Dynamics of Classical and Quantum Fields: An Introduction
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Path Integrals
Lecture - 30
Path Integrals - Harmonic oscillator

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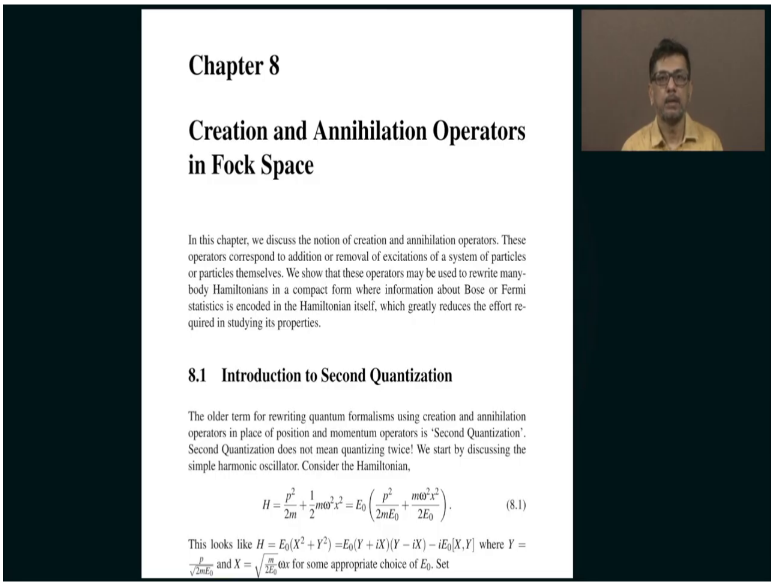

So, today let us discuss this new chapter and like I told you in the last class that today we are going to be discussing new approach to studying quantum mechanics. And, that is not using the usual type of operators, that you see in Schrodinger's quantum mechanics like momentum and position and that sort of thing. But, instead we will be using the method of raising and lowering operators; that means, creating and annihilating.

Initially, we will be creating and annihilating excitations that is the quanta which correspond to the excitations of the system. But, later on we will reinterpret these ideas or rather generalize these ideas to accommodate not merely excitations of an underlying system, but we will be introducing operators that correspond to creating and annihilating particles themselves rather than merely excitations ok.

So, to understand how to do that, let us start with very familiar situations. So that means, we will start with a very familiar example. As usual it is either free particle or harmonic oscillator. So, those are the you know the standard work horses of quantum mechanics and even classical mechanics. So, sometimes we get the impression that those are the only two models that physicist know how to solve and that would only be a slight exaggeration.

In fact, many of the examples that you encounter in physics are either approximated as free particles or approximated as harmonic oscillator. So, you might be wondering why that is that because of a lack of imagination in the physics community or is there a deeper reason. Actually, there is a deeper reason namely that you see the free particle is a prototype of a system which is unbound and whose wave functions are not localized; that means, its completely delocalized.

So, it represents a quantum particle that is not bound to anything. See whereas, the harmonic oscillator presents a situation which is extremely localized and the particle is bound to some other object. So, these are opposite extremes. So, in nature we encounter situations that are somewhere in between. So, either we are closer to one extreme or the other usually. So, it makes sense to pretend that you know as a first approximation, you pretend that you are either exactly at one of the extreme or the other.

So, its not really due to a lack of imagination among physicists, but rather it is a reflection of the fact that is how practically many systems in nature behave ok. With that preamble, let us start discuss the harmonic oscillator and let us try and understand how to study harmonic oscillators not using position and momentum. But, we are going to use position and momentum to derive representation or a picture, rework the harmonic oscillator in terms of creation annihilation operators.

So, you see the operators that correspond to creating and annihilating quanta have a very well defined mathematically precise definition in terms of the more traditional position and momentum operators. So, let us see how to do that practically speaking.

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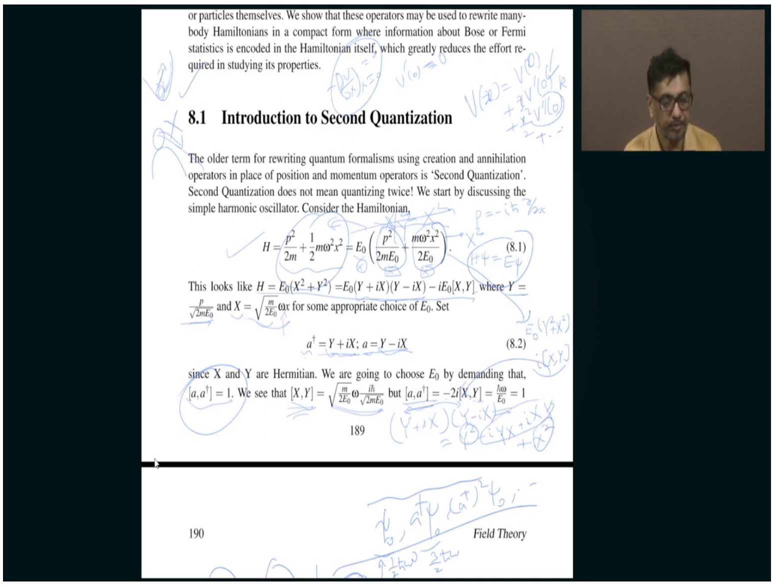

So, let us start with this familiar starting point, namely that the Hamiltonian of a harmonic oscillator clearly is given by this. So, it has the kinetic energy and potential energy which is of the form half k x squared. So, its basically where x is the displacement from some equilibrium position.

So, what I am going to do is that you see if I were doing Schrodinger's quantum mechanics, what would I do? I would find the stationary states by simply writing p as minus i h bar d by dx and I would simply write this as H psi equal to E psi and then I would solve for the stationary states. So, and I would be getting those Hermite polynomials and so on so forth.

And, I would be getting my eigenvalues as n plus half times h bar omega. So, that is all well understood from Schrodinger's approach, but I want to do it differently. So, to do it differently, I adopt the following approach. So, what I am going to do is I am going to multiply this by some number some constant which is non-zero. So, I am going to multiply and I am going to specifically select it to be positive because, it is up to me to decide what to multiply and divide with.

So, basically I am multiplying and dividing by this by the same constant. So, this so, if I expand this out, this will cancel out. So, with this see there is E 0 in the denominator, E 0 in the numerator. So, they will cancel out and I will get back this original. So, I am not done anything. So, if E 0 is positive; so, long as its non-zero, but I am selecting positive. So, if its positive I have not made any mistake at all, because E 0 is just a number.

But, what I am going to do is I am going to rewrite this object, I am going to call this X squared, sorry I am going to call this Y squared and I am going to call this X squared. So, I am going to call p squared by 2 m E 0 as capital Y squared and m omega squared x squared by 2 E 0 as capital X squared. So, this is going to be my capital X squared and this is going to be my capital Y squared.

So, now, the question is you see what is the form of this Hamiltonian? It has the form E 0 into Y squared plus X squared, I should have written Y squared plus X squared ok. So, it is E 0. So, this is basically writeable as Y squared plus X squared. So, what is Y? So, Y is going to be this one. So, it I define Y as p by under root 2 m E 0. So, you see because E 0 is positive and clearly m is positive, square root of 2 m E 0 is positive because the positive square root of a positive number.

And, p is operator which is Hermitian because p is momentum so, its Hermitian. So, similarly capital X is defined as square root of 2 m E 0 and m and E 0 are positive. So, again it is square root of a positive number which is positive times omega which is positive. So, times x, x is the position operator which is again Hermitian. So, then capital X is Hermitian and capital Y therefore, is also Hermitian.

So, you have two Hermitian operators which are linear in position and momentum and they are called capital Y and capital X. So, now, your Hamiltonian is basically E 0 into Y squared plus X squared. So, now, I am going to I want to factorize this. So, normally you see if Y and X were just some numbers, I could easily write that as a Y squared plus X square can easily be written as Y plus iX into Y minus iX. But, then Y and X are not numbers, they are actually Hermitian operators. So, I have to be little bit more careful.

So, if I see suppose I write Y plus iX and then I multiply by Y minus iX. So, what do I get? I get Y squared. So, that will be the first two product of the first two things, then I will get Y into minus iX. So, I will get minus I into YX. So, you see I have to be very careful about the order because, I cannot write X into Y when I mean Y into X, because X and Y do not commute.

See, because what is Y? Y is proportional to the momentum operator whereas, capital X is proportional to the position operator and we know that position and momentum do not commute. So, if in your expression Y comes before X you should make Y also, I mean you should never write X into Y when its coming out as Y into X. So, you see in this expression you will get once you will get minus i into Y into X, then next cross term you will get plus i into X into Y.

So, you will get both these things. So, if X and Y are just numbers instead of operators this would have cancelled out, because then Y into X would be equal to X into Y. But, now in this particular case these are not operate, I mean these are not numbers, they are operators; so, they do not cancel out. So, capital X is constant into position operator, capital Y is a constant into momentum operator.

So, they do not commute. So, X into Y is not equal to Y into X. So, then finally, the last term is plus i capital X into minus i capital X. So, plus i into minus i is plus 1. So, it is basically plus 1 into X square. So, you see this is what I will get. So, X squared plus or Y squared plus X squared can be written as. So, what I want is this, I want Y squared plus X squared. And, what is Y squared plus X squared? It is this one, then I have to make sure I subtract out whatever extra is here.

So, what is that extra thing? So, that extra thing is basically i into X commutator Y. Because, what is the definition of commutator? X commutator Y means X into Y minus Y into X. So, that is the definition of commutator. So, I take that commutator to the other side because then I want only Y squared plus X squared on one side and the remaining I want on the other side.

So, the other side already has Y plus iX into Y minus iX. So, then I have to make sure that I transfer the extra term which is this commutator on the other side. So, when I do that, I will now be successful in writing. So, I will be successful in writing the Hamiltonian as that constant which I have arbitrarily introduced which is positive times a Y plus iX into Y minus iX minus i times that constant into commutator X Y ok.

So, that is what I will end up doing. So, now, let us see so, what should we do next? So, the next thing I am going to do is the following. So, I am going to call Y plus iX as a dagger. So, I am just going to give it some name. So, I will call Y minus iX as a and clearly because Y and X are Hermitian, I can take the Hermitian adjoint; that means, I take dagger on both sides. If I take dagger on both sides, Y dagger is Y because Y is Hermitian.

But, then minus i becomes plus i because dagger will make the operators become Hermitian adjoint, but it will make complex numbers their complex conjugate. So, minus i will become plus i and capital X is because its Hermitian, it will remain capital X. So, then a dagger is Y plus iX whereas, a is Y minus iX. So, now, with that sort of a definition you can see that also further because X is defined like this, Y is defined like this; X commutator Y is basically this one ok.

So, now so, it is some constant, but what I am going to further demand is because you see I introduced this capital E 0 arbitrarily, I just said its positive and non-zero. I did not specify its value. So, now, I am going to specify this value indirectly. And, how am I going to specify the value? I am going to force the value to be such that the commutator of a and a dagger are is simple. And, the simplest commutator clearly the commutator of a and a dagger will be non-zero. So, the simplest non-zero number you can think of is 1.

So, I will force commutator of a and a dagger to be 1. So, when I force that to be the case, then you see this a dagger when I work it out, it will come out because a is defined like this, a dagger is therefore, this. So, the commutator of a and a dagger will come out as minus 2 i commutator X and Y. But, we can evaluate because X and Y are in terms of position and momentum. So, we can work out the exact answer for X commutator Y.

So, when you work that out and substitute, you will get commutator of a and a dagger to be h bar omega by E 0 which I have introduced. But, now I am forcing the commutator of a and a dagger to be 1, because you know I can force it to be any non-zero real number I want. But, I want to force it to be the simplest non-zero real number and the simplest non-zero real number is always 1. So, I am going to force it to be 1.

So, if I force it to be 1 so, but then a commutator a dagger I have just calculated and the answer comes out as h bar omega by E 0, that E 0 I have I have myself inserted into my equation. So, now, by demanding that the commutator of a and a dagger should be the simplest non-zero number which is 1; I can now calculate that E 0 has to be h bar omega. Then, only the commutator of a and a dagger will be the simplest non-zero number.

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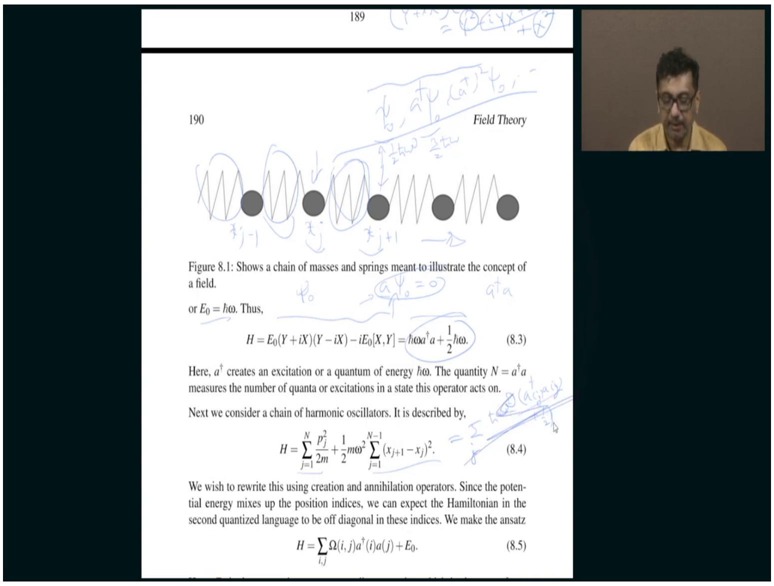

So that means, I can just go ahead and substitute all that here and I will be able to get this answer. So, h will come out as h bar omega into a dagger a plus half ok. So, now, this is familiar to us ok and we also know that. So, this is not really the end of the story. This is actually the beginning of the story. So, the next is you have to construct the ground state of the system. See a dagger a is clearly non-negative, because it has the form of z star z.

See, if z was a complex number, clearly z star z is a real number which is not negative. So, similarly a star a dagger a is a Hermitian operator which has non-negative eigenvalues. So, you can suspect that the if you can find a state which has the property that it is annihilated when a acts on it. Then, you can expect that that will be the ground state because, you cannot a dagger a cannot have an eigenvalue smaller than 0. So, clearly, if 0 is a possibility that should be the lowest eigenvalue.

So in fact, you can easily solve this and find that there is a acceptable psi 0 that comes out by solving this equation. And, that will be of the form of that Gaussian because, you remember what a is, a is in terms of Y and X. And, what is Y? Y is proportional to momentum and capital X is proportional to position. And, what is momentum is minus ih by d by dx. So, this will actually be a first order differential equation which you can solve. So, when you solve that, you will get basically a Gaussian.

So, all these things we can relegate to tutorials. Strictly speaking, these are part of prerequisite because it is you are supposed to know all this from quantum mechanics, somebody should have taught you. But, if you feel that your background is a little bit wanting and that your background is not up to the mark, then you can we can certainly have tutorials to rectify that.

So in fact, there are some subtle issues here which you have to address; for example, sure I mean you can construct a bunch of states like this. So, if from here you get the ground state, but then you can construct another state called psi a 0. And, you can show that the eigenvalue of this one means the a Hamiltonian. So, all these will be eigenstates of the Hamiltonian. So, the eigenvalue of the Hamiltonian corresponding to the ground state will be half h bar omega whereas, it will be 3 by 2 h bar omega here.

So, what is not clear is that these states are complete; that means, you have to show that there are no other eigenstates other than these and that is not at all obvious. So, the all you can show is that these are certainly eigenstates of the Hamiltonian, that is easy to show. But, what is not easy to show is that you know these are the only possible eigenstates, that there are no other eigenstates of h. So, that is not at all easy to show.

And when I taught this course in IIT, well I taught a different course. But, I had asked this question in one of the tutorials and sadly no student was able to answer it. So, the question is that how do you know that these are the only eigenstates right? So, you have to show that these are complete. And so, that is something we can also relegate to tutorials, because that is especially not discussed in any of the undergraduate or even MSc courses, in at least in India it is not discussed.

So, its worthwhile looking into that because you see these NPTEL courses, especially the courses that I teach are typically those that are meant to fill in gaps in your education that might otherwise exist as a result of you know the courses, that you learn from your MSc syllabi and so on so forth. So, I think given that that is the aim of these courses, it is worthwhile to include these types of somewhat challenging issues to the tutorials.

So, keep that in at the back of your mind, that we will have to at some stage show that these states not only are; obviously, eigenstates that is easier to show, what more difficult and important is that these are the only possible eigenstates ok.

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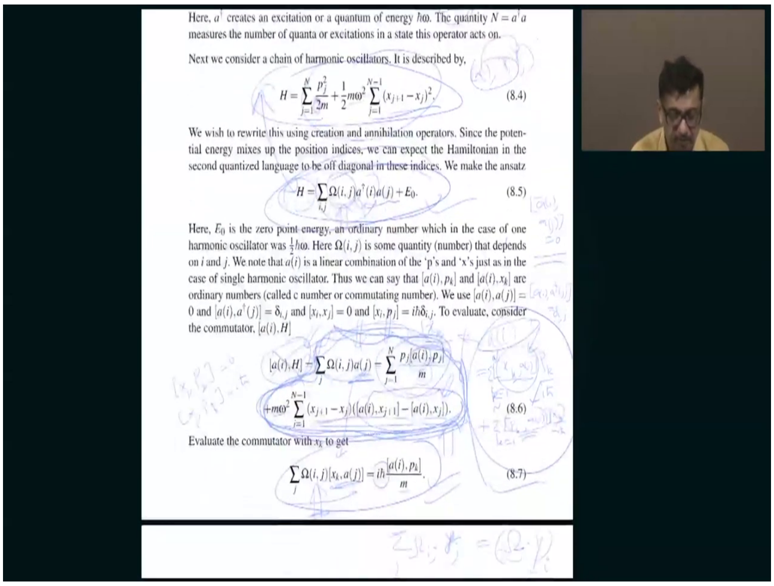
Here, $a^\dagger$ creates an excitation or a quantum of energy $\hbar\omega$. The quantity $N = a^\dagger a$ measures the number of quanta or excitations in a state this operator acts on.

Next we consider a chain of harmonic oscillators. It is described by,

$$H = \sum_{j=1}^{N} \frac{p_j^2}{2m} + \frac{1}{2} m\omega^2 \sum_{j=1}^{N-1} (x_{j+1} - x_j)^2. \quad (8.4)$$

We wish to rewrite this using creation and annihilation operators. Since the potential energy mixes up the position indices, we can expect the Hamiltonian in the second quantized language to be off diagonal in these indices. We make the ansatz

$$H = \sum_{i,j} \Omega(i,j) a^\dagger(i) a(j) + E_0. \quad (8.5)$$

Here, E_0 is the zero point energy, an ordinary number which in the case of one harmonic oscillator was $\frac{1}{2}\hbar\omega$. Here $\Omega(i,j)$ is some quantity (number) that depends on i and j. We note that $a(i)$ is a linear combination of the 'p's and 'x's just as in the case of single harmonic oscillator. Thus we can say that $[a(i), p_k]$ and $[a(i), x_k]$ are ordinary numbers (called c number or commutating number). We use $[a(i), a(j)] = 0$ and $[a(i), a^\dagger(j)] = \delta_{i,j}$ and $[x_i, x_j] = 0$ and $[x_i, p_j] = i\hbar\delta_{i,j}$. To evaluate, consider the commutator, $[a(i), H]$

$$[a(i), H] = \sum_j \Omega(i,j) a(j) = \sum_{j=1}^{N} \frac{p_j [a(i), p_j]}{m} + m\omega^2 \sum_{j=1}^{N-1} (x_{j+1} - x_j)([a(i), x_{j+1}] - [a(i), x_j]). \quad (8.6)$$

Evaluate the commutator with x_k to get

$$\sum_j \Omega(i,j)[x_k, a(j)] = i\hbar \frac{[a(i), p_k]}{m}. \quad (8.7)$$

So, anyway let us relegate that to a later date and we will discuss that at a later date. So, now, let us generalize what we have been discussing. You see so, what we have been discussing so, far is basically a mass tied to a spring ok. So; that means, what is the harmonic oscillator? It is just a mass tied to a spring and we are now imagining that mass basically obeys the laws of quantum mechanics.

Of course, you might be wondering that what sort of a microscopic system will have a spring. But, spring is actually a metaphor; that means, it is basically it is not literally a spring, it is basically a potential energy that goes through a minimum. So, any potential energy that goes through a minimum can be Taylor expanded around that minimum.

And, it can mimic that of a spring, you see if x, x is the displacement from an equilibrium.

So, any function of x can be written like this right. So, you can write this as x V dash 0 plus x squared by 2 factorial V dash dash 0 and so on so forth. So, this is Taylor series. So, you see if x equal to 0 is equilibrium then the force; so, what is V dash? Minus dV by dx at x equal to 0 is the force acting on the particle. So, if x equal to 0 is equilibrium, then the force at equilibrium is 0. So, V dash 0 should be 0.

So, the linear term will be absent at equilibrium, if x equal to 0 is indeed the equilibrium position of that particle then there is no linear term. So, the potential energy close to the equilibrium is going to have this form. It will be some constant plus x squared by 2 into V dash dash 0.

But, then if you want the equilibrium to be stable, you want V dash dash 0 to be positive. Because, then the potential energy goes through a minimum, if it goes through a minimum its stable. So, if the potential energy is like this, then you know if you displace the ball like this, it will come back here. But, if the potential energy is like this, you displace the ball and it will run away. So, you want the potential energy to look like this, but not like that.

So, if it has to look like this V dash dash 0 should be positive. So, if V dash dash 0 is positive, I can call this plus k. So, it will be half k x squared. So, basically any potential energy that goes through a minimum close to the minimum will look like, it will look as if the mass is tied to a spring even though there is no real spring. So, usually these subatomic particles which are in equilibrium, they are actually acted upon by forces due to other subatomic particles in their vicinity and they keep them at their equilibrium position.

So, if you try to displace this subatomic particle which will of course, obey quantum mechanics, it will come back to its equilibrium position as if some spring is pulling them so, or pulling that particular quantum object. So, there is no actual physical spring. It is just that it mimics the forces the forces between subatomic particles under certain circumstances mimic that of a spring ok. So, given that preamble we can now model.

So, I want to model a collection of masses and springs in a linear fashion. So, imagine you have a mass and spring mass and a spring mass in a spring. So, this is meant to mimic typically a one-dimensional solid. So, usually what happens in a solid is that you have these atoms at their lattice positions and then you try to displace any one atom, it will the atom to the left will pull it towards it where the atom towards the right will push it away from it.

So, that it will the atom that is trying to move away from its equilibrium position will be pulled back to towards its equilibrium position. So, therefore, the forces acting on each subatomic particle will mimic that of a spring, but notice that. So, now, let us try and write down the Hamiltonian. So, Hamiltonian of these particles. So, we have to pretend that there are large number of them ok.

So, if there are large number of them; suppose the capital N of them. Then, clearly they are going to have kinetic energy, but they are also going to have a potential energy. So, now, what is the potential energy that we can expect between I mean what is the total potential energy? See, the total potential energy is the potential energy stored in all these springs because; so, we have to understand by how much these springs are stretched.

So, suppose x j is the displacement of this particular say atom. So, from its equilibrium position and this is the displacement of this particular, the next atom and this is the displacement of the earlier atom. So, now, what will be the amount by which this string is stretched? It will clearly be x j plus 1 minus x j, at least I mean magnitude wise it will be the magnitude of x j plus 1 and x j. Because, you see even if suppose x j is equal to say 1 1 nanometer and if x j plus 1 is also 1 nanometer, then the spring is not stretched because both.

So, it is like you know shifting the spring in this direction, without actually stretching it or compressing it. So, spring will stretch or compress only if these two are different. So, the amount by which the spring in between j and j plus 1 will stretch or compress will depend upon the difference between x j plus 1 and x j. So, the potential energy stored in that spring between j and j plus 1 is clearly half m omega squared or basically half k into the amount by which the spring is getting stretched whole squared.

So, what is the amount by which the spring is getting stretched is x j plus 1 minus x j whole square. So, you see same with this spring and so on and so forth. So, what you do is you have to add up all the springs. So, starting from plus 1 to N minus 1 because, we stop at j plus 1 will become N when j is equal to N minus 1 ok. So, this is the amount of potential energy that is stored in all the springs. And, then the kinetic energy only the masses have, the potential energy only the springs have.

So, put together this is the entire Hamiltonian of the system ok. So, now, I want to be able to see just like I wrote this Hamiltonian in terms of a dagger a, is not it? So, there was a p squared by 2 m plus half k x squared, I wrote that like this. So, I also want to be able to write now you see there was only one p and one x here, because it was one mass tied to one spring. But, here I have so many masses and so many springs.

So, clearly I have to have that many a’s and that many a daggers and also you see because this is all getting mixed up. So, if I expand this out, I will get x j into x j plus 1 and so on so forth. So, the Hamiltonian of the jth mass is not unrelated to the Hamiltonian of j plus 1, because they are all getting mixed up. So, I cannot simply write this as something like this, I cannot write like this. I mean it would be nice if I could write like this, but I cannot write.

I mean I cannot write like this because that is only if j and j plus 1, do not talk to each other right. If j and j plus 1 do not talk to each other, then I can write as h bar omega a dagger j a j plus half. So, then that would be too easy. But, in this case clearly j and j plus 1 talk to each other because there is spring connecting them right. So, that same spring connects both so; obviously, they talk to each other. So, if they talk to each other, it is not separable like this. We cannot separate the Hamiltonian in that way ok.

So, what should you do instead is the question ok. So, what you should do instead is the following. So, you should allow for a more general description. We still want it to be something like a dagger a, but we do not want it to be necessarily diagonal; that means, we do not want we do not expect it to be a dagger i into ai because; that means, that would be only if i is does not talk to i plus 1.

So, what will happen is that you see because i talks to i plus 1 because of spring connecting i and i plus 1, but then there is also a spring connecting i plus 1 and i plus 2. So, indirectly i will talk to i plus 2 also because through i plus 1, i will also talk to i plus 2 and so on and so forth. So, indirectly i and j will talk to each other. So, we should be allowing for that possibility. So, in general we should allow for the possibility that the Hamiltonian is of something like a dagger a.

But, it is not a dagger i into a i or a dagger i or into i plus 1, but more generally it is a dagger i into a j times some appropriate coefficient which depends on i and j and, then plus some constant which is at zero point energy. See, just like here also you had that constant. So, that is called the zero point energy which tells you that the lowest energy of a quantum mechanical system of this kind cannot be 0, because the usual reason given is that you know you have p squared and x squared.

So, you cannot simultaneously minimize, you cannot simultaneously make p squared and x squared both 0 because that will violate Heisenberg uncertainty principle. So, there is a compromise. So, there is a minimum value of p squared plus x squared and that is the zero point energy ok. So, similarly for in this case also we expect zero point energy for similar reasons.

So, now the big question is how would I fix this, so, that these two becomes mathematically the same things? See, I want this this one to be mathematically the same as this, I want them to be the same operators. So, the question is how would I do that? So, what I do clearly is that I am going to write ok. So, I am going to say the following that a is linearly related to p’s right. So, a i can always be written as right, I can always write this as p k into; so, I have skipped some steps.

So, let me explain to you what this is. So, basically I can write a i like this plus sigma k p k a i into x k by minus i h bar. So, why is that? See, I can always write like this because first of all this ensures that the a’s are linearly related to p and x. So, now, you see now we have to ensure that because look there are lots of p’s right. So, there are lots of p’s, it is not that there is only one p right; so, there are n number of p’s. So, similarly there are n number of x’s. So, this p k means k equal to 1 to N like that.

So, x k is equal to 1 to N. So, there are lots of p’s and lots of x, the capital N number of them. So, now, you see the a’s have to be linearly related to all the p’s and linearly related to all the x’s. So, I can always write like this because this is an identity. Why is this an identity? So, see if you take a commutator of this equation with say x k; so, take x k commutator of both sides, you will see that you will get an identity.

Because, x k commutator a i you will get on the left hand side, but on the right hand side you will get x k commutator p k and whenever that k does not hit, this outside k it will be 0. So, when its only that it will pick up that exactly that particular k, because you know x 1 will commute with p x 1 commutes with p 2, its only that x 1 p 1 is ih bar not means x 1 commutator p 2 is 0 right.

So, there is mass 1 and mass 2, do not talk to each other, I mean like commutator is 0. They talk to each other through springs, but their momentum and position commute because they are distinguishable particles, 1 and 2 are distinguishable ok. So, the point is that they are different particles so, their momentum and positions commute. It is only the momentum and position of the same particle do not commute, x x 1 commutator p 1 is ih bar.

So, so; that means, if you take x k commutator both sides, you will get ih bar when this k is frozen to be that outside k and ih bar ih bar cancels and you will get left hand side equals, I mean same thing equals same thing. So, left hand side is x k commutator a i, this is also x k commutator a i. But, this one is you will see that its clearly 0 because x is commute with all other x’s. So, the implication is that these commutators are actually numbers.

So, you do not have to further commute x with this commutator because, this commutator is already a number. So, it is just a complex number, some general complex number. So, therefore, a’s are linearly related to p and x. So, that is the implication. So, that if you specify x k commutator a i and p p k commutator a i, it is like specifying fully this transformation, that connects a to the p’s and x’s and vice versa, the reverse also. I mean you can write the p’s in terms of the a’s and a daggers.

So, because you know p has to have a Hermitian, so, a dagger has to be involved. So, bottom line is that you can now go ahead and you can go ahead and take the. So, now, how do you calculate? So, the bottom line so, now, the basic problem in front of us is this problem of pinning down these commutators. So, once you pin these down, you can then go ahead and explicitly write down. So, not only you can explicitly write down the connection between the a's and p's, you can also pin down what this omega has to be.

So, you will be able to explicitly say what should be this omega. So, now how do you go about pinning this down? So firstly, we will start with this ok. So, now, take the commutator of a i with H ok. So, if you take the commutator of a i with H, you will get this because it will pick up this particular a i only. Because, you see commutator of a i and a j is 0, commutator of a i and a dagger j is also 0, unless a i equals j.

So, in other words this is the rule we are using; a i commutator a j is always 0, irrespective of what i and j are. But however, a i commutator a dagger j is 0, unless i equals j and if i equals j the commutator is 1 ok. So, if i equals j commutator is 1 because, that is what we did for this one mass tied to one spring. So, you see a commutator a dagger, if there was only one mass and one spring, it is just a 1 commutator a 1 a dagger 1, a 1 commutator a dagger 1 which is 1.

But, here there are many springs and many masses so, it is a i and a j. So, clearly mass i has nothing to do with mass j; so, they will commute. So, except when i and j are the same, then the commutator is not 0. It is so, instead it is the simplest non-zero quantity which is 1. So, we are going to use that idea and we are going to rewrite. So, the commutator of a i and H ok is now going to be this one. So, it is going to be omega i comma j times a j summed over all j.

So, that is what is its going to be, that is the left hand side. So that means, if I start like this, the commutator is going to be that. So, now, that is should be equal to right hand side which is so, instead of this H if I use this H; so, what is that going to look like? So, that is going to look like this. So, you get first a i commutator p j squared. So, you will get a i commutator p j which is a number into p j.

But, once a i commutator p j is a number, it does not matter whether you put it on the left or right. So, you have to do that you see you have to write this as p j into p j which is p j squared. So, a i commutator p j p j is first you do this commutator, then you do this commutator. So, those two gives you the same answer because, a i commutator p j is a number. So, whether you do it on the right side or left side, it is the same because it has become a number. So, p j multiplying the some number to the left hand, same number to the right is the same thing.

So, you will get twice the same answer and that twice will cancel the 2 in the denominator here ok. So, instead of 1 by 2m, it will become same thing into 2 divided by 2m. So, that 2 2 will cancel, you will get 1 by m. So, same thing will happen here also. So, it is the same thing twice. So, that half m omega squared or half k x squared will become just k x, basically it just become k. Instead of half k, it will become k because it is the twice of this, I mean the same thing twice.

So, it is so, you will get this one ok. So, this is what that is ok. So, now, this is one equation. So, this is our equation. So, this is what we got. So, now, we can further take commutator with respect to x k. Suppose so, now I so I got this one generic equation. Now, I take commutator of this equation, once with x k and then once with p k. So, if I take commutator of this equation which I am circling here with respect to x k, what I will get? Left hand side will become like this because so, forget this now, I mean this much is the equation.

So, I take x k I mean left hand side commutator x k. So, this will become this one ok. So, this after taking commutator. So, after taking x k commutator, clearly this all will go away because this is a number. So, this much has become a number because the commutator of a and x. So, commutator of a and x and a and p these are all numbers, some complex number.

So, now, if I take x k commutator this one; so, this is going to become 0, because x k is a position operator and x k commute commutes with all other x's ok. So, what about this one? So, this one will not become 0, because x k commutator p j is 0 unless j equals k. So, if j equals k then it becomes x k commutator p k. So, what is x k commutator p k? It is ih bar and ih bar so; that means, what will survive is a i commutator p k, because j has become equal to k now.

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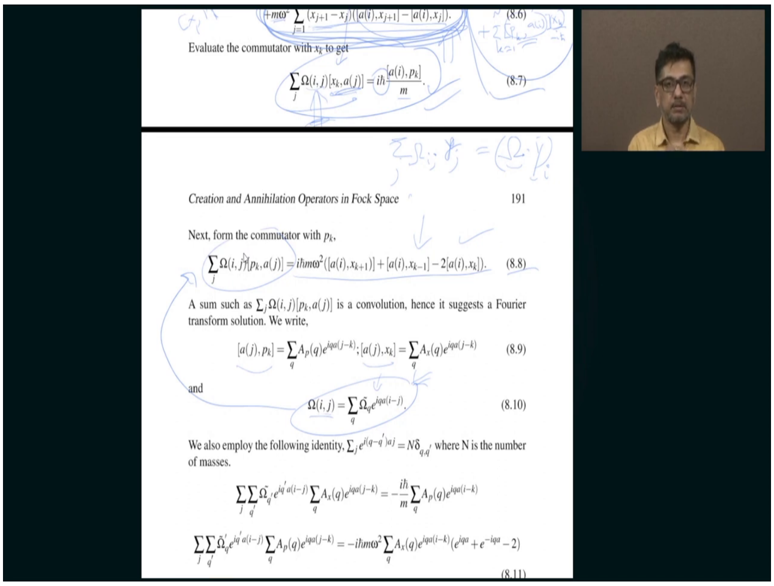

So, you will get a i commutator p k into ih bar divided by m. So, that is one equation. So, that is if you decide to do x k commutator this thing. Now so, that is after doing that we got this. Now, second thing what we do is we do p k commutator this whatever I have circled here. So, if I do x k commutator what I have circled here, I get this one. So, if I do p k commutator what I have circled here, what you will get.

So, left hand side will become this, but right hand side this will go away because this is already a number. This commutator a i p j is already a number. So, if I do p k commutator so, the it will be p k commutator p j which is always 0 anyway, because both are p's only. So, now, only thing which will not be 0 is the second term now. So, now, what you will get if I if you do this carefully is you will get this, I will not actually do it, but you can do it. So, if p k commutator x j plus 1 will be equal to minus ih bar, if j plus 1 equals k and 0 otherwise.

So, you will get all this type of things. So, you can work this out, you will get this ok. So, do not take my word for it, actually work it out. So, you will get these two equations. So, see 8.7 and 8.8 are our basic equations now. So, we have to solve this. So, you see these are so, these are somehow you are like unknowns. This x x k commutator a j is an unknown, then a commutator p k's these are unknowns. Basically, x k commutator a j and p k commutator a j, you can think of that. Those are your unknowns.

So, I have to solve that. So, these are simultaneous equations, 8.7 and 8.8, these are simultaneous equations for those commutators. So, now, you see so, now, this seems quite hopeless because, it is especially seems hopeless because I do not even know what this omega is. So, the question is how would you how on earth would you be able to even solve this 8.7 and 8.8, especially when you do not even know what omega is.

So, but then we should not despair because you see this has the form of a matrix multiplication. See, what is if somebody tells me say omega i j into some y j sigma j. So, this is as if some matrix m is getting multiplied by some vector and I am taking the ith component. So, it is as if I am doing that right. So, it is as if this is a to some m n by n matrix and this is some column vector and I am multiplying taking ith.

So, now specifically you see this this I can always think of this matrix element i comma j to be a function of i minus j, because you will see that when I do that, this will have this summation will have the form of a convolution. So, so if you do not believe it, you do not have to fret too much. Because, what I will do is I mean I will allow you to first make this assumption and its like a posteriori justification; that means, you assume that this is ok and then substitute and you will see that it will work. So, it would not lead to horrible contradictions.

So, then you can work backwards and convince yourself that it could not have been in any other way. So, I do not want to spend too much time you know trying to rigorously justify why this is the only approach that makes any sense. But it is in fact, the only approach that makes any sense. So, the bottom line is that this will be a function of i minus j.

First of all, this is extremely plausible, because you see this system especially if it extends forever in both directions is translational invariant; that means, that any physical quantity that is a function of i minus j will clearly only depend on i minus j; that means, it depends on how far apart j and i are. So that means, it only depends on the distance between i and j. It does not depend on exactly where i is or j is because, if I shift both by the same amount, the system will still look the same.

Because, you see the system extends to infinity in both the positive as well as negative direction. So, there is no sense of this being the origin or that being the origin. So, I can start anywhere. So, but if I have two points, the only thing that determines a physical quantity which depends on two points is the distance between those points. It does not depend on exactly where any one point is because, there is no sense of any origin.

There is no reference point that you can associate any particular point two. So, if you have two points, the only reference is the other point. So, therefore, it is the distance between them is the only reference that you have ok. So, if that is the case, then omega i j which is a physical quantity is now a function only of the difference between i and j ok. So, then the thing then things simplify enormously.

So, what we are going to do is we are going to rewrite this in the Fourier, in the Fourier sense that we can always rewrite a function which depends on i minus j in terms of its Fourier component. So, now, your q is some kind of a Fourier component of omega tilde q will be a Fourier component, q is your Fourier transform variable.

So, if i and j were your direct variables, q is your Fourier transform variables. So, similarly we are going to rewrite your commutators also in terms of Fourier, because after all these commutators are also functions of j and k. So, it should only be a function of the distance between j and k j and not j and k independently, for reasons that I just told you. So, it is going to be a function of the distance between j and k. So, then you simply go ahead and substitute here and then you will be able to do lots with this.

So, I want to stop now because I want to you know walk you through this somewhat slowly. It is easy for me to simply scroll down this slide and then say we are all done. But, then its it would be doing injustice to this course because, many of these descriptions are not readily available in the books. Especially, the steps that I have written down are many times glossed over in many of the books and they do not properly explain them.

So, this is the first course that I have seen on YouTube or anywhere else where many of the steps that are hidden in many of the books are explicitly explained. So, I think it is important for me to spend some time going through many of these steps; so, that you will feel comfortable about learning the course properly ok. So, I am going to stop here. In the next class, I am going to continue with this step.

So, it is even though it is just algebra, it is worthwhile doing it properly, because this sort of algebra appears again and again in solid state physics and many other subjects, when you are trying to understand how to diagonalize Hamiltonians involving many particles ok. I am going to stop here.

Thank you and hope you will join me for the next class.