}{\sqrt{N+1}}\sum_{P_{cyclic}} s^{|P_{cyclic}|}\psi_s(\mathbf{r}_{P_{cyclic}(1)},\mathbf{r}_{P_{cyclic}(2)},...,\mathbf{r}_{P_{cyclic}(N)}) \times\delta(\mathbf{r}_{P_{cyclic}(N+1)} - \mathbf{r}'), \quad (8.73)$$

where P_{cyclic} refers to a cyclic permutation of $N+1$ vectors where a smaller index is mapped to the next larger one and the largest one is mapped to the first one. For example, for three number $1,2,3$, $P_{cyclic}(1) = 2$, $P_{cyclic}(2) = 3$ and $P_{cyclic}(3) = 1$. The assertion in Eq. (8.73) rests crucially on ψ_s being already s-symmetric. This redefinition (Eq. (8.73)) is made plausible, for example, by setting N = 1.

$$c^\dagger(\mathbf{r}')\psi_s(\mathbf{r}_1) = \frac{1}{\sqrt{2}}\sum_{P_{cyclic}} s^{|P_{cyclic}|}\psi_s(\mathbf{r}_{P_{cyclic}(1)})\delta(\mathbf{r}_{P_{cyclic}(2)} - \mathbf{r}') = \frac{1}{\sqrt{2}}[\psi_s(\mathbf{r}_1)\delta(\mathbf{r}_2 - \mathbf{r}') + s\,\psi_s(\mathbf{r}_2)\delta(\mathbf{r}_1 - \mathbf{r}')]$$

This is the same as what one would get, e.g., from the definition Eq. (8.63). We list the next one $N = 2$ and leave the general proof to the exercises.

$$c^\dagger(\mathbf{r}')\psi_s(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{\sqrt{3}}\sum_{P_{cyclic}} s^{|P_{cyclic}|}\psi_s(\mathbf{r}_{P_{cyclic}(1)},\mathbf{r}_{P_{cyclic}(2)})\delta(\mathbf{r}_{P_{cyclic}(3)} - \mathbf{r}')$$

$$= \frac{1}{\sqrt{3}}\psi_s(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}_3 - \mathbf{r}') + \frac{1}{\sqrt{3}}\psi_s(\mathbf{r}_2,\mathbf{r}_3)\delta(\mathbf{r}_1 - \mathbf{r}') + \frac{1}{\sqrt{3}}\psi_s(\mathbf{r}_3,\mathbf{r}_1)\delta(\mathbf{r}_2 - \mathbf{r}')$$

200 *Field Theory*

Reverting to the general case,

So, maybe you use some inductive reasoning, you know it is not merely asserting that it works for two particles. Therefore, it had better work for all other number of particles; I mean we are not like demanding anything. So, but then, inductive proof means that you successively you assume you first prove that it works for 2, 3 and then, you assume it works for N and then, you prove that it also because it works for N particles, then you go ahead and prove that it works for N plus 1 particle.

So, that is called inductive reasoning. So, that is called mathematical induction which you must have learned in school; at least I learned in school, when I was studying mathematical induction.

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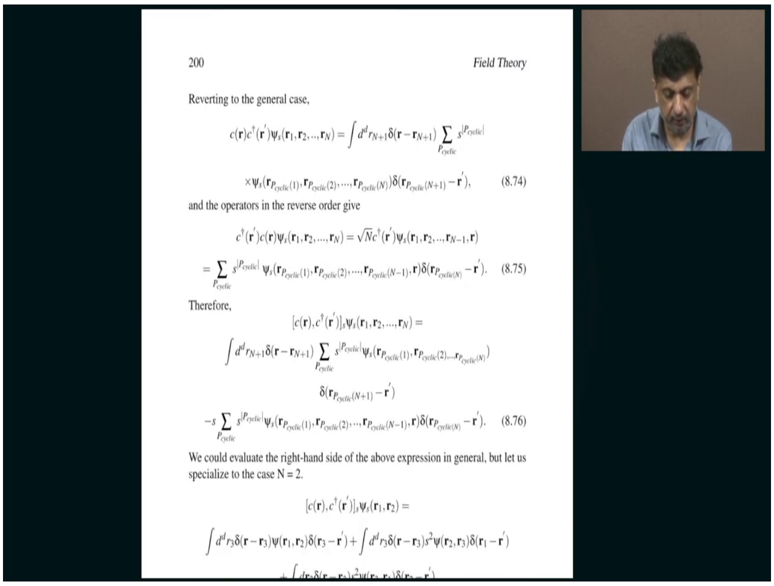
200 *Field Theory*

Reverting to the general case,

$$c(\mathbf{r})c^{\dagger}(\mathbf{r}')\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N) = \int d^d r_{N+1}\delta(\mathbf{r}-\mathbf{r}_{N+1}) \sum_{P_{cyclic}} s^{|P_{cyclic}|}$$

$$\times \psi_s(\mathbf{r}_{P_{cyclic}(1)},\mathbf{r}_{P_{cyclic}(2)},...,\mathbf{r}_{P_{cyclic}(N)})\delta(\mathbf{r}_{P_{cyclic}(N+1)}-\mathbf{r}'), \quad (8.74)$$

and the operators in the reverse order give

$$c^{\dagger}(\mathbf{r}')c(\mathbf{r})\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N) = \sqrt{N}c^{\dagger}(\mathbf{r}')\psi_s(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_{N-1},\mathbf{r})$$

$$= \sum_{P_{cyclic}} s^{|P_{cyclic}|}\ \psi_s(\mathbf{r}_{P_{cyclic}(1)},\mathbf{r}_{P_{cyclic}(2)},...,\mathbf{r}_{P_{cyclic}(N-1)},\mathbf{r})\delta(\mathbf{r}_{P_{cyclic}(N)}-\mathbf{r}'). \quad (8.75)$$

Therefore,

$$[c(\mathbf{r}),c^{\dagger}(\mathbf{r}')]_s\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N) =$$

$$\int d^d r_{N+1}\delta(\mathbf{r}-\mathbf{r}_{N+1}) \sum_{P_{cyclic}} s^{|P_{cyclic}|}\psi_s(\mathbf{r}_{P_{cyclic}(1)},\mathbf{r}_{P_{cyclic}(2)}...\mathbf{r}_{P_{cyclic}(N)})$$

$$\delta(\mathbf{r}_{P_{cyclic}(N+1)}-\mathbf{r}')$$

$$-s\sum_{P_{cyclic}} s^{|P_{cyclic}|}\psi_s(\mathbf{r}_{P_{cyclic}(1)},\mathbf{r}_{P_{cyclic}(2)},...\mathbf{r}_{P_{cyclic}(N-1)},\mathbf{r})\delta(\mathbf{r}_{P_{cyclic}(N)}-\mathbf{r}'). \quad (8.76)$$

We could evaluate the right-hand side of the above expression in general, but let us specialize to the case N = 2.

$$[c(\mathbf{r}),c^{\dagger}(\mathbf{r}')]_s\psi_s(\mathbf{r}_1,\mathbf{r}_2) =$$

$$\int d^d r_3\delta(\mathbf{r}-\mathbf{r}_3)\psi(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}_3-\mathbf{r}') + \int d^d r_3\delta(\mathbf{r}-\mathbf{r}_3)s^2\psi(\mathbf{r}_2,\mathbf{r}_3)\delta(\mathbf{r}_1-\mathbf{r}')$$

So, something like that has to be possibly employed here.

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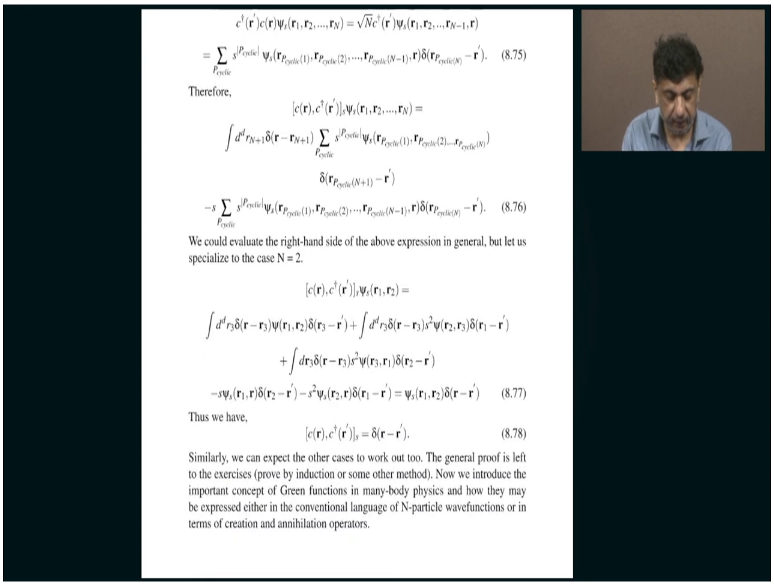
$$c^{\dagger}(\mathbf{r}')c(\mathbf{r})\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N) = \sqrt{N}c^{\dagger}(\mathbf{r}')\psi_s(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_{N-1},\mathbf{r})$$

$$= \sum_{P_{cyclic}} s^{|P_{cyclic}|}\ \psi_s(\mathbf{r}_{P_{cyclic}(1)},\mathbf{r}_{P_{cyclic}(2)},...,\mathbf{r}_{P_{cyclic}(N-1)},\mathbf{r})\delta(\mathbf{r}_{P_{cyclic}(N)}-\mathbf{r}'). \quad (8.75)$$

Therefore,

$$[c(\mathbf{r}),c^{\dagger}(\mathbf{r}')]_s\psi_s(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N) =$$

$$\int d^d r_{N+1}\delta(\mathbf{r}-\mathbf{r}_{N+1}) \sum_{P_{cyclic}} s^{|P_{cyclic}|}\psi_s(\mathbf{r}_{P_{cyclic}(1)},\mathbf{r}_{P_{cyclic}(2)}...\mathbf{r}_{P_{cyclic}(N)})$$

$$\delta(\mathbf{r}_{P_{cyclic}(N+1)}-\mathbf{r}')$$

$$-s\sum_{P_{cyclic}} s^{|P_{cyclic}|}\psi_s(\mathbf{r}_{P_{cyclic}(1)},\mathbf{r}_{P_{cyclic}(2)},...\mathbf{r}_{P_{cyclic}(N-1)},\mathbf{r})\delta(\mathbf{r}_{P_{cyclic}(N)}-\mathbf{r}'). \quad (8.76)$$

We could evaluate the right-hand side of the above expression in general, but let us specialize to the case N = 2.

$$[c(\mathbf{r}),c^{\dagger}(\mathbf{r}')]_s\psi_s(\mathbf{r}_1,\mathbf{r}_2) =$$

$$\int d^d r_3\delta(\mathbf{r}-\mathbf{r}_3)\psi(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}_3-\mathbf{r}') + \int d^d r_3\delta(\mathbf{r}-\mathbf{r}_3)s^2\psi(\mathbf{r}_2,\mathbf{r}_3)\delta(\mathbf{r}_1-\mathbf{r}')$$

$$+\int d\mathbf{r}_3\delta(\mathbf{r}-\mathbf{r}_3)s^2\psi(\mathbf{r}_3,\mathbf{r}_1)\delta(\mathbf{r}_2-\mathbf{r}')$$

$$-s\psi_s(\mathbf{r}_1,\mathbf{r})\delta(\mathbf{r}_2-\mathbf{r}') - s^2\psi_s(\mathbf{r}_2,\mathbf{r})\delta(\mathbf{r}_1-\mathbf{r}') = \psi_s(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}-\mathbf{r}') \quad (8.77)$$

Thus we have,

$$[c(\mathbf{r}),c^{\dagger}(\mathbf{r}')]_s = \delta(\mathbf{r}-\mathbf{r}'). \quad (8.78)$$

Similarly, we can expect the other cases to work out too. The general proof is left to the exercises (prove by induction or some other method). Now we introduce the important concept of Green functions in many-body physics and how they may be expressed either in the conventional language of N-particle wavefunctions or in terms of creation and annihilation operators.

So, I have given you some indications of how to do that using cyclic permutation and so on and so forth ok.

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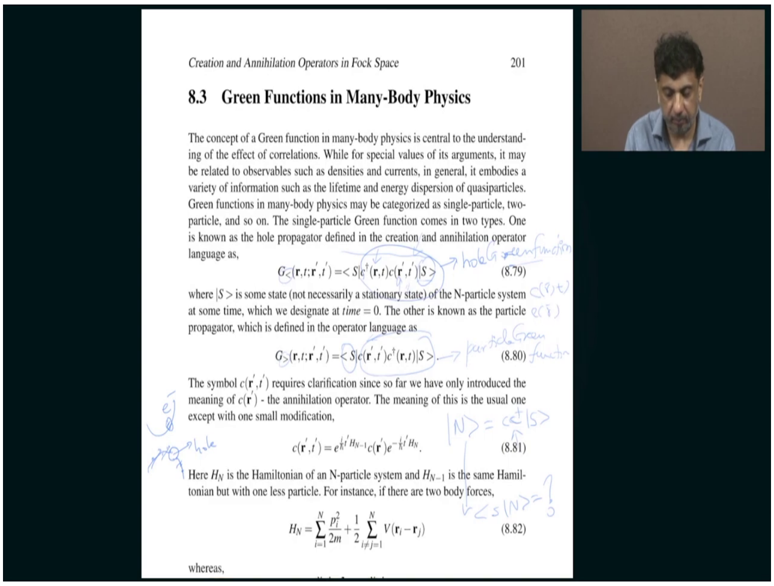

So, now that we have proved the important commutation rules obeyed by the creation and annihilation operators, we are now perfectly equipped to answer this important question; namely of what possible utility is the introduction of these operators going to be. In other words, why did we go ahead and introduce these operators.

So, the reason is because like I told you firstly, in relativistic systems, you actually do create particles out of nothing; namely, just have to pump energy into vacuum and you create particle anti particle pairs. But a very analogous situation exists even in condensed matter; namely, you know if you imagine a semiconductor at 0 temperature an undoped semiconductor you have this valence band that is perfectly fully filled and you have an empty conduction band. So, all these are filled.

And so, now, if you pump energy into the system, there is going to be an electron here and a hole is left behind. So, basically what is going to happen is that. So, a hole is going to be left behind, but they are all electrons here. So, this is a hole and this is an electron ok. So, actually you end up creating a particle and an anti-particle in this. This hole actually is basically absence of an electron, but that absence of an electron itself behaves like a particle and it is it moves around the material as if it is a particle ok.

So, the bottom line is that you can in fact even in condensed matter create particles and anti-particles out of nothing. So, which is the reason why you should learn how to study a system of many particles, where the number of particles is not fixed, where you can go ahead and create and annihilate ok. So, one of the important physical quantities that are going to be of interest in our study of many particle system is what is called the one particle greens function.

So, in this in the remainder of this lecture, I am going to just introduce this concept and in the next lecture, I am going to study more of its properties. So, this is called the hole Green's function; the hole Green's function. So, Green has to be with capital G because Green is the name of a mathematician; it is not the colour green. It is Green is the name of a individual.

So, its G is capital and this is called the particle Green's function. So, what this does is. So, why are we studying this? But let us first understand what this is, then you will understand immediately why we are studying it. So, the point is that what this hole Green's function concept is that imagine you start with some state.

So, there is a state of N particle. So, there are N particles minding their own business. So, that means, those N particles are in some quantum mechanical state called s. Now, what you do is you come along and remove a particle at position r dash. So, that is what this c does. It annihilates or removes, it removes a particle at r dash at time t dash. So, when you remove a particle, what you are doing is basically you are leaving behind a hole; that means, you are creating a hole.

So, you remove a particle, you are leaving behind a void or a hole. So, now, that hole is going to propagate in the system. So, what that means is basically that hole is going to be filled by other particles and those particles that fill this void will themselves leave behind another hole. So, it will be as if that hole itself is moving here and there. So, that is pretty much what is going to happen. So, now, that hole is going to keep wandering off until you decide to finally put back the whatever you have taken away at position r at time t.

So, you insert the particle that you had removed back into the system, but at a different position at a different obviously later time. So, you insert it back into the system. So, now, what is going to happen is that you see when you removed a particle, you are basically disturbing the system tremendously. So, now, the wave function, you see the system had N particles, now it has one fewer. So, when it had N particles, it was properly anti symmetrized. There was mutual.

Well, it was either properly anti symmetrized or symmetrized under the interchange of any two particles; but all of a sudden, when you remove one particle as one fewer particle, now the system has to scramble to again properly symmetrize or anti symmetrize itself. Now, what that means is basically that once you remove a particle from the system, the state of the system is no longer going to be stationary.

So, that means, it would not be a stationary state, in the sense of quantum mechanics; it is going to have dynamics, it is going to evolve. So, it is going to obey the time dependent Schrodinger equation, it is going to evolve according to that. So, now, once it evolves according to the time dependent Schrodinger equation, it will evolve until you again decide to further disturb the system by re-inserting the particle that you had removed. So, namely, you reinsert the particle at position r at time t.

So, when you do that you end up with a system with the same number of particles you had when you started off with. So, you started off with N particles, now you end up with a system which also has N particles ok. So, now, the question is now it is completely meaningful to ask the following question. You see you started off with the state s, so you removed a particle and then, later on you reinserted a particle.

Now, you have got a state with the same number of particles as you started off within s. So, now, the question is what is the overlap of this state? So, this state is what you obtained; what I have circled here is the state, you obtained after doing this procedure of creating a hole and filling the hole with the particle again.

So, you have the same number of particles. So, now, you have created a completely new state. So, the question is now the valid question is what is the overlap between that state and the original state. Because you could ask many questions; in fact, you can ask the what is I mean the; obviously, the most general question you can ask is what is this new state that itself is a valid question.

But a more limited less ambitious question is what is the overlap the quantum mechanical overlap between this state and the starting state. So, that is a less ambitious less informative question; but it is a valid question to ask and if you ask that question the answer to that question is basically called the hole Green's function.

So, it is the one particle hole Green's function. It is one particle because you are removing one particle and then, replacing the same particle into the system. So, similarly, you can create you can imagine a particle Green's function, where you have an N particle system s to begin with. Now, rather than removing a particle and creating a void, you insert a new particle of the same kind into the system.

Now, you insert it at position r at time t and then, you watch the system evolve because then, it is as usual it is not going to be a stationary state because that new particle is going to again disturb the symmetry of the wave function, it does a whole bunch of other things and even if you see the point is that it is not as if you know the new particle has to physically interact with the other particle.

It is not as if say for example, they all have to be charged particles, if you insert them, they will repel each other, not like that. So, even if they are you know inert towards the presence of the others classically speaking; but quantum mechanically, the mere fact that they are there as a collective system of particles will force them to sense each others presence.

In the sense that there is something called the statistical interaction. So, that is the technical term that physicists use. What that means is there some apparent interaction that comes about because of the need to properly symmetrize or anti symmetrize the wave function. So, the need to symmetrize or anti symmetrize the wave function creates an apparent effect of interaction. So, that means, it creates effect as if there is an interaction between the particles.

So, even though classically there is none. So, this is a purely quantum mechanical effect, the statistical interaction. So, you do have. So, as a result, once you create one more particle you induce statistical interaction in the system and the system is no longer stationary and evolves according to the time dependent Schrodinger equation. So, it evolves up to a certain time, until you then decide to remove another particle of its own kind because they are all perfectly indistinguishable.

So, you cannot really keep track of which particle you added. So, all the best you can do is simply remove another particle because they are all identical. But then you remove the other particle at position r dash at time t dash; whereas, you inserted the original particle at position r at time t. So, having done all that you end up with a new state. Now, that new state as usual is going to have the same number of particles that you started off with; namely N particles.

So, in this particle Green's function you first insert a particle which was not there at all from the outside and then, you wait for some time and then, you remove another particle of the same kind and then, you end up with a new state and that new state has the same number of particles as s. So, now, it makes perfect sense to ask the question what is the quantum mechanical overlap of this new state with the original state s.

So, if this is the new state, so if this is your new state, so this is the particle Green's function. So, it makes perfect sense to ask what is this. So, this is called the particle Green's function because this is called particle because you first create a particle and then, you annihilate it. So, that is called the particle Green's function. It is defined with a greater subscript and for reasons that I will tell you later, described by less subscript when it is a hole Green's function ok.

So, but clearly in this way of doing things, there is an implication that I am working in the Heisenberg picture because you know that in the Heisenberg picture, the operators are time dependent; whereas, the states are time independent. So, there is the implication here that. So, I said you create at time t. So, I have to explain to you what; see I have not defined c r t. I have only defined c of r in my earlier see in all this earlier section, I only successfully defined c of r and c dagger of r. I have not told you what this means. So, this is completely unacceptable.

So, I have to first explain to you what c of r comma t is. So, in other words, I have to tell you how c of r changes with time. So, that is going to be done using the Heisenberg's interpretation of operators and their time dependence, which I am going to relegate to the next lecture.

So, I hope you will join me to understand these ideas in the next lecture. So, see you next time.

Thank you.