# Dynamics of Classical and Quantum Fields: An Introduction

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## Quantum Systems
## Lecture - 26
## Perturbation theory

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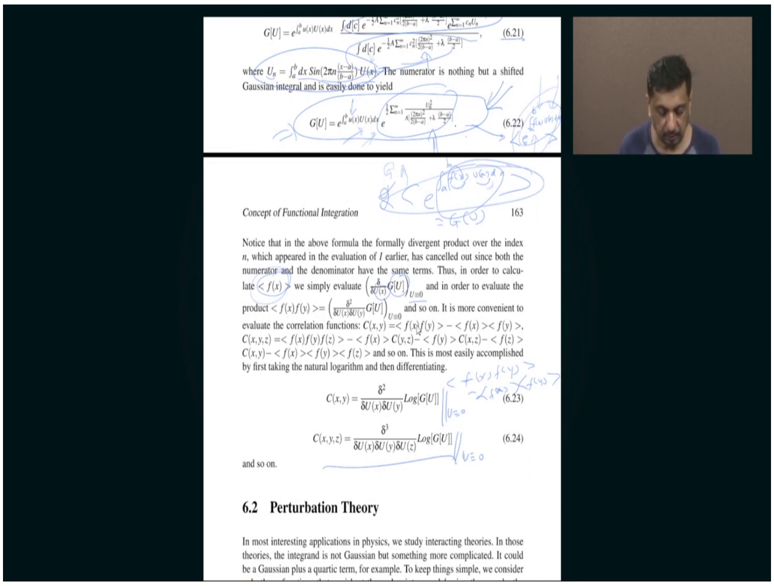


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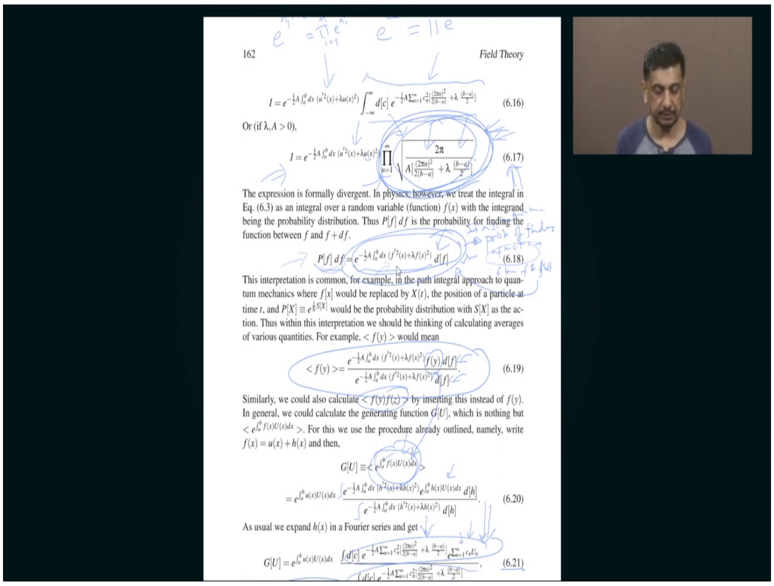

Ok. So, let me continue where I left off in the last lecture. So, if you recall I was trying to explain to you the meaning of integrating over function spaces. So, I was also explaining that the reason why we need to do this. The reason why we have to introduce this concept of integrating over spaces of functions is because in many applications in physics, we are usually called upon to integrate over all possible paths particle can take and you see the path is itself some function, it could be a function of time for example.

So, that the position as a function of time is a path. So, in you could be called upon to integrate over all possible paths or in stat make it could be possible configurations of you know the phase space of the system. So, basically, you are typically called upon to integrate over all possible paths that a particle can take. Now, so because the paths and so on are functions, you have to know how to integrate over functions. So, and not only that, the integral over paths are associated with some weight. So, that means, not all paths are equally probable.

Some paths can be more probable or less probable and typically, those the probability of given path being taken by the particle is typically given by some kind of a probability distribution like the 6.18. So, that means, the interpretation of the left hand side P f d f is that is the probability that the part particles path is between f and f plus d f; where, f is a path that it might take. So, in other words, different paths have different probabilities given by this. So, that is typically how the problem description is posed in physics.

So, of course, I have not reached; I am not motivated the physical aspect of this. So, you might think that this, where does this all come from because I have never come you, might say I have never come across this before. Because you know in classical mechanics, there is no question of probability of some path. Basically path is unique; I mean, it is determined by the equations of motion. So, mostly, I have in mind quantum mechanics, where the path taken by a particle is not fixed; that means, that different all paths are possible with, but some are more probable compared to the others.

Well, I have to specifically say the probability amplitude associated with a given path is different compared to other paths. So, the question is whatever it is that I should finally, be able to make sense out of integrating over paths with or without some weights typically with weights such as this; 6.18. So, with that sort of an application at the back of our mind, what I succeeded in doing was I have succeeded in proving to you that even though the you know.

So, so if there is some quantity depends on the path; obviously, that particular quantity is not well-defined because the paths can change mean that there is no fixed path that different paths all paths are possible. But some are more probable than others. So, because of that a quantity that depends on the path will not have a well-defined value because all different paths are equally possible; if not equally probable.

So, now, the question is so even though a quantity that depends on the path itself does not have a specific value, it is perfectly legitimate to talk of the average of that quantity. So, in other words, if there is a quantity that depends on a path and that path changes with different probabilities. Then, you can always speak of the average of that quantity as the paths change. So, that sort of. So, typically if that quantity that you are interested in is of this sort; where f is your path and U is some arbitrary fixed function of x.

So, now, if you average over all possible paths, where f is the path you average over all possible paths, the final answer has to be something which depends only on U and of course, the inputs that are there in the probability distribution itself. So, if you remember that I selected my probability distribution specifically to be the 6.18 which has the lambda and a and b and all that in it. So, now because that is the choice of the probability distribution and this is the choice of the quantity, whose average we want to calculate.

Of course, you might be wondering why am I interested in calculating the average of this particular quantity. So, I told you already I am going to refresh your memory very soon. But suppose you really wanted to calculate the average of this quantity or you follow a series of steps which involves basically rewriting it in terms of some shifted meaning you shift you shift your f to a value such that the end points are 0 rather than some y 1 and y 2. And then, because the end points are now both equal, you interpret that as indication of periodicity.

So, in other words, you realize that; that means, it is the that shifted function is periodic and then, you do a Fourier series and then, having done a Fourier series, now the coefficients now are discrete; but infinitely in number, but at least discrete. So, you can integrate over each one of those discrete coefficients in one by one and when you do that you end up with this result ok. So, and where U n is described. So, that U of x was some externally given function that you decided to insert in that definition of the quantity whose average you are interested in.

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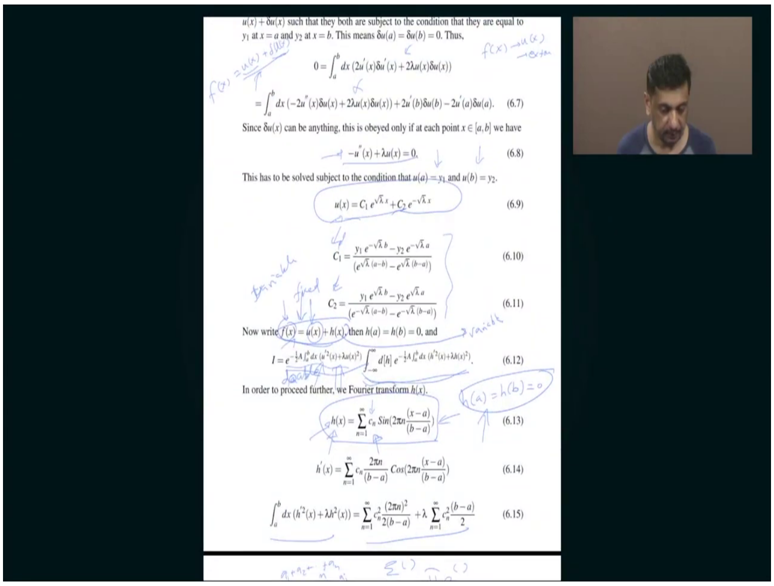


So, that is given, but then small u is not something randomly chosen, but it is basically the function which extremises your probability distribution. So, in other words, the most probable value of the path. So, U of x is the most probable value of the path and this is your answer to this average that you are looking for ok. So, we spent a considerable amount of effort and derived this. So, this is the unique exact answer to that question you. So, long as the probability distribution is nice and simple like this. In other words, it is a Gaussian ok.

So, I also told you that the reason why we are interested in the average of a quantity like this is because you can easily calculate quantity such as average of f of x by simply differentiating with respect to U once and bringing down an f of x and then, putting U equal to 0. So, if you want average of f of x into f of y, you differentiate with respect to U of x and then you differentiate with respect to U of y that will bring. So, each time you differentiate, you bring down an f. If you differentiate U of x, you bring down an f of x; you differentiate U of y, you bring down an f of y and then, you set U equal to 0.

So, you end up calculating the average of f of x into f of y. So, like that you can calculate average of f of x, average of f of x into f of y, average of f of x, f of y, f of z like that. So, you can calculate all those so called moments of the; so that random variable is the path that f itself is that random variable. So, you can calculate all the moments very conveniently and more specifically, if you first take the logarithm of that this G.

So, this is what we have been calling G, take the logarithm and differentiate, you get something which is typically called the correlation function. That means, it is so I will tell you what correlation function is suppose. So, it is the, so the two point correlation function corresponds to the difference between the average of f of x into f of y minus average of f of x into average of f of y.

So, in other words, what this two point correlation function tells you is how the average of f of x into f of y differs from. So, typically you see if f of x and f of y are unrelated random variables, finding the average of their product is same as finding the average of each of them separately and then, multiplying them. But then these two f of x into f of y, f of x is not I mean it is the same random path evaluated at x and f of y is the same random path evaluated at y. So, they are not unrelated.

So, what this difference; does is it tells you how much the average of f of x into f of y differs from the average of f of x times average of f of y? So, that is why it is called the correlation function. So, that means, it tells you to what extent f of x is correlated with f of y. So, you expect them to be correlated because basically they are just quantities evaluated at on the same path at different points; but because they are on the same path, you expect them to be correlated.

So, the degree of correlation may differ as you vary x and y. So, typically if x and y are far apart you expect less correlation compared to when they are closer together. But whatever it is this tells you the correlation function. So, it tells you how they differ. So, you can have a higher order correlation also. So, you can ask yourself how does this differ from various ways of pairing you know the fs and with each other. So, you can either like separately evaluate all of them or you pair two of them together, like that.

So, what this tells you is it kind of subtracts out all the different ways of pairing f and even after subtracting it out, it will still not be fully faithful representation of what that is. So, the difference between the actual average between f of x times f of y into f of z and the difference between that and the different ways of pairing the fs is basically the correlation function because it is kind of what is left over after you have subtracted out all the different ways in which you can pair it up and different ways in which you can pretend that they are all uncorrelated ok.

So, after subtracting them out, something will still remain and that whatever remains is basically called the correlation function. So, that is pretty much again a summary of what I explained earlier in a different way of saying it.

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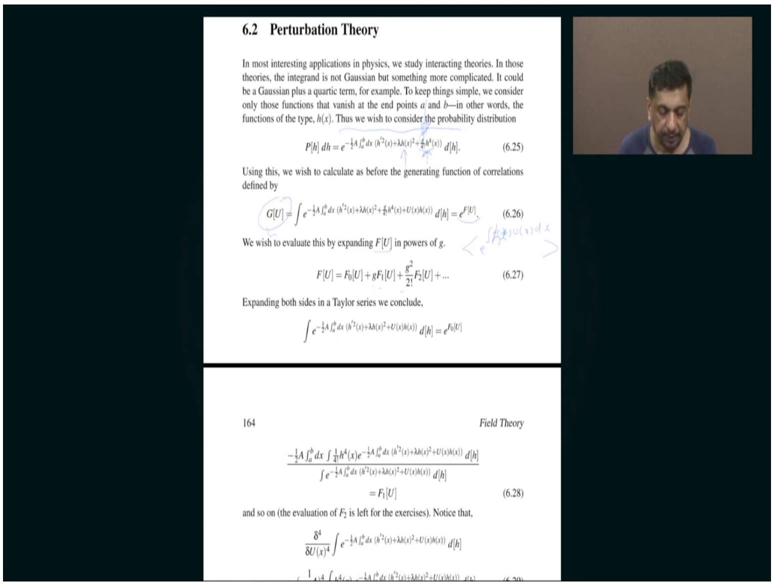

6.2 Perturbation Theory

In most interesting applications in physics, we study interacting theories. In those theories, the integrand is not Gaussian but something more complicated. It could be a Gaussian plus a quartic term, for example. To keep things simple, we consider only those functions that vanish at the end points $a$ and $b$—in other words, the functions of the type, $h(x)$. Thus we wish to consider the probability distribution

$$P[h]\,dh = e^{-\frac{1}{2}A\int_a^b dx\,(h'^2(x)+\lambda h(x)^2+\frac{g}{4!}h^4(x))}\,d[h]. \quad (6.25)$$

Using this, we wish to calculate as before the generating function of correlations defined by

$$G[U] = \int e^{-\frac{1}{2}A\int_a^b dx\,(h'^2(x)+\lambda h(x)^2+\frac{g}{4!}h^4(x)+U(x)h(x))}\,d[h] = e^{F[U]}. \quad (6.26)$$

We wish to evaluate this by expanding $F[U]$ in powers of $g$.

$$F[U] = F_0[U] + gF_1[U] + \frac{g^2}{2!}F_2[U] + ... \quad (6.27)$$

Expanding both sides in a Taylor series we conclude,

$$\int e^{-\frac{1}{2}A\int_a^b dx\,(h'^2(x)+\lambda h(x)^2+U(x)h(x))}\,d[h] = e^{F_0[U]}$$

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$$\frac{-\frac{1}{2}A\int_a^b dx\int\frac{1}{4!}h^4(x)e^{-\frac{1}{2}A\int_a^b dx\,(h'^2(x)+\lambda h(x)^2+U(x)h(x))}\,d[h]}{\int e^{-\frac{1}{2}A\int_a^b dx\,(h'^2(x)+\lambda h(x)^2+U(x)h(x))}\,d[h]} = F_1[U] \quad (6.28)$$

and so on (the evaluation of $F_2$ is left for the exercises). Notice that,

$$\frac{\delta^4}{\delta U(x)^4}\int e^{-\frac{1}{2}A\int_a^b dx\,(h'^2(x)+\lambda h(x)^2+U(x)h(x))}\,d[h]$$

So, now, I am going to tell you some slightly more advanced extension of whatever I was talking about because this will be useful later on. So, remember that in quantum mechanics, you typically deal with perturbations; that means, that usually you will have a model which is exactly solvable and there will be some extra terms which cannot which will spoil exact solvability. But does not mean that the problem cannot be tackled, it just means that you have to expand in powers of that extra term. So, that is called perturbation theory.

So, even here in this context you can imagine perturbation theory. For example, if your probability distribution is not exactly Gaussian, but it has this additional term which is quartic which is fourth power. So, you might think why did I choose a fourth power, I will allow you to think about it. Because third power will create some problems because second power is already here. So, the next higher power is third power which you will have to explain to me in the exercises why that is not acceptable.

So, the after the third power, it is just the fourth power. So, that is the perturbation that I introduce into the problem and now, I want to again ask myself the same question what is G of U. That means, what is the average of this quantity f of x U of x d x or in this particular case, I have chosen instead of f; I have chosen h because I do not want the distraction of end points being y 1 and y 2. So, I have chosen y 1 equals y 2 equals 0. So, in which case that f becomes h.

So, you remember what the distinction was between f and h; basically, f is the that path where the end points are y 1, y 2; but h is the path where you subtract out the most probable path and you end up with you know 0 0 as the end points. So, I mean already there is this complication of perturbation. So, I do not want those well understood distractions to complicate matters further. So, I start with h instead of f ok. So, the end points are now 0. So, now, the question is what I want to do is I want to treat this problem perturbatively.

So, I write G as e raise to F and now, I expand this F in powers of this the coefficient this the cups called coupling constant. So, I expand in powers of g which basically tells you how strong the non-Gaussian nature of that integrand is. So, bottom line is that if I first set g equal to 0, I end up with this and this is easy to do right because we have done this earlier and F of 1 is basically when you expand in powers of. So, I will allow you to think about how. So, you just insert all you do is insert this expression here because this is F of U.

So, instead of F of U you insert this expansion ok and then, basically you expand both sides in powers of g and you compare both sides. So, when g, so the g to the power 0 term is this one. So, the g to the power 1 term will give you this and so, the higher order terms are similarly more complicated, I will allow you to calculate it yourself.

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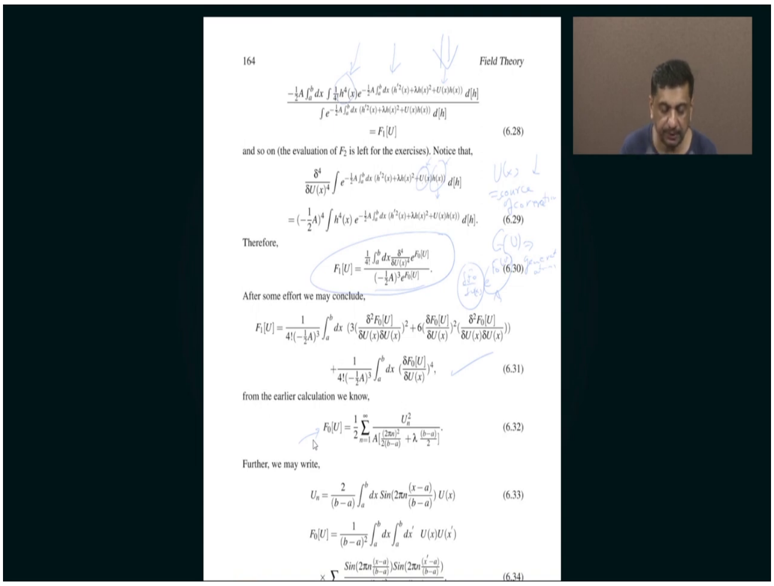


But bottom line is that the first order term is already somewhat challenging and we will have to do it. So, you see the point is that the first order term this F 1. So, you see this F of U is basically the whatever it was when that quartic perturbation was absent. This is the unperturbed F, which we already know from the earlier section what that is. But all now we have to calculate this first order correction to F so that happens to be this one ok.

So, now you see it is the same as calculating the average of h to the power 4 in terms of including this. So, called source term. So, this U is in sometimes called as source term because you are introducing the source of correlations. So, what capital U of x does is it introduces a source for correlations and it. So, in other words, it that source generates the correlation function. So, this U of x is a source, it is source of correlations.

Because it allows you to extract the correlations through it is introduction and G of U is basically the generating function because by suitably differentiating with respect to the source, you are generating all the correlations ok. So, that is what that is. So, now, because h to the power 4 may be written as the fourth derivative of the source because you see if you take the fourth derivative with respect to U.

So, each time, you take a derivative, you bring down an h. So, 4 times you bring down 4 hs which is what you need here. Because that is what it is the at that level. So, you see this F 1 has the simple form namely this ok. So, you can simply differentiate this 4 times. So, if you take a you know you just go ahead and differentiate; each time you differentiate, so if you differentiate first time, you will get d f 0 d u x.

Again, times e FF 0, then again if you differentiate once more, you have to differentiate this first, then times this, then again this first times that. So, you will get this whole squared plus second derivative like that. So, if you do that 4 times this is what you end up with and then, you divide by this quantity. So, when you do that you get this result ok and from our earlier calculation, we know what that F of U is in terms of the Fourier series that we have been successful in writing.

So, when you go ahead and so, you can re substitute the coefficient U of n in terms of the original Us because it is the is U of ns are related to U of Us through a inverse transform and when you substitute that you get this expression of F of U in terms of the source in real space ok in terms of the source U of x and U of x dash. So, it involves, so basically this unperturbed function.

So, basically, what is this F of 0 is the logarithm of the generating function for the unperturbed case. So, that function has the form, the source at x times source at x dash times some function which depends on x and x dash and that is what I have called W ok.

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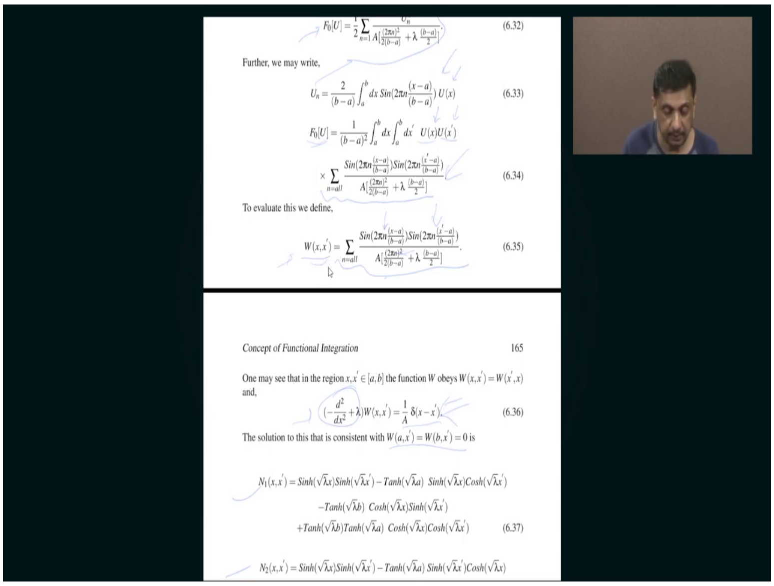


So, now this W happens to so now, it is important for us to evaluate this W because you have to evaluate this summation because it is a well well-defined question you know there is an n there; there is an n there and there is an n here. So, it is a perfectly valid question that if you sum over all possible ns starting from minus to plus infinity, what is the answer?

So, this I mean you one way of doing this is to simply insert this in some symbolic algebra package like Mathematica or MATLAB and allow the computer to tell you the answer, which is of course, what is typically done these days; nobody ever takes pride in doing tedious calculations anymore because that is what computers are there for. So, I strongly encourage you to learn symbolic algebra, computer packages like Mathematica or MATLAB. My favourite is Mathematica because I find it especially useful for tedious physics related calculations.

But on the other hand, if you are more old-fashioned and you do not have access to software rather than just throwing your hands up and say I cannot do it, because I do not have software you should try to learn how the software itself does it. After all, you know how does the software do it? It is does it because it knows a systematic method invented by mathematicians several hundred years ago because after all this calculus is all very old subject right I mean.

So, bottom line is that this is how the ancients would do it and this is how your software would also do it. It is just that we would not; if we had software, we would make the software do it and how would the software do it, the way the ancients did it and the way the ancients did it is that they convert the summation to a differential equation. So, in other words, if you operate on W, this operator you will see that it actually becomes a Dirac delta function and not only that it is subject to these boundary conditions that at a and b the Ws are 0 which is exactly what we expect.

So, now, so you can go ahead and solve this and the W that is consistent with these constraints. So, when x and x dash are you know between a and b and so that is what that is ok. So, and this N 1 and N 2 are; so, you can explicitly write down the answer for W. So, this was your F of; so, in other words, this F of 0 which is the unperturbed value for the logarithm of the generating function, now can be expressed in terms of source at x times source at x dash times W x x dash integrated over all x and x dash, so in other words, this ok.

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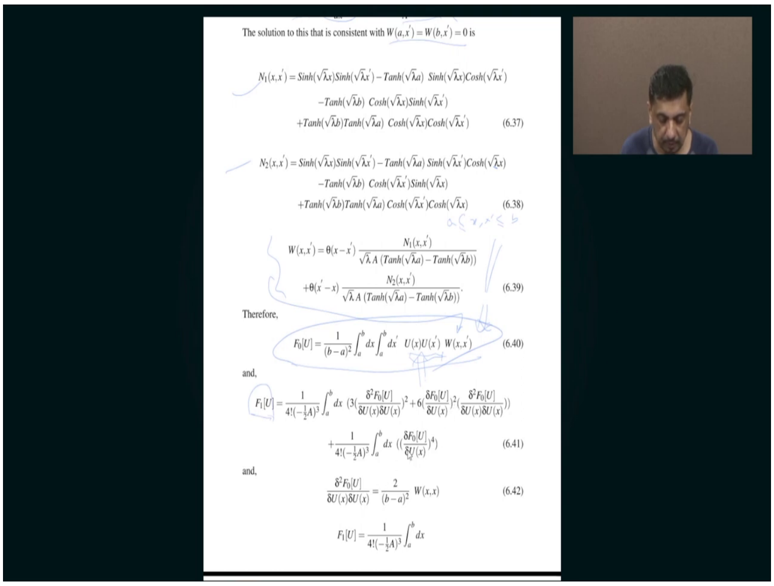


So, now that you know what W is, you can go ahead and insert that here and you can evaluate that; but that is assuming you know what the sources are. You cannot really proceed any further because the sources are very general U of x. But you can leave it like this because that is, this is as far as we are interested in going anyway. So, the point is that we got the W that is the key. So, now, we go ahead and find the first order correction to the basically what is called the well, you it is just the logarithm of the generating function yeah.

So, the for the first order correction to the logarithm, the generating function. In stat (Refer Time: 24:04), it would be called free energy you know if you think of g as your partition function, F would be the free energy. You do not have to call it that because we are not talking about physics now, we are talking about I mean general mathematics with probability distributions that are functionals alright. So, now, all we have to do is go ahead and calculate this F 1. So, the F 1 simply means that you just have to differentiate with respect to x x dash whatever. So, remember what F 1 is; it is this.

So, now, that you know what F 0 is you will simply perform the required differentiation and then finally, integrate over x ok.

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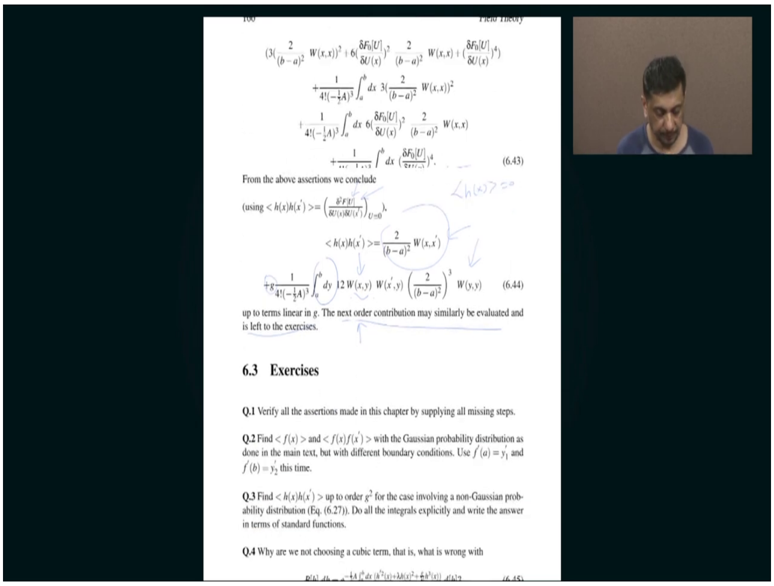


So, when you do all that you end up getting, yeah I have skipped a step. So, you will end up getting some horrendous expression; but finally, when you try to calculate what you are interested in which is the second moment that is the correlation function because remember that h of x on an average is 0 because you know the end points are 0. So, it is see on an average, it is going to be 0 because it starts at 0, ends at 0.

So, on an average h is 0. Since the end points are 0, the average is 0. But however, the average of h x into x h x dash will not be 0 because that is the so called correlation function and the correlation function is simply given by this and when you evaluate this, it comes out to be this. So, in other words, if there is no perturbation, the correlation function is just W times some constant. But if there is a Non-Gaussian term namely g, so that correlation function acquires a first order correction of this sort.

So, now, somebody has to go ahead and do this integral over y which I will leave you to the exercises ok. So, it says well, the next order is left to; even this itself is left to the exercises. Because I have to integrate over y because W is known. So, to see what W is. W is explicitly this, where N 1 is this, N 2 is that. So, I know what W is. So, this is do able ok. So, bottom line is that is what it is. So, this whole activity, this whole section was meant to introduce you to integrating over function spaces and it tries to motivate the integration over function spaces by pointing out that.

The reason why we do that is because we interpret the function as some kind of a path and in quantum mechanics, all paths are possible although some are more probable than others. So, that means, that you are forced to introduce a probability distribution associated with each path and if those probability distributions are Gaussian, then you can go ahead and perform certain calculations exactly, you can find the average of the path or the correlation between one path and the neighbouring path. Yeah, so that sort of thing.

So, what I did was explain to you how to think of these concepts, introduce them, evaluate them and so on. But then, I also pointed out that typically in applications, the integrands are not Gaussian; especially, when you have interaction with mean particles. It is typically more complicated than a Gaussian. So, the simplest non-Gaussian is one where there is a quartic term. So, that quartic term can be analysed through perturbation theory and I evaluated the first order term and I invited you to generalize this to higher orders ok.

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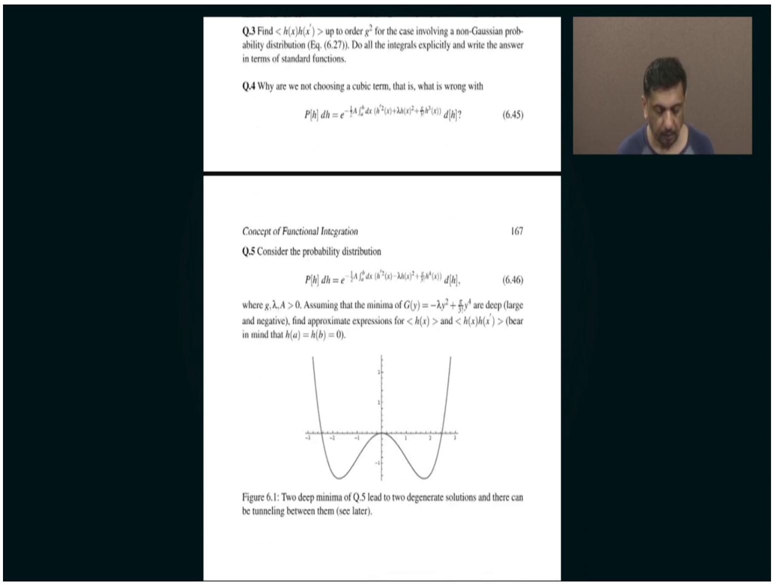


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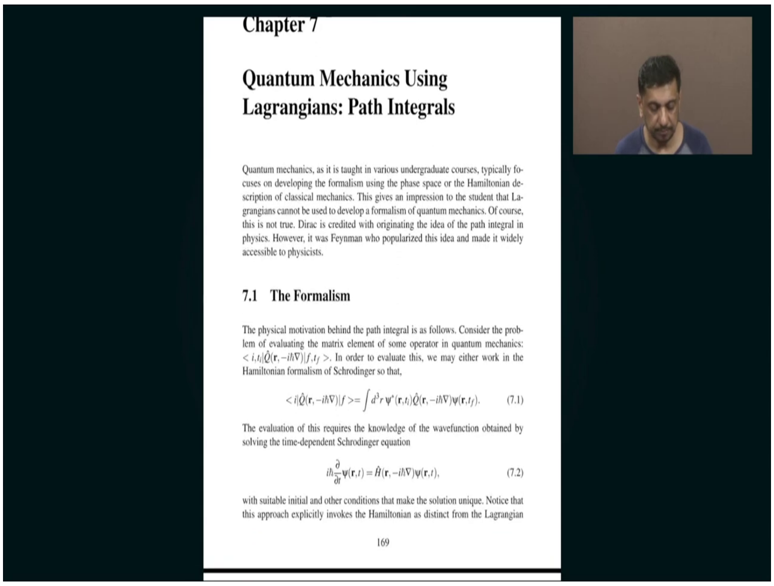


So, that sort of completes my introductory discussion of integrating over function spaces. So, now, I come to the physics motivation of that. So, the earlier section was fully mathematical. So, in other words, I introduced the mathematical notion of integrating over function spaces. So, it has no relevance to physics, until I make such a connection right. So, those of you who are not particularly mathematically-minded will probably have got impatient by now. So, many of you will be wondering why am I discussing so much of tedious mathematics, where is the physics ok.

So, usually, many physicists learn math as you know as a last resort, they are they operate on a need to know basis. So, in other words, they will learn only that maths that is necessary to answer the physics question. So, that is neither good nor bad; but it is certainly how most of us operate. So, because of that it is important that I explain to you what the physics behind these mathematical subjects are. So, namely, the integrating over function spaces. So, in order for me to explain the physics behind this, I have to introduce some formalism ok.

So, I will just briefly mention the starting idea behind this you know this whole idea of path integral approach to quantum mechanics, which is basically motivates the need for integrating over function spaces. So, I am going to just start because time is running out. So, in the next lecture, I will begin and properly explain the whole idea behind path integral approach to quantum mechanics. So, you see this let me read off the starting paragraph of this chapter.

So, it says 'Quantum mechanics, as it is taught in various undergraduate courses, typically focuses on developing the formalism using a phase space or Hamiltonian description of classical mechanics. So, this gives an impression to the student that Lagrangian's cannot be used to develop a formalism of quantum mechanics. Of course, this is not true. So, it is Dirac who is credited with originating the idea of the path integral in physics. However, it was Feynman who popularized this idea and make it made it widely accessible to physicists'.

So, that is the whole idea you know the idea that you can do quantum mechanics using Lagrangian's was already known to the pioneers of quantum mechanics like Dirac; but the next generation of physicists like Feynman, who came after the pioneers, they are the ones who popularized this idea of path integrals. So, I will have to explain to you where exactly this idea of integrating over paths or integrating over function spaces enters into the subject because the way you study quantum mechanics, it is all about operators, expectation values, matrix elements and so on.

So, these are the types of ideas we are familiar with in quantum mechanics. So, somehow I have to introduce, I have to start with those familiar concepts; namely, operators, matrix elements, expectation values and then, I have to manipulate those expressions until I naturally encounter integrating over function spaces or integrating over paths ok. So, that is what I am going to do in the next lecture.

So, I will allow you to stare at this slide and I hope you will study ahead and if you have access to the textbook, you should definitely study ahead so that you will be it will be easier for you to follow the lectures also. Yeah, so that is important. So, when you whenever you are studying or whenever you are trying to learn a difficult subject, studying ahead helps a lot. So, in other words, you study; so, if the teacher is teaching some chapter now, you should have studied that chapter three days ago on your own.

Of course, you would not have understood much. But whatever you little you understand will be helpful in understanding the actual lecture that comes three days later. So, I hope you will do that and I will meet you for the next class, where I will explain to you very properly how quantum mechanics can be studied using Lagrangian's and that naturally introduces the concept of the path integral or integral over paths or in other words, integral over spaces of functions ok.

Thank you.