Dynamics of Classical and Quantum Fields: An Introduction

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Creation and Annihilation
Lecture - 31
Creation and annihilation operators - Excitations

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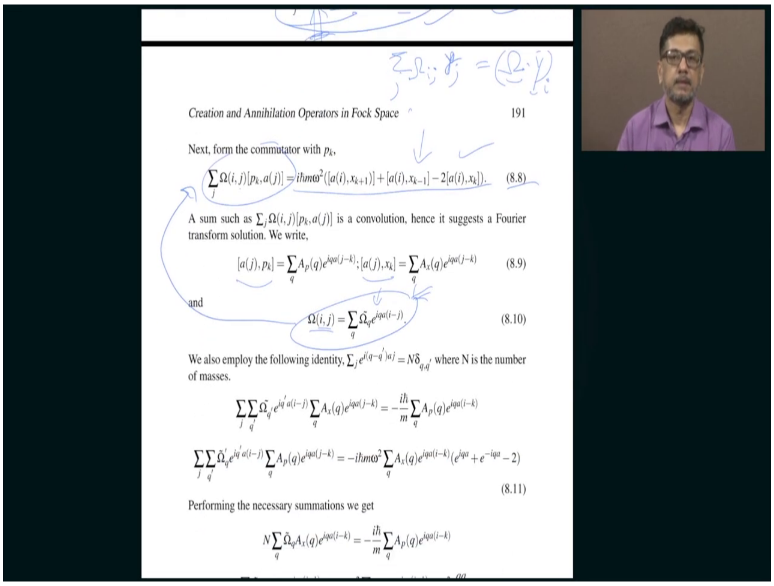

So, in today's class let us continue our discussion of this system of masses and springs. So, if you recall in the last class I told you that we are going to be studying you know fields in the sense of having a dynamical system with many degrees of freedom, but which corresponds to physically something resembling a crystalline solid. So, the idea was that first you study a very simple system namely a mass tied one mass tied to one spring and then you generalize to a system consisting of a mass followed by a spring then followed by a mass and indefinitely that way.

So, that this particular indefinite sequence of masses and springs alternating is basically simple prototype of a one dimensional crystal. So, the implication there is the masses in that model correspond to the atoms of the solid and these springs are a metaphor or basically they signify the potential energy the you know the electromagnetic potential energy that exists between the atoms of a solid. So, something like Van der Waals perhaps.

So, whatever it is that potential energy basically goes through a minimum. So, the idea is that any potential energy that goes through a minimum will be of the form of a spring potential energy meaning its as if there is a spring tied between the masses and I explained to you in the last class why that is. So, the reason why we look for a potential energy that goes through a minimum is because we expect that in the ground state the system that we are studying will correspond to the forces acting on the atoms to be actually 0.

So; that means, that forces act only when you displace an atom from its equilibrium original lattice position, but that force is of a restoring kind. In the sense that if you push it by some distance in one direction the remaining atoms will try to pull it back to its original location to its equilibrium location. So, there is a restoring force and that restoring force is proportional to the displacement of the atom.

So, we know that such a force is basically a spring force. So, even though there is no physically there is no spring, but it basically corresponds its as if there is some spring between the masses. So, we did all that and then I showed you how to basically the idea was to recast or rework this Hamiltonian which was in terms of the momenta of the masses and displacements of the masses in terms of creation and annihilation operator just like we did for a single mass and a single spring.

So, when you have many masses and many springs. So, you will have many such annihilation and creating creation operators and because the displacement of each mass.

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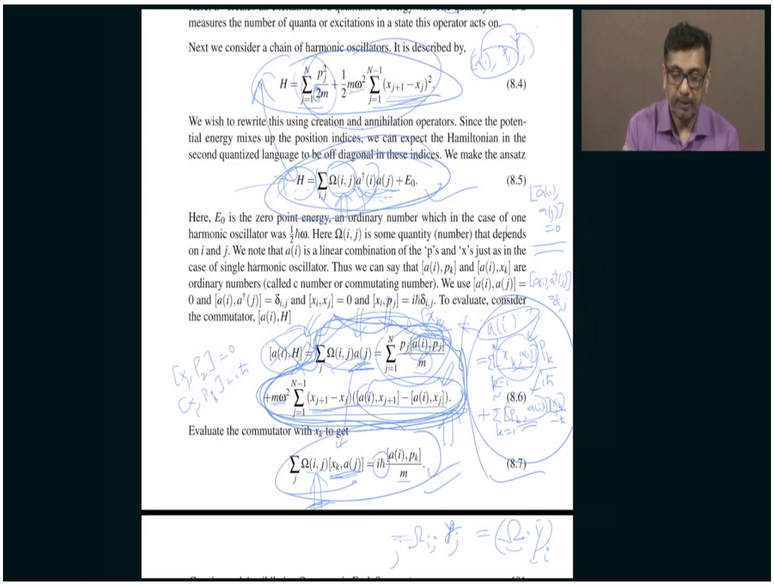

So; that means, the given that the potential energy is basically stored in the spring. So, the Hamiltonian has two parts one is the kinetic energy of all the masses that is this one and the potential energy which is only in the spring, but the potential energy stored in the spring between masses at j and j plus 1 is basically the difference in the displacement squared.

Its half k delta x squared where delta x is basically the amount by which the spring has stretched or compressed. So, that is basically x j plus 1 minus x j. So, now, this implies that therefore, that the masses are not all independent see because a jth mass is pushed and pulled by j plus 1th mass which is in turn pushed or pulled by j plus 2. So, indirectly j will be influenced by j plus 2 and so on and so forth.

So, they are all linked up. So, you cannot really separate them. So, that is the reason why when we write down the Hamiltonian in terms of creation annihilation we are forced to introduce an off diagonal form like this. So, it would not be a dagger i a i because if that were the case; that means, that ith mass is independent of i plus 1 because they all get separated as a summation, but we know that can cannot happen. So, it has to be a dagger i a j where j can be anything because i is indirectly influenced by j regardless of what j is.

So, we did all this and we would demanded that these two operators that is 8.4 and 8.5 have to be mathematically the same operators. So, to do that we had to introduce this idea that the annihilation and creation operators are linear in the position and momenta. So, because they are linear the coefficients will basically be the commutators of the position and the momentum operators commuting with the annihilation or creation operators.

So, as a result you will end up with 2 eigen value equations as it were. So, these are the two equations 8.7 and 8.8. So, now, we make the assertion that basically because there is a translational symmetry in the problem; that means, that if you have a mass at i and another mass at j you know the forces between them.

Or you know the properties that are linked to a mass at i and mass at j depend only on the distance between i and j it does not matter exactly what i is because you know you can always shift i by 1 unit let to the left or right basically the system looks the same because the system is indefinitely extending to the right and left.

So, they all look the same at all points. So, its only that if you have. So, there is no particular reference point. So, if you have a mass at i and mass at j clearly the physical properties will depend only on the difference between i and j. So, given that fact we can always write any function which depends on i and j as function which involves only the difference between i and j.

So, same with j and k. So, it only involves the difference between j and k. So, and the coefficients are basically the Fourier components and then you go ahead and insert these simplifications into 8 7 and 8.7 and 8.8.

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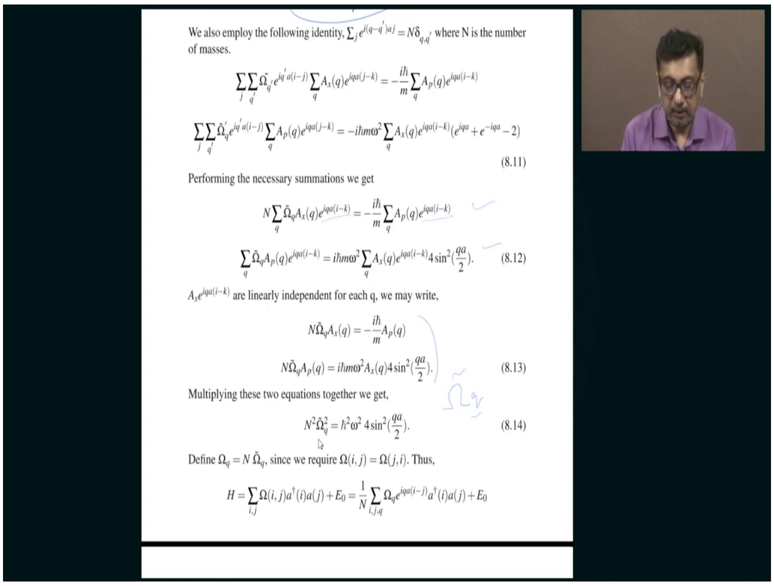

And you will end up with these equations and these equations because these functions are all linearly independent. So, different qs they are all linearly independent. So, you they have to be term wise equal. So, the summation goes away and you get these equations. So, this will tell you that basically omega has to be a some very specific quantity see what is omega? Omega tilde is related omega tilde is the Fourier transform of this omega.

So, basically its some kind of an energy of excitation of the system. So, because you see in the earlier case there was only one omega as it was because there was only one mass and one spring. So, but here you have many possible omegas for a given q. So, it that is why its called a field. So, you have a dispersion relation. So, omega now starts to depend on q ok.

So, this is see in the earlier case for a given quantum. So, if you fix n then the energy was h bar omega into n plus half. So, that omega is only one particular omega which is square root of k by n, but here this this omega is not one omega, but it depends on q. So, you have to specify q also q is basically the inverse of the wavelength of the excitations that are propagating in the system ok.

So, because there are large numbers of masses followed by springs. So, the excitations. So, even if there is one quantum there you will still have different modes basically. So, different qs will have different energies. So, its a continuum analog of one mass tied to one spring. So, its basically a that is why its called a field. So, there are infinitely many degrees of freedom. So, bottom line is that is what it is ok.

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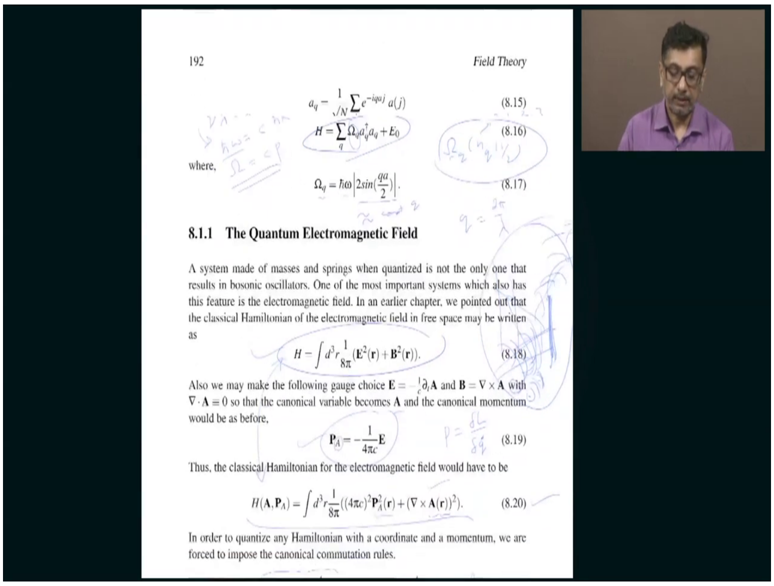

So, you will end up having an energy which depends on q and where q is basically the inverse of the wavelength of the modes that are propagating in the system. So, then of course, your actual energy of the system will basically be omega q into n q plus one half where n q is now 0, 1, 2, 3 for a given q.

So, if you fix q that is if you fix the wavelength of the modes that are propagating in your system you still have to tell me how many quanta there are for that particular wavelength. So, if you tell me that also then I can tell you what is the energy of the system I mean of the entire system. So, that is still going to be this and the peculiar thing is that this is not h bar omega it involves the inverse of the wavelength which is q; q is strictly 2 pi by wavelength ok.

Its the its proportional to the inverse of the wavelength. So, its basically thumb is the wave number. So, bottom line is that yeah. So, this leads to description in terms of. So, if the wavelength is long; that means, if q is small. So, if you have modes that are propagating in the system that have long wavelengths; that means, q is small then you will see that this approximately is some constant times q.

So, it is basically the energy is proportional to the wave number. So, that is typical of systems like sound waves you know. So, you know that nu lambda equals c this is for both light and sound. So, then your h bar omega is basically c into. So, c into h bar k. So, so this is basically your energy and this is c p. So, its basically energy is speed into momentum. So, this would be the case. So, this is dispersion less excitation.

So, that basically when q is small you will see that it resembles sound waves that you typically see in your high school textbooks. So, what is the physical meaning of q being small? q being small means wavelength of the modes that are propagating are large. So, if the wavelength of the modes are propagating a large they sort of are unable to see that the underlying system is actually discrete. So, it is made of these discrete atoms that are separated by this constant distance called a small letter a. So, the modes cannot make out. So, lambda is much greater than a.

So, they cannot make out that there are these underlying discrete modes. So, that is the reason why they will start to resemble this continuum sound wave like description ok. So, that is as far as that is how you would go about studying the fields that are propagating basically the. So, this is a field description of the excitations found in a solid. So, this is the specifically is the quantum field description.

So, its a field because you have infinitely many masses and springs. So, you have infinitely many degrees of freedom and further its a quantum field because the underlying degrees of freedom are fully quantum mechanical. So, we studied that using commutators and so on so; that means, that we studied that the modes that are propagating in the solid using quantum mechanics. So, the next example I am going to study or introduce is basically very important and that is the quantum electromagnetic field.

So, in the several lectures ago I had explained to you the classical electromagnetic field specifically I had explained how Maxwell’s equations may be thought of as the Euler

Lagrange equations of some suitable Lagrangian and we had done lots actually. If you think about it we had derived something called the stress energy tensor and we had figured out so, many things about the electromagnetic field the fact that the equations are consistent with special relativity rather than Galilean relativity and whole bunch of other things.

So, the only ingredient that we have missed is treating the electromagnetic field quantum mechanically. So, in nature of course, you know the this subject of what really is light has a long and sort of tortuous history that people were thinking of light as made of particle Newton was the first one to postulate that you know light is made of particles just like you can just throw a bunch of grains of sand into some tube and then they will they will carry energy the kinetic energy of those particles will propagate.

So, Newton naively thought the that is how light is. So, of course, he had no proof supporting his claim. So, he just assumed that because he could not think of anything else, but; however, later experiments by Huygens Young and all those other people who are proponents of the wave theory of light they actually performed experiments which sort of showed reasonably convincingly. In fact, quite convincingly that the that light exhibits the properties that you would commonly associate with waves.

For example, waves one of the characteristics of waves is that they bend across obstacles. So, if you put a obstacle here. So, imagine that you this is an you put a wall and this is full of water here and imagine you create waves. So, imagine you create waves here. So, if these are just you know bunch of particles going like this you would not find anything here they will all get reflected like this, but if the waves you will see that some of them will actually bend around and then they will even go in this direction.

So, waves by nature are delocalized. So, they are disturbances of some medium that are propagating. So, even if you start off creating that disturbance in a localized way they will not remain localized they will spread out and then they will go around obstacles. So, that is the nature of waves.

So, you see the whole idea is that you know in in our school days we are always taught I mean the way your school teachers teach you know is light a particle or a wave, but the more fundamental question they do not ask is like why should it be one or the other you know.

What is so, special about a particle or a wave? You know why cannot it be a you know unicorn or some other why does it have to be a particle or a wave what does that even mean? Why did they think of those two possibilities? So, of course, your school teachers do not tell you that because the syllabi do not mention that clearly. So, the answer to that is basically because the underlying implication is that particles are actually localized entities; that means, that they exist only in some finite region.

And both particles and waves are carriers of energy. So, they transport energy from one location to another, but particles transport energy by virtue of their own motion; that means, they themselves move and the particles are actually localized objects and if you have many particles they are all in their localized positions and they collectively move and transport energy whereas, waves are exactly opposite. So, these are two opposite extremes. So, that is the reason why people talk of particles or waves.

So, the implication is that these are two opposite extremes. So, waves are completely delocalized objects and because they are delocalized they cannot be any material particles, but rather they are actually disturbances of some medium, but then those disturbances also carry energy. So, just like particles carry energy particles are completely localized; that means, they exist in some finite region of space whereas, waves are exactly the opposite they are completely delocalized and yet both waves and particles carry energy.

So, because light also is in phenomenon where energy is transported from one place to another it makes sense to ask is light more resembling you know is light made of entities that are completely localized which then physically transport themselves to some other location thereby transferring energy from one point to another or is it completely made of or is it made of completely delocalized entities such as disturbances of some medium which then propagates.

And energy gets transported from one place to another. So, these are two extreme viewpoints and its. So, that is a legitimate question to ask because clearly its one or the other or something in between see because its can be. So, the physical content of light can be either extremely localized or it can be extremely delocalized or something in between. So, that is a valid question to ask. So, that is the reason why this question is posed in that way.

So, in the time after Newton. So, when Newton was alive he was pretty much considered the authority on pretty much everything because of his teacher as a result of his being able to explain celestial phenomena through his theory of gravitation and his formulation of laws of mechanics. So, people assume that he was an expert on everything else also. So, specifically they did not want to question his opinion on what light is made of.

So, we will assume that Newton has to be right because he said that light is made of corpuscles which are basically localized entities that physically propagate from one point to another. But after his death many scientists began to critically examine those claims and performed experiments and then they realize that the truth was closer to the exact opposite of what Newton believed that light appeared to bend around obstacles and it was able to do things that you normally associated with waves.

So, for several centuries after Newton’s death people for valid reasons understandably came to the agreement that light was made of waves, but in the early part of the 20th century. So, there was this phenomenon of the photoelectric effect and that seem to be completely at odds with the wave nature of light and also nearly simultaneously there was this observation of the black body spectrum and any attempts to derive the black body spectrum through you know by assuming that light was a electromagnetic wave also failed miserably.

So, these two ideas immediately led to a kind of crisis in physics and that crisis was resolved by Einstein who realized that the there is a link between these two seemingly unrelated crises and then he postulated that light actually the energy contained in light is not continuous, but it comes in discrete packets of course, just like Newton he had done no dependent reason to believe that.

So, but; however, unlike Newton it was not merely an opinion that he was able to explain that postulate was enough for him to explain observed physical phenomena namely the photoelectric effect in the black body radiation yeah. So, Planck was the one who guessed that there must be something analogous to quanta, but it Planck incorrectly assume that the walls of the container are actually responsible for.

So, Planck assume that light still consists of electromagnetic wave with energy continuous. So, Planck assumed that it is the walls of the quanta that refuse to observe and emit continuously. So, they observe and emit only discrete quanta. But it was Einstein who realized that it is the electromagnetic field itself which is quantized; that means, the energy contained in the electromagnetic field itself comes in discrete packets and that sort of a realization was enough for Einstein to explain not only black body radiation, but also the photoelectric effect.

But however, that sort of a ad hoc assumption explaining some phenomenon is a success no doubt, but that success is necessarily short lived in the sense that once that euphoria that excitement of having explained these phenomena wears out then the fundamental question will still surface namely you will be asking you know where did this quantum electromagnetic field come from because they certainly do not come from Maxwell's equations because there is no room for Maxwell's equations to produce discrete energy.

So, you see the energy contained in the electromagnetic field we had derived was this. So, E can be continuously anything E is electric field at position r and B is electric magnetic field at position r and they can be continuously anything you want them to be and the total energy is just that integral. So, there is no room for making a claim that the energy contained is discrete in the electromagnetic field.

But of course, that is only apparently because then you realize that Maxwell's theory is basically classical in the sense that it involves treating electric and magnetic fields as just numbers that you can measure and you can just go ahead and find the energy and so on. So, what one realizes is that the quantum nature of the electromagnetic field since it is necessary now that Einstein has told us that it is necessary to describe electromagnetic field quantum mechanically.

It becomes necessary for us to find out a way to rework Maxwell's equations so, that its consistent with the principles of quantum mechanics. So, you know that how you do that in quantum mechanics that basically you identify generalized coordinates and you have a Lagrangian. So, that is typically how its done. So, that is the reason why I introduced the Lagrangian. So, you typically have a Lagrangian and then you have a generalized you identify generalized coordinates.

And then from the generalized coordinate and the generalized and the Lagrangian you can derive generalized or you can write down the generalized momentum. So, this is the prescription that is universally followed. So, this is at even at the classical level. So, you have a classical Lagrangian and you identify a generalized coordinate with respect to which the Euler Lagrange equations of that particular classical Lagrangian results in the Maxwell equations.

So; that means, we have already done that we have already shown that the suitable Lagrangian is basically E squared B squared integrated over all space and the generalized coordinate is basically the vector potential at least for the electromagnetic field without sources and you can even work out the generalized momentum. And the generalized momentum conjugate to the generalized coordinate which we have selected to be A the vector potential is basically the electric field apart from a constant.

So, if you rewrite this Hamiltonian in terms of the generalized momentum in the generalized coordinate you end up being able to write it like this ok. So, now, this is all classical even 8.20 is classical because. 20 is just rewriting H in terms of q and p right. So, its basically 8.20 is rewriting 8.18 in terms of generalized coordinate which is a and generalized momentum which is P A.

So, there is nothing quantum yet, but; however, we now can immediately realize how to study the electromagnetic field quantum mechanically and what; that means, is that now you start pretending that the generalized coordinates and generalized momenta they are not numbers, but they are operators.

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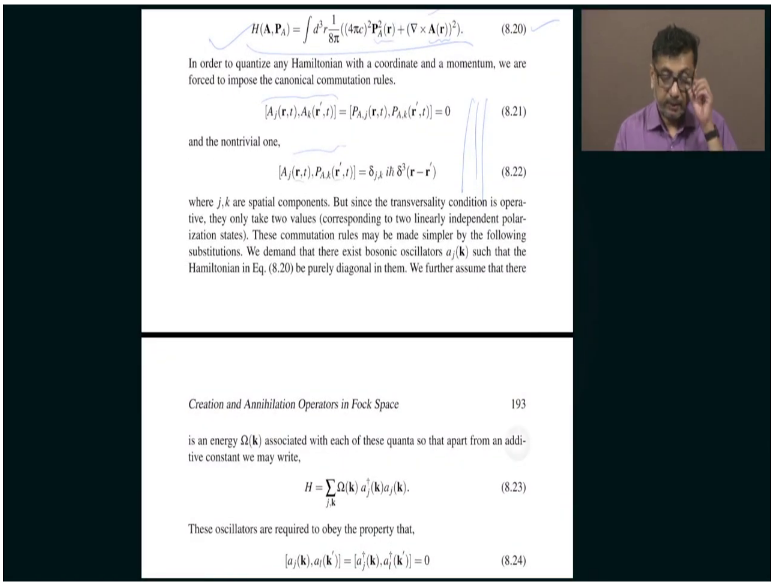

And they are operators in the sense that they obey these commutation rules. So, that if you look at the commutator between the two generalized coordinates at the same time of course, then they are 0 same with generalized momenta. But; however, coordinate commutator with momentum should be ih bar and because these are fields.

So, that. So, its ih its 0 if r and r dash are not the same because clearly I mean at different points they do not talk to each other and if they are the same points then the position and momentum are at the same point. So, I told you already that you see the position in space takes on the role of some kind of a index which counts the number of degrees of freedom.

So, if you have say n number of generalized coordinates you would write that as q 1 q 2 q 3. So, here this r is taking on that role of 1, 2, 3 like that. So, instead of 1, 2, 3 is now continuous r vector. So, now, clearly q i q 1 will commute with p 2 where q is the generalized coordinate of the first particle and p is the generalized momentum of the second particle q commutator of q 1 commutator p 2 is always 0.

So; that means, that unless r equals r prime the commutator of the generalized coordinate and generalized momentum are actually 0, but if they are the same you should have a delta function ok. So, that is the idea behind. So, the chronicle deltas naturally become

Dirac deltas when you go from a discrete situation to a continuum situation ok. So, that is the thumb rule. So, you are replacing things by f distributions rather than discrete quantities.

So, the idea is that now we can go ahead and reinterpret 8.20 as a quantum mechanical Hamiltonian provided we make these postulates that the commutators. So, you replace the generalized coordinates and generalized momenta and you reinterpret them as operators obeying certain commutation rules namely 8.21 and 8.22 ok. So, now, the idea is that you see just like in the earlier case we were successful in rewriting our Hamiltonian like this.

So, for mass type mass spring mass spring mass spring that type of system which physically described the one dimensional solid that is basically the sound waves propagating in one dimensional solid. So, here you have light waves propagating its not clear in what because you see even classically the electromagnetic waves are not actually disturbances of any medium they are actually disturbances of some abstract thing called the electromagnetic field.

And that electromagnetic field is not any physical quantity its a mathematical construct, but you have a abstract mental construct which also can suffer disturbances. But in sound waves you see it is the physical medium like air or so, you have sound waves in water its the is the liquid itself that is undergoes disturbances and the disturbances propagate. If it is sound waves in air its the pressure and density variations that propagate as sound waves, but in case of light you see light propagates in a vacuum and it does so, with greater speed than when it is propagating in a medium.

So, the question is that how can it propagate in a vacuum because there is nothing that is getting disturbed vacuum by definition is the absence of matter, but then absence of matter does not mean absence of things that can get disturbed. So, the electromagnetic field is itself an entity that exists even in a vacuum that can actually suffer disturbances.

So, the electromagnetic field is not made of matter its just an abstract construct it always exists in a vacuum and that electromagnetic field can suffer disturbances which can then propagate and those disturbance are what we call light. So, just like in the case of solid you can have sound waves which you can treat classically, but when you treat them quantum mechanically you get this quanta of sound in fact, these are called phonons by the way.

See the quanta of sound waves which we got here this this sort of thing. So, this is these are called phonons. Phonons is basically the quantum mechanical description of sound. So, just like you could do that you should be able to using this procedure that I just outlined namely start with classical 8.20 and then recast that quantum mechanically and you should be able to study the electromagnetic waves quantum mechanically.

So; that means, you should be able to study photons just like phonons the subscript on refers to the quantum mechanical version of the classical starting point. See phonon phone means sound. So, its a prefix which represents sound on is a suffix that represents quantization. So, a sound quanta ok phonons see similarly photon on is basically the quantum description photo is light.

So, photon would be the quantum description of light. So, bottom line is that is what we should be successful in doing if we start with the classical 8.20, but then now reinterpret the generalized coordinate A as being not a number, but as an operator and the generalized momentum also as an operator obeying these commutation rules namely 8.21 and 22.

So, just like we were able to finally, write down the Hamiltonian of sound waves in terms of these modes. So, you have the energy of modes and then you have these quanta of excitation. So, this is the number of quanta with wave number q here also we expect in a very similar vein we should be able to write this.

So, now, of course, there is an additional index j simply because this you see in this particular problem it was one dimensional. So, there was no room for any further index. So, that it was just a one dimensional chain of mass spring mass spring mass spring, but here in the electromagnetic field its three dimensional. So, you expect one more index at least. So, and that is what that is. So, the bottom line is that we are going to postulate canonically that you should have these commutators these as should obey these commutators.

Now, of course, I mean there is an implication that you know you put stuff in a box. So, that k s become discrete you know you know that if you have a particle in the box your position is continuously from can change from 0 to l within the box, but the momentum is n pi by l where momentum k is discrete or the wave vector is discrete.

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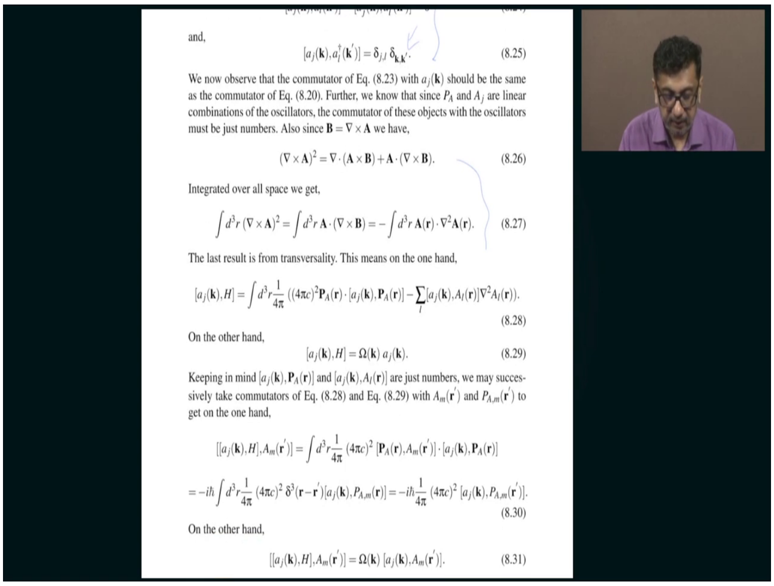

So, that is why you have a chronicle delta when you are dealing with. So, this is true whether regardless of whether the underlying particle is a photon or a material particle basically its just one quantum particle in a box you would immediately have a k that is quantized. So, that is the underlying mental picture that you our radiation trapped in a box may be a cubicle box of side l.

So, therefore, the ks become discrete alright. So, if you allow me these kinds of optimistic assertion then I can go ahead and recast my Hamiltonian. So, I do the same thing I did earlier. So, I want to be able to rewrite the as in terms of the ps and the I mean the creation the annihilation operators lower case a in terms of the generalized coordinate upper case a bold face; that means, the vector potential

So, I should be able to write the annihilation operator in terms of the vector potential and the corresponding generalized momentum which is basically the electric field. So, I will allow you to go through this algebra which is tedious, but straightforward.

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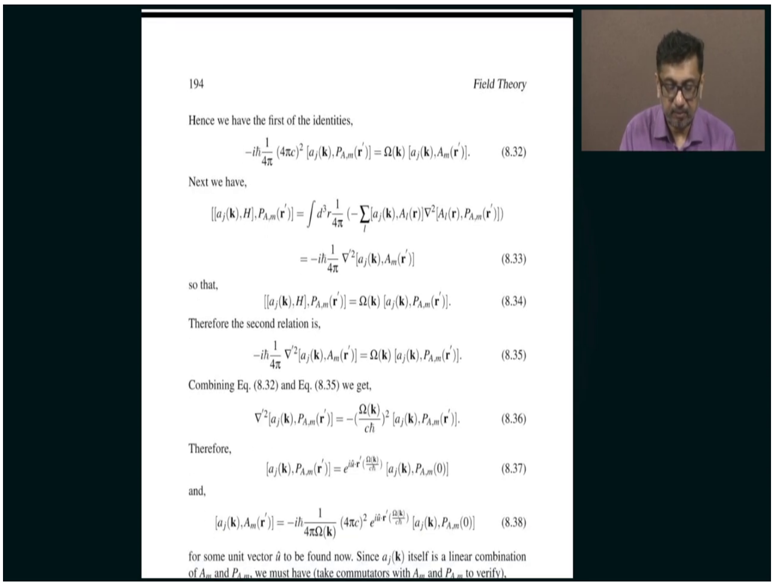
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Hence we have the first of the identities,

$$-i\hbar\frac{1}{4\pi}(4\pi c)^2[a_j(\mathbf{k}),P_{A,m}(\mathbf{r}')] = \Omega(\mathbf{k})\,[a_j(\mathbf{k}),A_m(\mathbf{r}')]. \quad (8.32)$$

Next we have,

$$[[a_j(\mathbf{k}),H],P_{A,m}(\mathbf{r}')] = \int d^3r\frac{1}{4\pi}(-\sum_l[a_j(\mathbf{k}),A_l(\mathbf{r})]\nabla^2[A_l(\mathbf{r}),P_{A,m}(\mathbf{r}')])$$

$$= -i\hbar\frac{1}{4\pi}\nabla'^2[a_j(\mathbf{k}),A_m(\mathbf{r}')] \quad (8.33)$$

so that,

$$[[a_j(\mathbf{k}),H],P_{A,m}(\mathbf{r}')] = \Omega(\mathbf{k})\,[a_j(\mathbf{k}),P_{A,m}(\mathbf{r}')]. \quad (8.34)$$

Therefore the second relation is,

$$-i\hbar\frac{1}{4\pi}\nabla'^2[a_j(\mathbf{k}),A_m(\mathbf{r}')] = \Omega(\mathbf{k})\,[a_j(\mathbf{k}),P_{A,m}(\mathbf{r}')]. \quad (8.35)$$

Combining Eq. (8.32) and Eq. (8.35) we get,

$$\nabla'^2[a_j(\mathbf{k}),P_{A,m}(\mathbf{r}')] = -(\frac{\Omega(\mathbf{k})}{c\hbar})^2[a_j(\mathbf{k}),P_{A,m}(\mathbf{r}')]. \quad (8.36)$$

Therefore,

$$[a_j(\mathbf{k}),P_{A,m}(\mathbf{r}')] = e^{i\hat{u}\cdot\mathbf{r}'(\frac{\Omega(\mathbf{k})}{c\hbar})}[a_j(\mathbf{k}),P_{A,m}(0)] \quad (8.37)$$

and,

$$[a_j(\mathbf{k}),A_m(\mathbf{r}')] = -i\hbar\frac{1}{4\pi\Omega(\mathbf{k})}(4\pi c)^2 e^{i\hat{u}\cdot\mathbf{r}'(\frac{\Omega(\mathbf{k})}{c\hbar})}[a_j(\mathbf{k}),P_{A,m}(0)] \quad (8.38)$$

for some unit vector $\hat{u}$ to be found now. Since $a_j(\mathbf{k})$ itself is a linear combination of A_m and $P_{A,m}$, we must have (take commutators with A_m and $P_{A,m}$ to verify),

And you can just go ahead and maybe should I spend some time perhaps not yeah its just tedious and, but straight forward.

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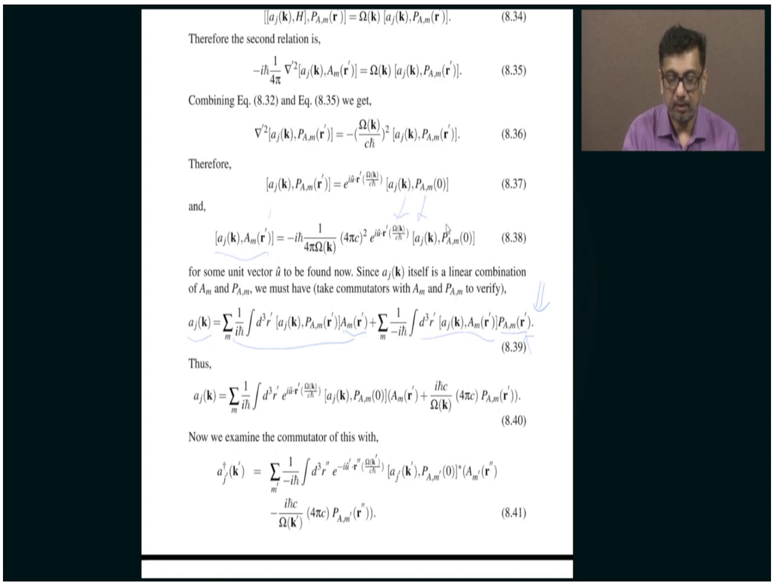
$$[[a_j(\mathbf{k}),H],P_{A,m}(\mathbf{r}')] = \Omega(\mathbf{k})\,[a_j(\mathbf{k}),P_{A,m}(\mathbf{r}')]. \quad (8.34)$$

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$$[a_j(\mathbf{k}),P_{A,m}(\mathbf{r}')] = e^{i\hat{u}\cdot\mathbf{r}'(\frac{\Omega(\mathbf{k})}{c\hbar})}[a_j(\mathbf{k}),P_{A,m}(0)] \quad (8.37)$$

and,

$$[a_j(\mathbf{k}),A_m(\mathbf{r}')] = -i\hbar\frac{1}{4\pi\Omega(\mathbf{k})}(4\pi c)^2 e^{i\hat{u}\cdot\mathbf{r}'(\frac{\Omega(\mathbf{k})}{c\hbar})}[a_j(\mathbf{k}),P_{A,m}(0)] \quad (8.38)$$

for some unit vector $\hat{u}$ to be found now. Since $a_j(\mathbf{k})$ itself is a linear combination of A_m and $P_{A,m}$, we must have (take commutators with A_m and $P_{A,m}$ to verify),

$$a_j(\mathbf{k}) = \sum_m\frac{1}{i\hbar}\int d^3r'\,[a_j(\mathbf{k}),P_{A,m}(\mathbf{r}')]A_m(\mathbf{r}') + \sum_m\frac{1}{-i\hbar}\int d^3r'\,[a_j(\mathbf{k}),A_m(\mathbf{r}')]P_{A,m}(\mathbf{r}'). \quad (8.39)$$

Thus,

$$a_j(\mathbf{k}) = \sum_m\frac{1}{i\hbar}\int d^3r'\,e^{i\hat{u}\cdot\mathbf{r}'(\frac{\Omega(\mathbf{k})}{c\hbar})}[a_j(\mathbf{k}),P_{A,m}(0)](A_m(\mathbf{r}') + \frac{i\hbar c}{\Omega(\mathbf{k})}(4\pi c)\,P_{A,m}(\mathbf{r}')). \quad (8.40)$$

Now we examine the commutator of this with,

$$a_{j'}^{\dagger}(\mathbf{k}') = \sum_{m'}\frac{1}{-i\hbar}\int d^3r''\,e^{-i\hat{u}'\cdot\mathbf{r}''(\frac{\Omega(\mathbf{k}')}{c\hbar})}[a_{j'}(\mathbf{k}'),P_{A,m'}(0)]^*(A_{m'}(\mathbf{r}'')$$

$$-\frac{i\hbar c}{\Omega(\mathbf{k}')}(4\pi c)\,P_{A,m'}(\mathbf{r}'')). \quad (8.41)$$

So, you will eventually be able to if you follow this somewhat messy algebra you will immediately be able to show that this the annihilation operators are linearly related to the As and Ps. So; that means, just like you know in the k even for the one mass and one spring a was just some constant into x plus some other constant into p.

So, that is what even here you have the same thing the annihilation operator for a photon is related to the generalized coordinate times some something some constant plus generalized momentum into some other constant, but then you have to add up over all the positions because now you have a field instead of just one mass and one spring you have a field actually.

So, you have to integrate or basically you have to add up all the degrees of freedom you should not miss all that any of the degrees of freedom. So, we will be able to show we have shown in the earlier steps that these commutators can be the r dependence and can be extracted out this way and so, you can simply rewrite. So, the r dash dependence can be extracted out because this depends on r dash in only in this way ok.

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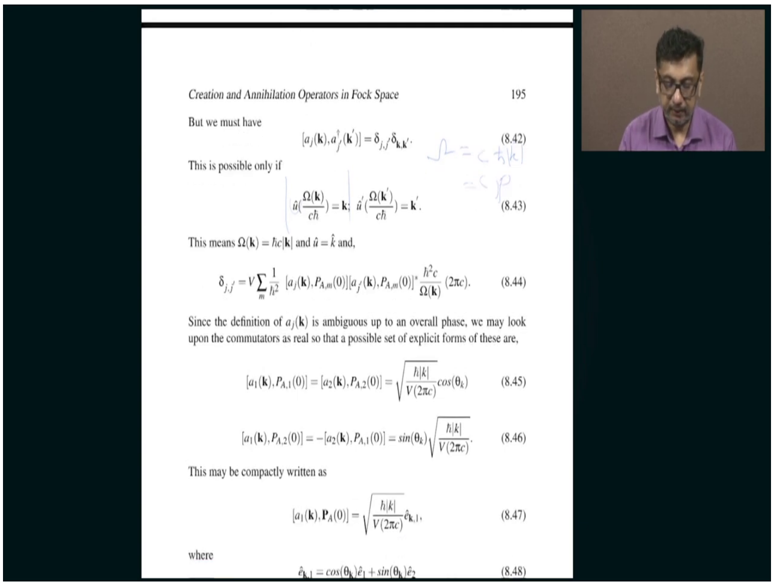
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But we must have

$$[a_j(\mathbf{k}), a^{\dagger}_{j'}(\mathbf{k}')] = \delta_{j,j'}\delta_{\mathbf{k},\mathbf{k}'}. \quad (8.42)$$

This is possible only if

$$\hat{u}(\frac{\Omega(\mathbf{k})}{c\hbar}) = \mathbf{k}; \quad \hat{u}'(\frac{\Omega(\mathbf{k}')}{c\hbar}) = \mathbf{k}'. \quad (8.43)$$

This means $\Omega(\mathbf{k}) = \hbar c|\mathbf{k}|$ and $\hat{u} = \hat{k}$ and,

$$\delta_{j,j'} = V\sum_m \frac{1}{\hbar^2} [a_j(\mathbf{k}), P_{A,m}(0)][a_{j'}(\mathbf{k}), P_{A,m}(0)]^* \frac{\hbar^2 c}{\Omega(\mathbf{k})}(2\pi c). \quad (8.44)$$

Since the definition of $a_j(\mathbf{k})$ is ambiguous up to an overall phase, we may look upon the commutators as real so that a possible set of explicit forms of these are,

$$[a_1(\mathbf{k}), P_{A,1}(0)] = [a_2(\mathbf{k}), P_{A,2}(0)] = \sqrt{\frac{\hbar|k|}{V(2\pi c)}}cos(\theta_k) \quad (8.45)$$

$$[a_1(\mathbf{k}), P_{A,2}(0)] = -[a_2(\mathbf{k}), P_{A,1}(0)] = sin(\theta_k)\sqrt{\frac{\hbar|k|}{V(2\pi c)}}. \quad (8.46)$$

This may be compactly written as

$$[a_1(\mathbf{k}), \mathbf{P}_A(0)] = \sqrt{\frac{\hbar|k|}{V(2\pi c)}}\hat{e}_{\mathbf{k},1}, \quad (8.47)$$

where

$$\hat{e}_{\mathbf{k},1} = cos(\theta_{\mathbf{k}})\hat{e}_1 + sin(\theta_{\mathbf{k}})\hat{e}_2 \quad (8.48)$$

So, you will be able to do that and eventually what you will be able to show is basically that this omega ok. So, this omega will come out as. So, the important thing is that energy. So, what will that energy be? So, it will actually. So, this is a unit vector. So, if you just take the modulus of both sides. So, this omega will come out as c h bar k.

So, basically that is what this is its C P. So, this is C P. So, that is what we expect for light. So, we expect the energy to be proportional to the momentum right. So, we expect energy to be C P. So, that is what we expect for a massless particles and we know that photons are massless particles.

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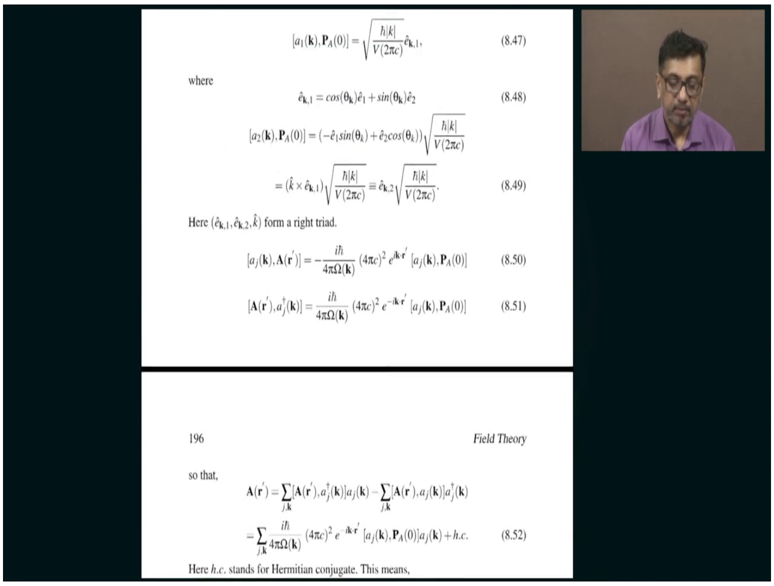

So, we get that. So, bottom line is that. So, you get that and you are also able to express the generalized coordinate in terms of the annihilation operators. And similarly the electric field. So, the bottom line is that you are able to now account for the fact. So, now, you have a rigorous mathematical description of the electromagnetic field in terms of quanta. So, that was what Einstein suspected all along that there should be a consistent mathematical description of the electromagnetic field in terms of quanta rather than in terms of electromagnetic waves.

So, there was this earlier wave description and you can always rework that wave description in terms of quanta. So, of course, you might be thinking that you know does that mean that the wave description has now completely been proven wrong. The answer is no because you see we know that say if you have a projectile say if you have a cricket ball and you throw it from your you know 4 boundary towards say some bowler.

So, it will you know it will move in a parabolic trajectory until it reaches the hands of the bowler. So, now, that parabolic trajectory is perfectly described by classical mechanics its described by Newton second law and the trajectory and all that. So, whereas, we know that is still an approximate description of nature.

So, the question is if you study the motion of the ball quantum mechanically; that means, if you write down the wave function of that ball and you find the rate at which means you would solve the time dependent Schrodinger equation for the wave function of the ball. Are you able to also recover the same results that you get from studying the motion of the ball as a projectile in using Newton's second law.

The answer is yes, but it is rarely done in the books, but you can actually show that you get the same answers especially because the ball is macroscopically large I mean the Planck's constant is overwhelmingly small compared to the corresponding dimensions of an actual cricket ball. So, as a result the classical description is overwhelmingly accurate. So, the quantum description is almost I mean its completely superfluous except conceptually.

So, you can actually rework or rederive Newton's second law starting from Schrodinger equation I know that should be properly described in the quantum mechanics courses, but many instructors do not spend enough time proving that or explaining that carefully and that is probably worth doing at some stage, but it can be done you can show that the motion of a projectile can be thought of as the solution of a Schrodinger equation also.

So, similarly here the mere fact that we have described the quantum electromagnetic field in terms of photons does not mean that the classical description has now been invalidated. So, the Young's double slit experiment this Huygens and Fraunhofer diffraction all those things continue to be valid for the same reason why the projectile motion is still a valid description of how a cricket ball moves in a cricket field even though it is being described using the imperfect classical mechanics.

So, even though the correct theory of nature is quantum mechanics you will still get a overwhelmingly accurate description of a motion of a projectile by studying it classically similarly its only under very special circumstances that will force you to study the electromagnetic field quantum mechanically. So, for the most part you can get away by pretending that the electromagnetic waves are actually I mean the electromagnetic field the disturbances in the electromagnetic field are actually waves and those waves carry continuous amount of energy.

So, the only situation where that description is breaks down just like in the case of projectile it breaks down when the mass in question is microscopic. So, similarly here also it breaks down when the intensity of light is extremely small so, that the wave description becomes less and less accurate and the photon description becomes more and more accurate. So, then the individual packets of light actually start to matter.

So, if you. So, its just like you know if you have a water gushing out of your faucet from your tap its really silly to think of the water that is coming out of your tap as made of water molecules is better to think of that as a fluid that is coming out. But however, if only a small number of molecules of water come on I mean if the tap is like say has a diameter of the size of microns or even less then you cannot ignore the fact that water is made of molecules because then actually one molecule after another will come out rather than water coming out.

So, you cannot ignore the fact that molecules come out rather than just a fluid of water. So, similarly with light also if the intensity of light is very large you can simply ignore the wave I mean you can ignore the quantum nature of light you can ignore the fact that its made of photons, but if you turn down the intensity then then light comes one after I mean the quanta of light arrives at the detector one after another and the discrete nature of light of the electromagnetic field becomes quite apparent ok.

So, I am going to stop now. So, the mathematical details are important and in fact, the mathematical details also show you that there are two modes of polarization; that means, that in the case of electromagnetic field in free space you have light that propagates in a certain direction and the electric and magnetic fields point in a perpendicular direction in the perpendicular direction is a plane. So, it has two independent directions. So, those are the directions of polarization anyway.

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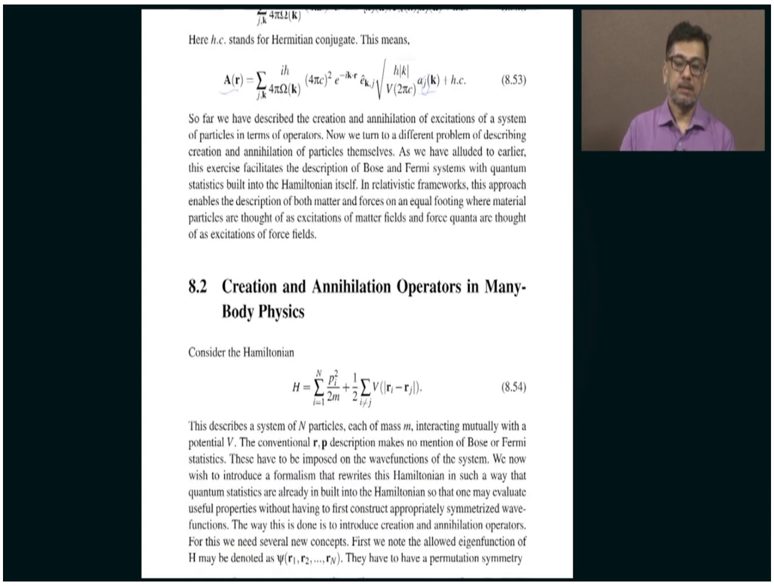

So, those are also there in classical description of the electromagnetic field and they carry over to the quantum description which is why you have this additional index j which actually represents the two polarization directions ok. So, in the next class I am going to be describing the creation and so, what all did I do just now?

So, I describe the creation annihilation operator first of all for one mass tied to one spring then I described the creational annihilation operator treatment of a system of mass tied to a spring followed by another mass then spring. So, that was supposed to be a caricature or representation of one a dimensional crystalline solid.

So, the basically the description of that system in terms of creation annihilation operator basically meant that I was describing phonons that is the quantum description of sound waves that are propagating in that solid. So, then today's class I explained how to study a very similar system, but namely electromagnetic field in terms of a quantum creation and annihilation operator description.

So, that would correspond to describing the quantum of the electromagnetic field which are called photons and I explained how the energy contained in the electromagnetic field is discrete and how to write the quantum electric and magnetic field in terms of these creation and annihilation operators.

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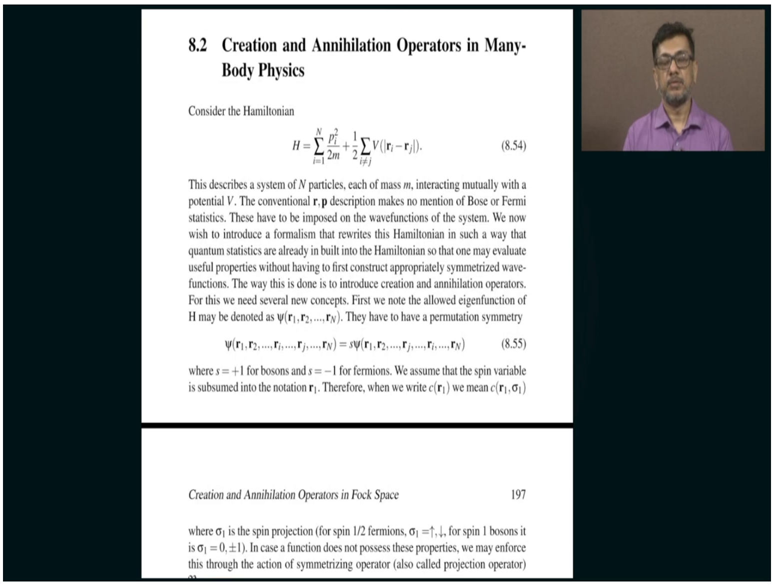
8.2 Creation and Annihilation Operators in Many-Body Physics

Consider the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i \neq j} V(|\mathbf{r}_i - \mathbf{r}_j|). \quad (8.54)$$

This describes a system of N particles, each of mass m, interacting mutually with a potential V. The conventional $\mathbf{r}, \mathbf{p}$ description makes no mention of Bose or Fermi statistics. These have to be imposed on the wavefunctions of the system. We now wish to introduce a formalism that rewrites this Hamiltonian in such a way that quantum statistics are already in built into the Hamiltonian so that one may evaluate useful properties without having to first construct appropriately symmetrized wave-functions. The way this is done is to introduce creation and annihilation operators. For this we need several new concepts. First we note the allowed eigenfunction of H may be denoted as $\psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N)$. They have to have a permutation symmetry

$$\psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_i, ..., \mathbf{r}_j, ..., \mathbf{r}_N) = s\psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_j, ..., \mathbf{r}_i, ..., \mathbf{r}_N) \quad (8.55)$$

where $s = +1$ for bosons and $s = -1$ for fermions. We assume that the spin variable is subsumed into the notation $\mathbf{r}_1$. Therefore, when we write $c(\mathbf{r}_1)$ we mean $c(\mathbf{r}_1, \sigma_1)$

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where σ_1 is the spin projection (for spin 1/2 fermions, $\sigma_1 = \uparrow, \downarrow$, for spin 1 bosons it is $\sigma_1 = 0, \pm 1$). In case a function does not possess these properties, we may enforce this through the action of symmetrizing operator (also called projection operator)

So, in the next class I will continue in a similar vein, but I will be studying not these fields like electromagnetic field and sound waves rather I will think of creating and annihilating not excitations. So, till now I was creating and annihilating excitations; that means, excitations corresponding to this harmonic oscillator excitations.

So, the mass was still one mass and one spring that was not going away, but its the excitations that I can create or destroy. So, similarly photons are excitations of the electromagnetic field that can spontaneously appear or disappear. So, however, you can also have a situation where you create and annihilate material particles. So, that is a somewhat an unusual re interpretation of creating and annihilating.

So, in the next class I will explain in what sense that has any relevance that why would we want to create and annihilate material particles and in what circumstances are they important ok. So, I am going to stop here now, I hope you will join me for the next class.

Thank you.