Dynamics of Classical and Quantum Fields: An Introduction

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Path Integrals
Lecture - 28
Path Integrals - Formalism

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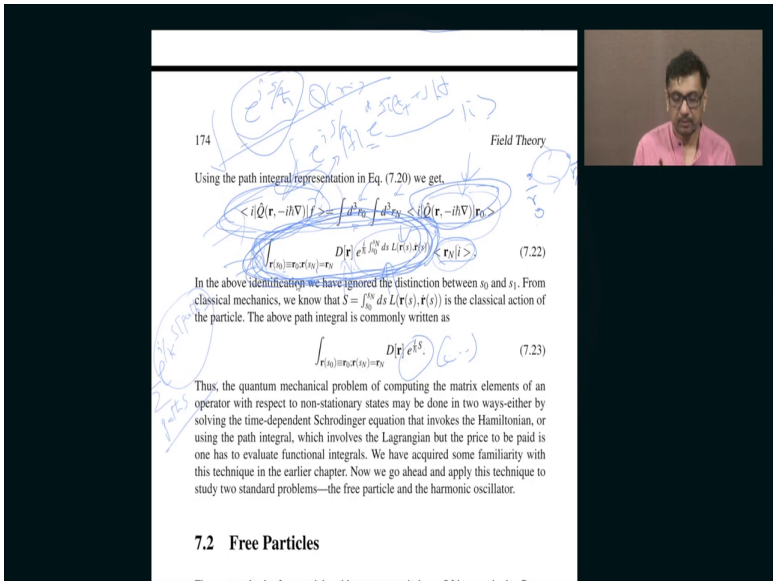

So, in today's class let us discuss the application of this Path Integral Formalism. So, if you recall the in the last class I had stopped here. Where I showed that you can find the quantum mechanical overlap between of any operator. So, suppose you want to find the matrix element of some operator called Q between some two states initial and final. And the final state is obtained by just time evolving the initial state from the initial time to the final time.

So, usually that type of answer to that type of a question in traditional quantum mechanics is obtained by first solving Schrodinger's equation that is the time dependent Schrodinger equation to obtain first the final state from the initial state. And then, just finding the this overlap between the initial state and the operator Q acting on the final state.

So, that is typically how it is done in traditional approaches. But, in the path integral approach which uses the Lagrangian instead of the Hamiltonian it just means that you find an answer to this type of question that is you integrate over all paths ok; starting from r 0 up to r N. So, your initial point so that means, you find all paths connecting a point called r 0, ok.

So, you find so there are two points r 0 r N which of course will also be integrated over finally, but to begin with you find the so, that means you find the integral of this quantity which is e raise to i by h bar into action. So, you add up all the so this is depends on the path, ok. So that means, you add up over all paths. So, this is called the path integral and the path starts from so that means, at the initial time the particle at is at r 0 final time it is at r N and all the paths start and end at the same point.

So, what you do is you first is find this quantity e raise to i by h bar into action for a given path, then you repeat it for a second path and so on and so forth and basically you add up all the paths. But, then there are uncountable infinity of paths so, you have to do a path integral. So, that is what this is that what I have circled here yeah I could have used a different color but it is.

So, but you know what I am talking about. So, this is that path integral. So, I have to integrate over all paths. So, now the answer after integrating over all paths the final answer to that will be a function of r 0 and r N. So, now, you multiply by the overlap between r N and the initial state which is given in the problem and then you also find the overlap between ah the initial state and this operator whose matrix elements you want to find acting on r 0, ok.

So, r 0 and r N are your initial and end point of the path integral. So that means, you even though you have summed over all paths, the final answer still depends on the starting point and ending point. So, r 0 is starting point, r N is ending point. So, now after integrating all r path you get a quantity which depends on r 0 and multiply that with this matrix element r N i, then multiply with the overlap between the initial state and Q acting on r 0.

And then, after that you integrate over all r 0and all r N. Because, the final answer clearly does not depend I mean r 0 and r N you introduce we introduced in between for our convenience. The left hand side just depends on the initial state final state. So, final state depends on the initial time final time and the Hamiltonian of the problem and Q is a given operator.

So, basically so if you integrate over r 0and r N after doing the multiplying the first one, second one, third one you integrate over r 0 r N. You finally get the answer to this question so, in other words you do not have to necessarily. So that means, there is no place where you are required to solve the time dependent Schrodinger equation and so on so forth.

But that is what you would have done if you had wanted to evaluate this matrix element using traditional quantum mechanics the left hand side, ok. So, that is how you would go about doing. So, I mean the answer to this question is a very general one is not it. So that means, most of the interesting questions in quantum mechanics can be posed as finding the you know matrix element of some operator between some initial and final state. So, most of the interesting questions are of that nature anyway.

So, what we have successfully done is that, we have recast the answer to that rather generic and general question that occurs repeatedly in traditional quantum mechanics; we have recast that in terms of a path integral. So, we have explained how to evaluate the I mean or compute the answer to that question using Lagrangian instead of Hamiltonians, ok. So now, let me go ahead and try to apply this to some particular problem. So, this was very general it was formalism. So, specifically what I am going to do is I am going to find what is called the propagator.

So that means, what I am interested in finding is basically imagine that the initial state is a state with well defined position of the particle. That means, in the initial state the particle has a well defined position called r 0 and in the final state the particle has a well defined position r N, ok. So, now I want to know the what is the overlap between these two states, because the initial state will be at time t i and final state will be at time. So in other words, basically I want to know this it is self whatever I am circling now.

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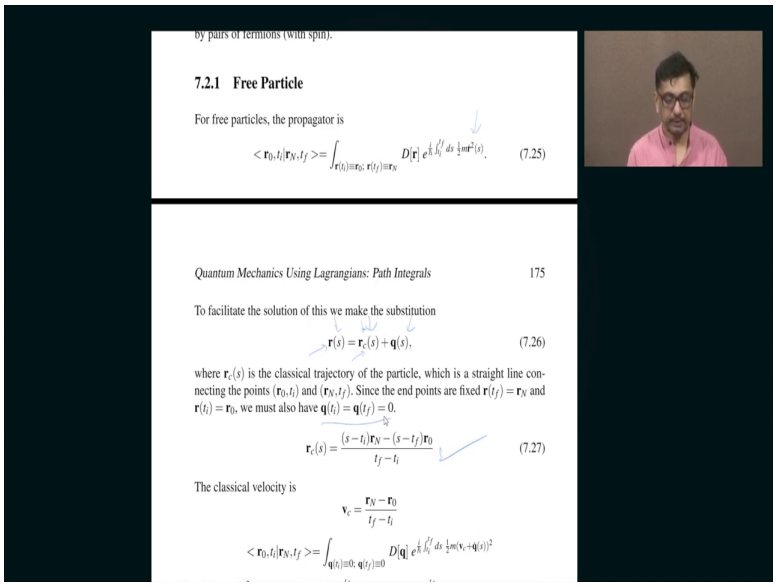

So, I really want that itself. So in other words, I want to know the path integral. So, that path integral clearly has the interpretation of an overlap between an eigen state of position at time t i and eigen state of position at time t f, ok. So, it is like this. So, if at time t i you make a measurement of the position and the answer for that measurement comes out as r 0. Then, you wait for a time t f minus t i and then again you make a measurement of the position. The answer comes out as r N t f.

So, the question is what is the probability or what is the overlap between these two states. That means, what is the probability that if the particle is at r 0 at t i the particle will end up at r N at t f. So, the answer to that question is this path integral. So now, if it is a free particle so, this is always true that is always the answer to the question. But now, if the particle is a free particle, then the Lagrangian is just the kinetic energy. So, now the kinetic energy is clearly half m r dot square. So now, what I am going to do is as usual I am going to split up this problem into two pieces.

So, I have to do a path integral. So that means, I have to do a path integral initial position is r 0 and r N. So, instead what I want to do as I have been doing earlier also I want to split this up in this path integral. That means, sum over all paths into a path which is basically the extremum or the most probable path the one which extremizes the action

and then, I want to say that the actual path is obtained by this extremum path plus a deviation.

So, this is extremum path is clearly the classical path. So, the path which extremizes the action is basically obeys the Euler-Lagrange equations. So, that is the classical path. So now, I am going to write a general path ok which is required in quantum mechanics, so that means, the quantum particle does not necessarily always take the classical path. So, it takes all possible paths and so, I can always write the general path as the classical path plus some deviation with respect to the classical path.

So, q s is my deviation. So, if I want to do a path integral I have to integrate over the deviation. So, the classical path is given but the deviation is the one which is getting summed over, because the original path has to be summed over. The classical path is a fixed unique path. So, there is nothing to sum over there. So, the path that is getting summed over is r therefore, it is also I means you are also required to sum over the deviation.

So, now clearly for a free particle the classical path is a straight line. And the straight line such that at t equals t i the position is r 0 and at t equals t f the position is so that means, s is my I have parameterized in terms of s instead of t, t for some reason. So, basically at s equals t i it is r 0 and as at s equals t f it is r N. So, that is obeyed and this is a straight line. So, it is linear in s, ok.

So, now because r N r N r c that is the classical trajectory and the actual path taken in by the quantum particle both have the same end points. Clearly, q has to have this property its end points has to be 0, because r and r c are the same end points.

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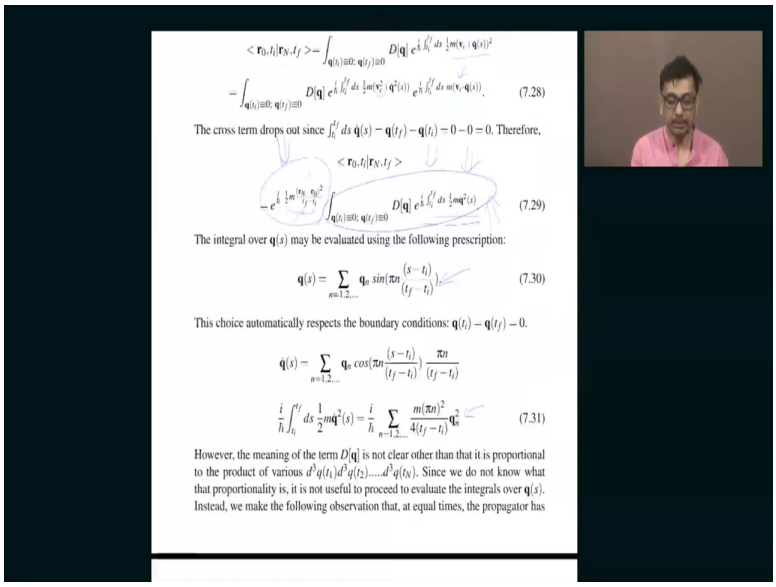

So, now the point is I mean so, when you actually recast all the original path integral in terms of q instead of this what you end up getting is basically the cross term cancels out and you get this term. So, you will get a classical term times this. So now, you know I told you the reason why we do this is because you see once you make the deviations the end point of the deviation both are 0 that immediately means I have periodicity. So that means, at q at t i is 0 and q at t f is also 0.

So that means, I can exploit periodicity and write decompose q in terms of a discrete Fourier series rather than some general transform which would not be useful. So, this is what I have done. So, you see here q bracket s has the property that when s is t i it is 0 when s is t f it is also 0 that is why it is sin pi n, ok. So, it is a discrete Fourier series. So, once I do this I can go ahead and rewrite the action in terms of the deviation in terms of the discrete components q n, ok.

So now, what I am going to do is that what we have to do is we have to integrate over all the q ns, but we also know that if see, but we would not be able to fully understand how to fix those proportionality.

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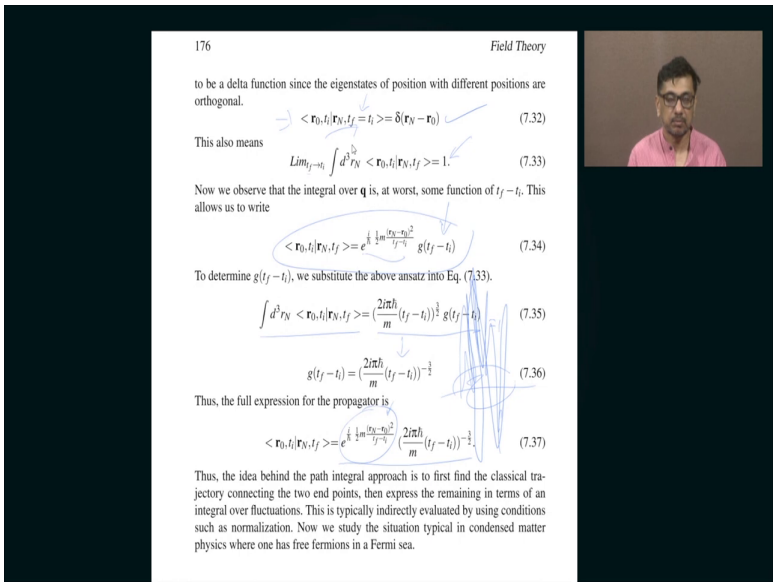

So, basically this is once you integrate over all the q ns it has the nature of a proportionality constant, but rather than fixing it that way I mean rather than struggling with that type of a calculation what we do is we make use of this clever observation and that is that if t f is equal to t i, right. So, if you have a free particle and you do not wait at all. That means, you start at t i, you are supposed to wait until you the time becomes t f before finding the overlap suppose you do not wait; that means, your starting time is t i ending time is also t i.

So, now what do you expect the overlap to be clearly it will be 0 unless the end value of r is same as the beginning value of r so, unless r N equals r 0. The overlap is 0, because the particle has not had time to move right initially it was at r 0 at time t i, but then t f is equal to t i that means, you do not give it enough time to move. So, clearly its position should still always remain r 0 only quantum mechanically as well as classically. So, there is no time for the wave function to spread or for the particle to move or whatever it is.

So, clearly this is valid. So that means, the quantum mechanical overlap is 0 unless in which case it is a Dirac delta function, ok. So, what you do is that basically rather than struggling with integrating over all these q Ns which is what it would be what you do is you multiply with this clearly we know that this integral this particular this one. This is

just a function of t f and t i. So, specifically it will be a function of t f minus t i because of free particle.

So, it is some function of t f minus t i and instead of struggling by actually doing that integral which is what you would have to do, what we do is we write this overlap as what we got earlier. So, just remember where this came from. This simply came from just substituting this kind of this kind of a relation with this into the original action and you end up with this one. That means, you end up with one half m v c square and v c square it was basically equal to this one, ok. So, that is v c square half m v c square into t f minus t i.

So, that is basically r N minus r 0 squared divided by t f minus t i squared into t f minus t i is half m v squared into t f minus t i but then, so one of the t f minus t i from the denominator cancels out so you get this. So, that is where that came from, ok. So, the rest of the thing is just the integral over deviations. So, the bottom line is instead of trying to struggle and find the actual answer by integrating over all the q ns. I do a shortcut. I realize that when t f equals t i its a Dirac delta. So, I just give it a name I just call it t f g of t f minus t i.

And so, this is clearly the most general answer to the overlap. So then, what I do is that I make use of this is obtained by integrate I mean this is the general observation then you simply integrate over all the r Ns and then take the limit you get one, because that is how it is. So, what we do is that here you simply integrate over all the r Ns because, right so and you make t f tends to t i it should give you 1. So, that is going to happen.

So, the after you integrating over all r N this what it is and this should be 1 when t f tends to t i. So, that is going to happen very easily if you choose this. So now, if once you choose this then it clearly has that property that as t f tends to t i this whole thing will become a Dirac delta function, ok. So, that is what that is.

So, intuitively also you can see that this is the case. So, suppose that t f is not equal to so, rather r N is not equal to r 0, but t f tends to t i, ok. So, if t f tends to t i what is going to happen is that this will oscillate so, you see t f tends to t i so it will have a large phase.

So, it will basically oscillate the average out to 0. So, basically if you plot this it will look like this.

So, it will actually look like this. So, on an average it will average out to 0, ok. So, however if r N is equal to r 0 then this will become 1 and then, if you first make r N equals r 0 and then take t f tends to t i you will get infinity because, that is what you are supposed to do, ok. So, if you first take r N equals r 0 and then my t f tends to t i you get infinity because, that is what.

So, if r N equals r 0 to begin with you are supposed to get infinity because of the delta function, but in general when r N is not equal to r 0 and you make t f tends to t i supposed to it 0. Because, when t f tends to t i r n not equal to r 0 means the chances of that are 0, so that means the overlap is 0. So, that is what we are finding here. So, intuitively that is correct, but mathematically also it is correct because, if you so, it is therefore proportional to the delta function and the proportionality is 1 because if you integrate over r N you get 1, ok, ok.

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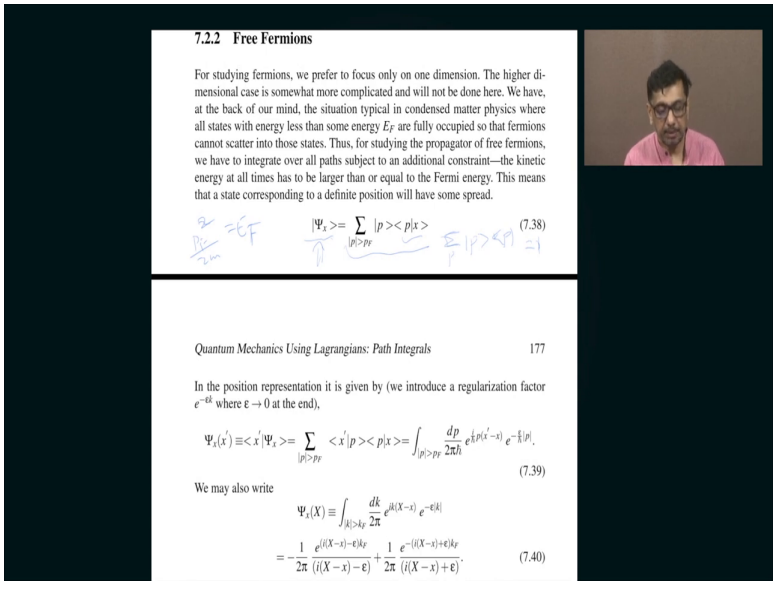

So, now so the next problem so this was free particles in the sense that is absolutely free and nothing is blocking anything. So, there are no potential energies and the particle can have any momentum it wants any energy it wants and so on so forth. So, that is what a

free particle is, but the next example is I have concocted this example which is not found in many books, but its also something you should not take to literally. So, the example is the following.

So, basically it I am trying to mimic the properties of an electron in a metal which has energy close to the Fermi energy. So that means, I am trying to mimic the behaviour of an electron in a metal whose energy is close to the Fermi energy. So, of course, I must I mean all these words will make sense if you have some background in solid state physics or stat mic.

So, that is one of the stat mic is a prerequisite for this course. So, that is the reason why I made that sentence. So, basically in a metal what happens is that you have a large number of electrons and they are all fermions they obey poly exclusion principle. So now, if you try to make the electron occupy one energy level so, you can at most occupy one more electron with that energy and after that you will have exhausted all the possible state; I mean you cannot accommodate more than two, because one has to have up spin the other is down spin.

So, then you have to go to the next level next level next level like that. So, if you have large number of electrons like you have conduction electrons in a metal. So, the ground state itself will have a huge energy. So, that huge energy is the called the Fermi energy. So, it will be fully populated up to some energy. So now, you know an electron with energy close to the Fermi energy cannot scatter to any state whose energy is less than the Fermi energy because, all states below Fermi energy are occupied.

So, what I am going to say is that, I am going to make this so, I am firstly I am going to focus in 1 dimension because that is of application interest also later on when we study routing or liquids and secondly also because it just serves to illustrate the main point without getting bogged down in integrals I mean 1 dimensional integrals are easy to do.

So, the bottom line is that normally you know if I invoke completeness what I would do is I would write like this. So, this would be my completeness, but then in this case what I do is that you see. So that means, normally a particle with a well defined position will

make sense for a free particle means like, if you have it makes sense to talk of a particle with a well defined position.

But in this example, because the particle that I am looking at. So, there is a imagine there is a fermion and it is trying to wander around, but then it is constantly aware that it is in the presence of a filled Fermi sea. That means, it is in the presence of a sea of fermions which is filled up to some Fermi momentum E F and which is determined by p F p F squared by 2 ms E F, ok. So, that is the Fermi momentum and it is related to Fermi energy like this.

So that means, there is a sea of electrons we filled up to Fermi energy. So, another electron comes wandering around and its completely aware of that its in the presence of this Fermi sea, because it has to obey Pauli principle; that means, it cannot go and sit on top of some other electron which is having energy less than Fermi energy. So, it is fully aware that it has so, what will happen is that it will then what this means is that it is impossible to create an eigen state of an electron with well defined position.

So, remember what that means. See, if a particle has a well defined position its momentum can be absolutely anything, right. If the position is strictly well defined, right. So that means, the spread in delta x is 0 so that means; Heisenberg principle is delta p is actually infinity right, because it can be anything. So, momentum can be absolutely anything but then, if there is a field Fermi sea its momentum cannot be absolutely anything it certainly cannot be less than the Fermi momentum because, those states are occupied by other electrons.

So, what happens therefore, is that this type of an interpretation forces us to conclude that it is not possible to create an electron in the presence of a filled Fermi sea to have a strictly well defined position. Because, a strictly well defined position implies completely arbitrary momentum, but that violates Pauli principle with respect to the other electrons. So, the best you can do is actually write this. So, this is the closest you can come to creating a particle with some well defined position.

So, what you do is, you find this overlap which is basically raise to ix i p x by h bar and you multiply by eigen state of position and you sum over all p such that its magnitude is

greater than p F. So, that you are staying away from filled states so, p greater than p F for all empty states. So, you are staying away from filled states so, and yet you are trying to mimic the idea that you are trying to find a particle with some position x. So, this is the closest you will come to finding particle with some position x, ok.

So, what will happen is that finally when you try to evaluate this you get this answer. So, if you try to find the overlap between this and a strictly eigen state of position.

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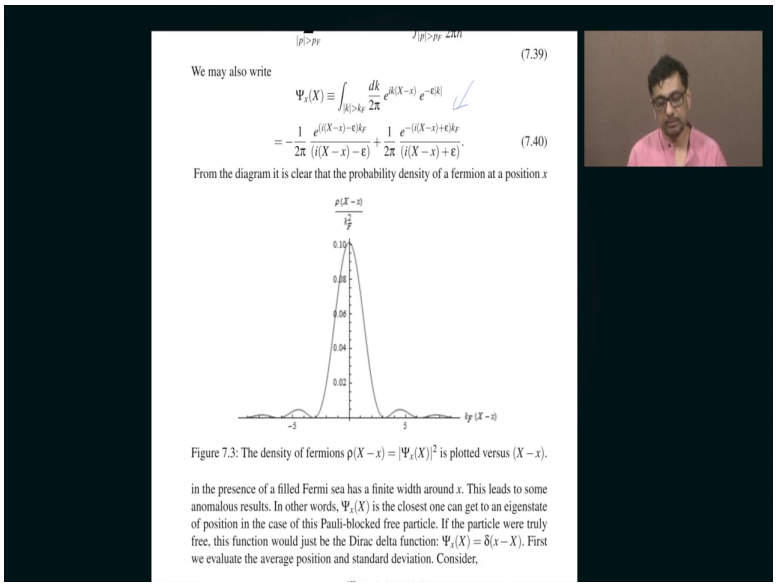

So, it basically tells you that it will actually not be a delta function as it would be if it were genuinely I mean if there was no Pauli principle involved. So, rather it will because there is a Pauli principle involved there is a spread. So; that means, that a particle with well defined position in the presence of filled Fermi sea is not possible, but it is possible approximately in the sense that you will get a distribution which is peaked at that position that you are looking for, ok.

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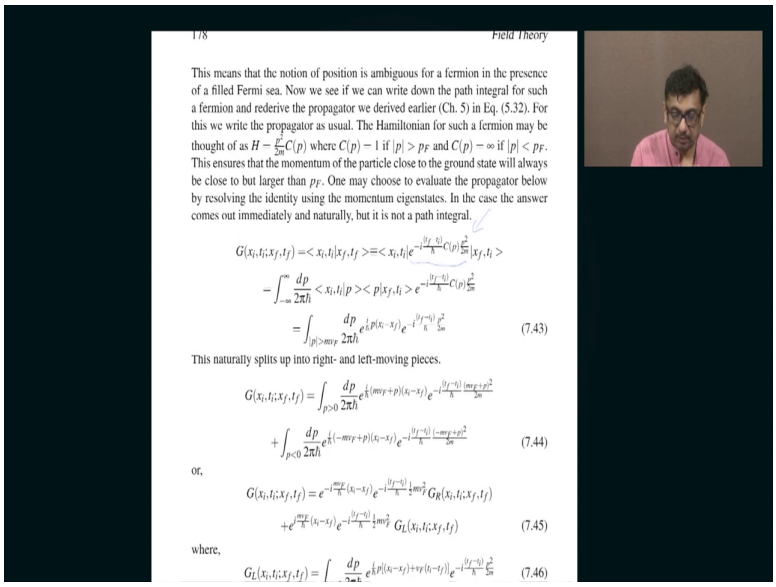

So, I am I do not want to make a big fuss about it, but I just thought I will let you know. So, now, the real question is the following. So, this is the propagator. So, this is called the propagator. So, this is what we got by just doing normal quantum mechanics there is no path integrals here, right. So, this is conventional quantum mechanics.

So, now my question is the following. So, why did I bring this up, because this chapter is about path integrals? So, the thing is that I want to be able to see if I can get this answer by a suitable path integral. So, what I am going what I have done here is that to do this I have postulated the following. So, imagine that the Hamiltonian so that means, now I have to reinterpret my problem in such a way that it captures the presence of this Fermi sea of electrons, ok.

Which is providing this Pauli principle Pauli exclusion principle effect which will prevent the an electron that is wandering around in the metal to have any momentum less than the Fermi momentum, ok. So, the way this is done is that you say that the Hamiltonian of that wandering electron is p squared by 2 m except when p squared by 2 m only when p is greater than p F; that means, magnitude of p is greater than p F.

But, when magnitude of p is less than p F the Hamiltonian is basically infinity in the sense that so, we say it is infinity to ensure that an electron can never be have that

momentum because, it is basically energetically unfavourable for the electron. So, I am forcing Pauli exclusion principle by making it energetically very unfavourable for the electron to have that energy less than the Fermi energy. So that means, I am making it impossible for the electron to have energy less than the Fermi energy by just introducing this cut-off function C p.

So, what I am saying is that if energy is less than Fermi energy the Hamiltonian blows up. So, it is energetically overwhelmingly unfavourable for the electron to have that energy, ok. So, now if you accept that point of view then it is easy to find the overlap. So firstly, you can find the overlap using conventional approaches so; that means, that this is your overlap this is the definition of the overlap this is the evolution unitary evolution.

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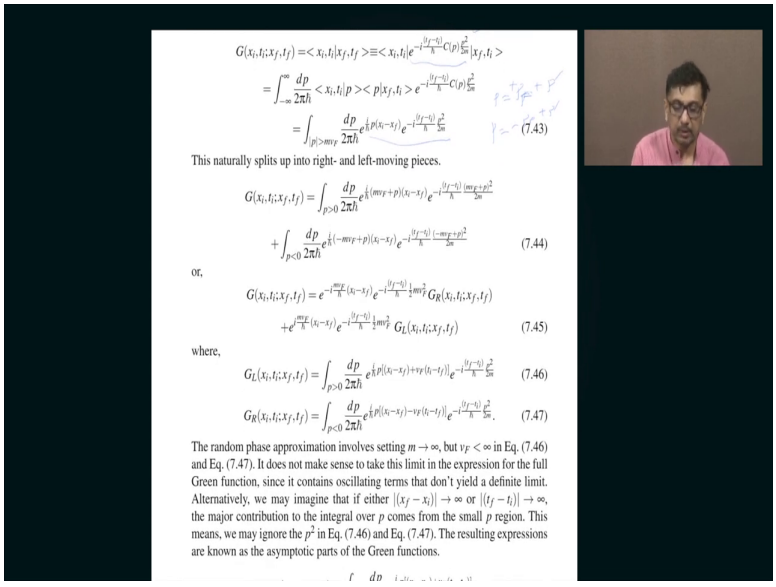

And then, you just go ahead and do this so you just have to evaluate this integral. So, then you split this up into right mover. So, p equals p F plus plus p F plus p dash and p equals minus p F plus p dash. So, you can split this up into two parts and then, you can go ahead and evaluate this, ok. So, you can actually get these what are called right mover and left mover by doing this, ok.

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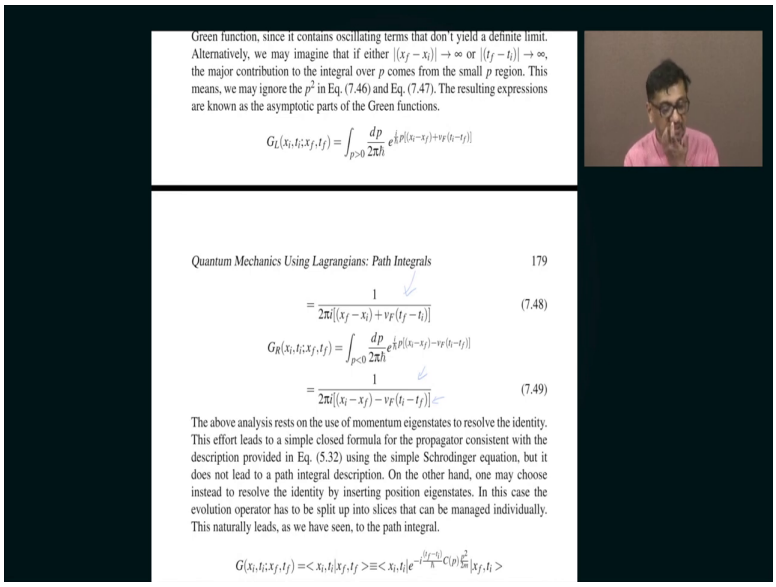

So, you will see that finally the answer is of this type. So, this is called the left mover this is called the right mover. So, that is because the pole of this function is when delta x is equal to plus v of delta t, ok. So, that is called the right movers. So, the because the velocity is positive and it is negative in this case. So, that is why it is called the left mover. So, the answer to this overlap is basically the sum of right and left moving Green functions.

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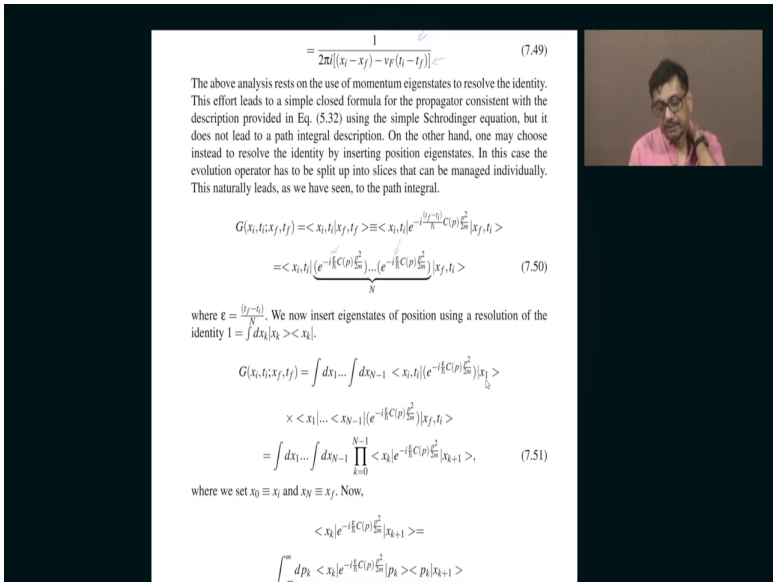

So, now the question is how would I reinterpret this in terms of path integrals. So, as usual what I do is I invoke this trotter product formula by dividing by N and making N copies. So, if I divide by N I get t f by minus t i by N is epsilon. So, I have introduced an epsilon then I make N copies of that. So, as usual when I make N copies I can insert a complete set of states with position x i in between.

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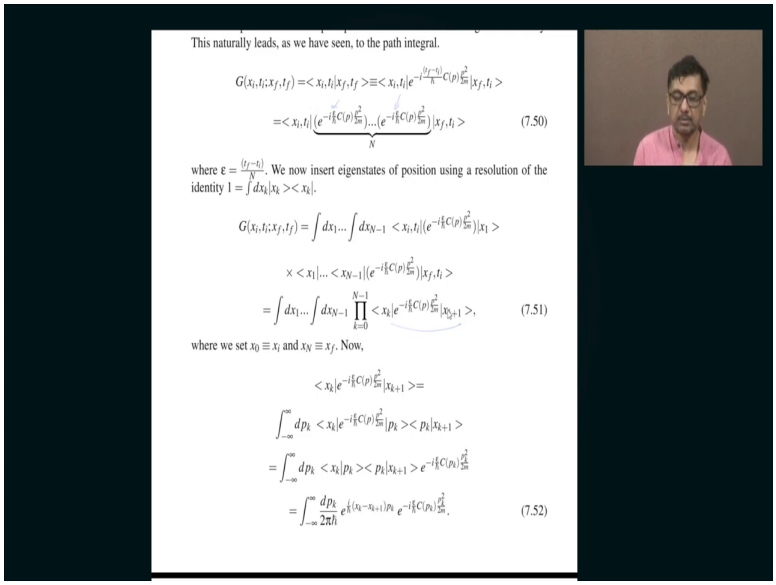

And then, I end up calculating this, ok. So, and here I do not have to use any trotter product because, actually there is no potential energy. So, it is I can just directly evaluate this, ok. So, I just directly evaluate this and I get this sort of thing, ok.

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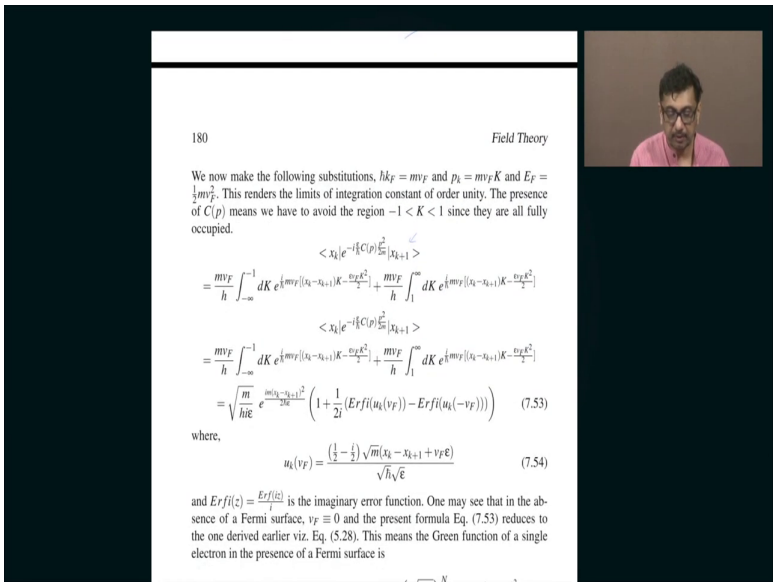

180 Field Theory

We now make the following substitutions, $\hbar k_F = mv_F$ and $p_k = mv_F K$ and $E_F = \frac{1}{2}mv_F^2$. This renders the limits of integration constant of order unity. The presence of $C(p)$ means we have to avoid the region $-1 < K < 1$ since they are all fully occupied.

$$< x_k | e^{-i\frac{\varepsilon}{\hbar}C(p)\frac{p^2}{2m}} | x_{k+1} >$$

$$= \frac{mv_F}{h}\int_{-\infty}^{-1} dK\, e^{\frac{i}{\hbar}mv_F[(x_k - x_{k+1})K - \frac{v_F K^2}{2}]} + \frac{mv_F}{h}\int_{1}^{\infty} dK\, e^{\frac{i}{\hbar}mv_F[(x_k - x_{k+1})K - \frac{v_F K^2}{2}]}$$

$$< x_k | e^{-i\frac{\varepsilon}{\hbar}C(p)\frac{p^2}{2m}} | x_{k+1} >$$

$$= \frac{mv_F}{h}\int_{-\infty}^{-1} dK\, e^{\frac{i}{\hbar}mv_F[(x_k - x_{k+1})K - \frac{v_F K^2}{2}]} + \frac{mv_F}{h}\int_{1}^{\infty} dK\, e^{\frac{i}{\hbar}mv_F[(x_k - x_{k+1})K - \frac{v_F K^2}{2}]}$$

$$= \sqrt{\frac{m}{i\hbar\varepsilon}}\, e^{\frac{im(x_k - x_{k+1})^2}{2\hbar\varepsilon}} \left(1 + \frac{1}{2i}(Erfi(u_k(v_F)) - Erfi(u_k(-v_F)))\right) \quad (7.53)$$

where,

$$u_k(v_F) = \frac{\left(\frac{1}{2} - \frac{i}{2}\right)\sqrt{m}(x_k - x_{k+1} + v_F\varepsilon)}{\sqrt{\hbar}\sqrt{\varepsilon}} \quad (7.54)$$

and $Erfi(z) = \frac{Erf(iz)}{i}$ is the imaginary error function. One may see that in the absence of a Fermi surface, $v_F \equiv 0$ and the present formula Eq. (7.53) reduces to the one derived earlier viz. Eq. (5.28). This means the Green function of a single electron in the presence of a Fermi surface is

So, I basically end up getting this type of. So, the overlap between successive points so; that means, so remember I am just going quickly, because I have done this earlier. So, you so the thing is the idea that you split up some path between initial and final points into pieces and these are called x k plus 1 like that and then you find this overlap and then, you rewrite this in terms of this integral like this.

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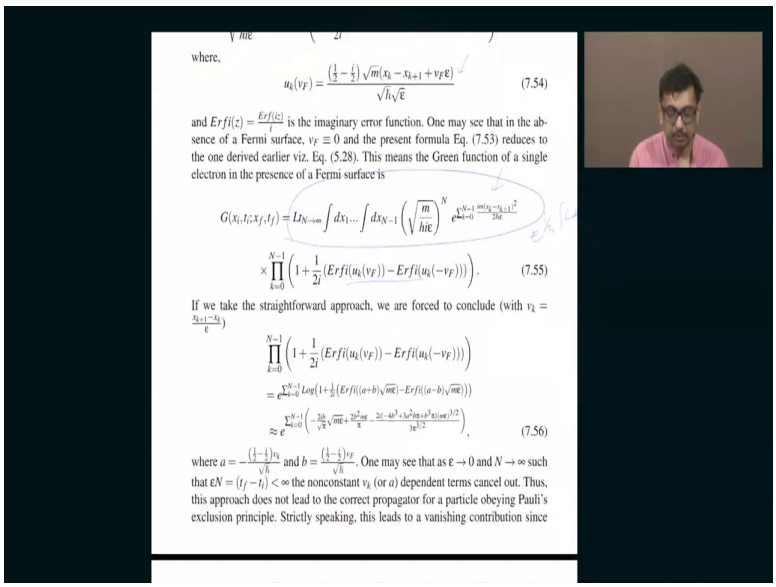

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$$G(x_i, t_i; x_f, t_f) = L_{N\to\infty}\int dx_1 \ldots \int dx_{N-1}\left(\sqrt{\frac{m}{i\hbar\varepsilon}}\right)^N e^{\sum_{k=0}^{N-1}\frac{im(x_k - x_{k+1})^2}{2\hbar\varepsilon}}$$

$$\times \prod_{k=0}^{N-1}\left(1 + \frac{1}{2i}(Erfi(u_k(v_F)) - Erfi(u_k(-v_F)))\right). \quad (7.55)$$

If we take the straightforward approach, we are forced to conclude (with $v_k = \frac{x_{k+1} - x_k}{\varepsilon}$)

$$\prod_{k=0}^{N-1}\left(1 + \frac{1}{2i}(Erfi(u_k(v_F)) - Erfi(u_k(-v_F)))\right)$$

$$= e^{\sum_{k=0}^{N-1} Log\left(1 + \frac{1}{2i}(Erfi((a+b)\sqrt{m\varepsilon}) - Erfi((a-b)\sqrt{m\varepsilon}))\right)}$$

$$\approx e^{\sum_{k=0}^{N-1}\left(-\frac{2b}{\sqrt{\pi}}\sqrt{m\varepsilon} + \frac{2b^2 m\varepsilon}{\pi} - \frac{2(-4b^3 + 3a^2 b)\pi_k + b^3\pi)(m\varepsilon)^{3/2}}{3\pi^{3/2}}\right)}, \quad (7.56)$$

where $a = -\frac{\left(\frac{1}{2} - \frac{i}{2}\right)v_k}{\sqrt{\hbar}}$ and $b = \frac{\left(\frac{1}{2} - \frac{i}{2}\right)v_F}{\sqrt{\hbar}}$. One may see that as $\varepsilon \to 0$ and $N \to \infty$ such that $\varepsilon N = (t_f - t_i) < \infty$ the nonconstant v_k (or a) dependent terms cancel out. Thus, this approach does not lead to the correct propagator for a particle obeying Pauli's exclusion principle. Strictly speaking, this leads to a vanishing contribution since

So then, you will end up with getting this Lagrangian. So, this is; so this is what you would have got if there was no Pauli principle. So, this is if there was no Pauli principle the path integral would have simply this. Because, it is just action right it is half m v squared, because Hamiltonian time Lagrangian are the same for free particle. So, it is just e raise to i by h bar into integral l d t because, d t is that epsilon.

So, l is half m v squared. So, it is x k minus because x k plus 1 minus x k divided by epsilon whole square is your x dot square. So, that gets multiplied by d t which is epsilon. So, you get this epsilon here, ok. So, that is what that was. So, bottom line is that you would have got this, but then the new ingredient is this one, ok. So, this is the new ingredient so, this is there because of Pauli principle.

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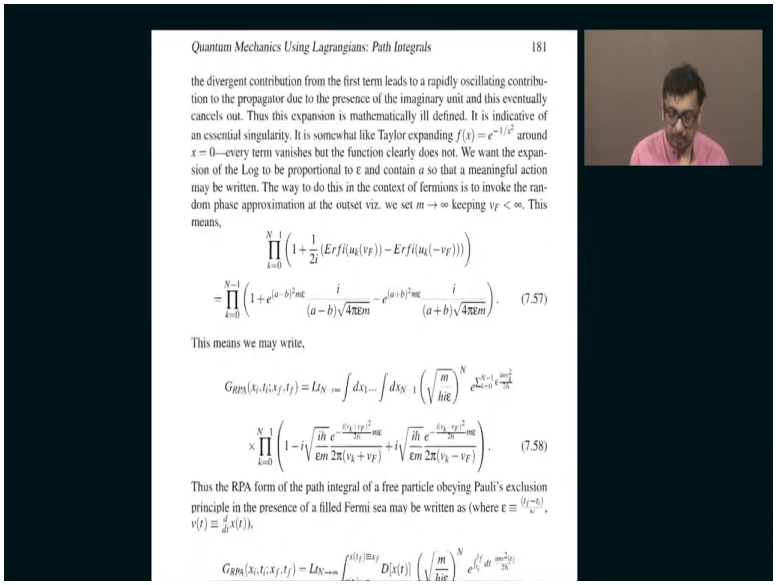

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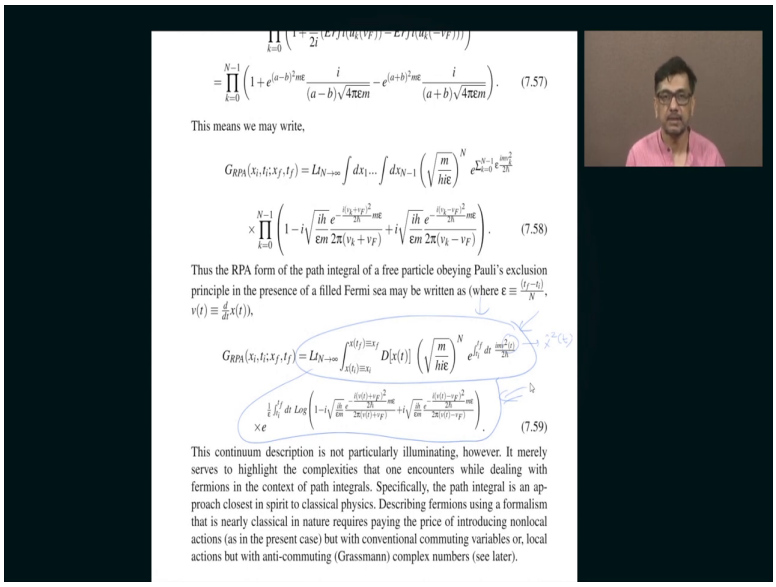

So, bottom line is that so if you follow this logic what you will do is you will end up getting this type of a path integral. So, this is the path integral. So, this is x dot squared I should have written x dot squared. So, if there was no Pauli exclusion principle; that means, if there was a free particle running around in 1 dimension and there was nothing else the answer for the propagator; that means, what is the probability if the particle was at you know x i at t i what is the probability.

Or what is the probability amplitude that the particle will be at x f at t f is simply given by this path integral; that is if that particle was minding it is own business there was no metal no Pauli exclusion principle thing this is would be the answer this path integral. So, just start at xi at t i and end at x f at t f, which we have already evaluated.

But now, if there is some something which is preventing it from moving freely like, there is a filled Fermi sea of electrons and this electron as wandering around has to be very conscious of the presence of this filled Fermi sea it has to make sure that its energy will never fall below the Fermi energy. So then, the path integral will have to be modified in this way.

So, you will have to introduce some additional terms like this, ok. So, this additional terms will be very complicated. So, it is basically the non local type of. So, bottom line is

that you might be wondering why did I give such a horrendous example. So, the reason is basically I just wanted to point out that in general if you introduce fermions in the problem there will always be some difficulties like this. So, this is a simple example where I showed you that introducing fermions will actually create problems in the sense that the path integrals will become very unusual.

So in fact, later on we will see in the context of what are called coherent state path integrals the you will be integrating over some very funny kind of versions of complex numbers called Grassmann numbers Grassmann variables. So, those are anti commuting complex numbers. So, those type of concepts will occur when you are dealing with fermions.

So, this is the first example I have introduced where introduction of fermions obeying Pauli exclusion principle even though the situation appears to be rather simple namely 1 dimension 1fermion trying to wander around from x i at t i to x f at t f, but then it is constantly being reminded that it is in the presence of this field Fermi sea of electrons up to Fermi energy. So, it has to constantly obey the Pauli Exclusion Principle as it moves from x i t i to x f at t f. So, the path integral will become extremely complicated because of that.

So, bottom line I am trying to say that once you try to study fermions basically things are very complicated and this is the first example where I have explicitly displayed that complication. So, later on we will find other examples where fermions will present it is own unique set of complications, but we have this is almost intractable, but in other examples especially in coherent state path integrals you will see that it is still tractable we can handle it, ok.

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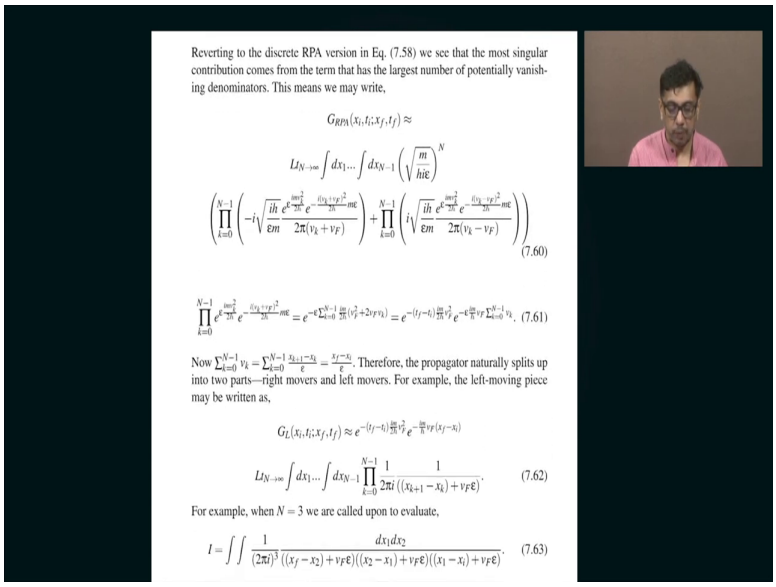

Reverting to the discrete RPA version in Eq. (7.58) we see that the most singular contribution comes from the term that has the largest number of potentially vanishing denominators. This means we may write,

$$G_{RPA}(x_i,t_i;x_f,t_f) \approx$$

$$L_{N\to\infty}\int dx_1 \ldots \int dx_{N-1}\left(\sqrt{\frac{m}{i\hbar\epsilon}}\right)^N$$

$$\left(\prod_{k=0}^{N-1}\left(-i\sqrt{\frac{i\hbar}{\epsilon m}}\frac{\epsilon^{\frac{im\epsilon v_F^2}{2\hbar}}e^{-\frac{i\epsilon v_k+v_F)^2}{2\hbar}m\epsilon}}{2\pi(v_k+v_F)}\right)+\prod_{k=0}^{N-1}\left(i\sqrt{\frac{i\hbar}{\epsilon m}}\frac{\epsilon^{\frac{im\epsilon v_F^2}{2\hbar}}e^{-\frac{i\epsilon v_k-v_F)^2}{2\hbar}m\epsilon}}{2\pi(v_k-v_F)}\right)\right) \tag{7.60}$$

$$\prod_{k=0}^{N-1}\epsilon^{\frac{im\epsilon v_F^2}{2\hbar}}e^{-\frac{i\epsilon v_k+v_F)^2}{2\hbar}m\epsilon}=e^{-\epsilon\sum_{k=0}^{N-1}\frac{im}{2\hbar}(v_F^2+2v_Fv_k)}=e^{-(t_f-t_i)\frac{im}{2\hbar}v_F^2}e^{-\epsilon\frac{im}{\hbar}v_F\sum_{k=0}^{N-1}v_k}. \tag{7.61}$$

Now $\sum_{k=0}^{N-1}v_k=\sum_{k=0}^{N-1}\frac{x_{k+1}-x_k}{\epsilon}=\frac{x_f-x_i}{\epsilon}$. Therefore, the propagator naturally splits up into two parts—right movers and left movers. For example, the left-moving piece may be written as,

$$G_L(x_i,t_i;x_f,t_f)\approx e^{-(t_f-t_i)\frac{im}{2\hbar}v_F^2}e^{-\frac{im}{\hbar}v_F(x_f-x_i)}$$

$$L_{N\to\infty}\int dx_1 \ldots \int dx_{N-1}\prod_{k=0}^{N-1}\frac{1}{2\pi i}\frac{1}{\left((x_{k+1}-x_k)+v_F\epsilon\right)}. \tag{7.62}$$

For example, when $N=3$ we are called upon to evaluate,

$$I=\int\int\frac{1}{(2\pi i)^3}\frac{dx_1dx_2}{\left((x_f-x_2)+v_F\epsilon\right)\left((x_2-x_1)+v_F\epsilon\right)\left((x_1-x_i)+v_F\epsilon\right)}. \tag{7.63}$$

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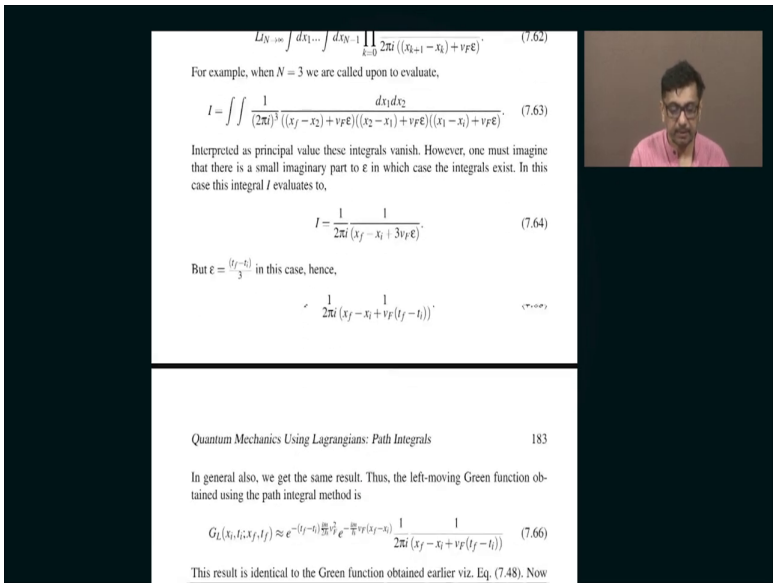

$$L_{N\to\infty}\int dx_1 \ldots \int dx_{N-1}\prod_{k=0}^{N-1}\frac{1}{2\pi i}\frac{1}{\left((x_{k+1}-x_k)+v_F\epsilon\right)}. \tag{7.62}$$

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Interpreted as principal value these integrals vanish. However, one must imagine that there is a small imaginary part to ϵ in which case the integrals exist. In this case this integral I evaluates to,

$$I=\frac{1}{2\pi i}\frac{1}{\left(x_f-x_i+3v_F\epsilon\right)}. \tag{7.64}$$

But $\epsilon=\frac{(t_f-t_i)}{3}$ in this case, hence,

$$\frac{1}{2\pi i}\frac{1}{\left(x_f-x_i+v_F(t_f-t_i)\right)}.$$

Quantum Mechanics Using Lagrangians: Path Integrals 183

In general also, we get the same result. Thus, the left-moving Green function obtained using the path integral method is

$$G_L(x_i,t_i;x_f,t_f)\approx e^{-(t_f-t_i)\frac{im}{2\hbar}v_F^2}e^{-\frac{im}{\hbar}v_F(x_f-x_i)}\frac{1}{2\pi i}\frac{1}{\left(x_f-x_i+v_F(t_f-t_i)\right)} \tag{7.66}$$

This result is identical to the Green function obtained earlier viz. Eq. (7.48). Now

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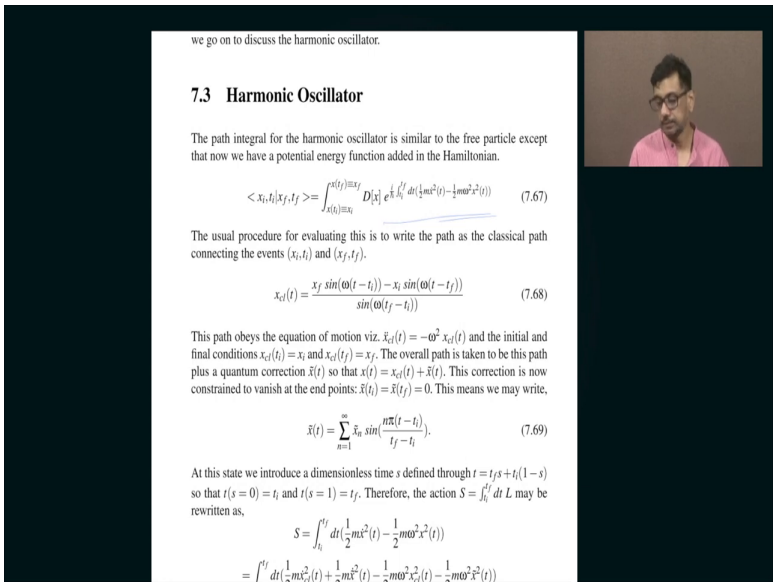

So, here for example, I have tried to evaluate that path integral by doing a bunch of things and I am getting back the answer that I got my traditional method. So, I do not want to bore you with this, but it is possible to struggle and evaluate this path integral by kind of doing something clever saddle point or whatever it is, then you can actually get back the original answer you got from more traditional methods.

So, I do not want to bore you with the details, but bottom line is that its possible to do this if you wish, but the main message is that if you are dealing with fermions be prepared to suffer some complications especially non local types of ideas will be quite common, ok. So, now the second more familiar example is that of a harmonic oscillator.

So, harmonic oscillator as usual so, same question what is the overlap between if the particle is at x i at t i what is the amplitude that you will end up at x f at t f. So, the answer is clearly same thing this path integral e raise to i by h bar into action, but then keep in mind the action will have two things one is the kinetic energy the other is potential energy.

So, as usual you will we will we will be doing the saddle I mean we will be finding the extremum the classical action and then we will expand around that and all that. So, I think I do not want to rush it. So, I am going to stop here and in the next class I will

continue with the harmonic oscillator ok, because I want to spend some time properly explaining it. It is pretty much very analogous to the free particle, it is just that I want to do it a little bit systematically, ok.

So, once we are done with that we will move to some other topic because this will give you enough practice in handling path integrals, ok. So, thanks for listening hope to see you in the next class.

Thank you.