Dynamics of Classical and Quantum Fields: An Introduction

Prof. Girish S. Setlur
Department of Physics
Indian Institute of Technology, Guwahati

Quantum Systems
Lecture - 24
Functional Integration - I

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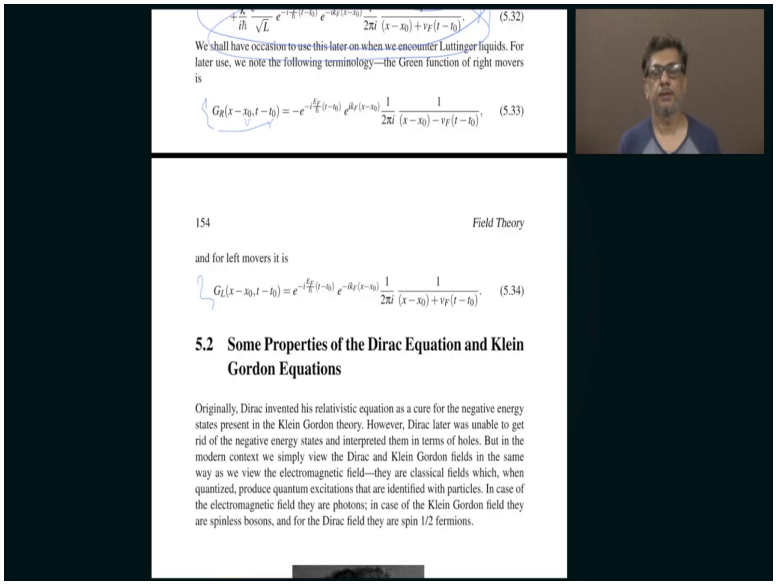

Ok, so let us continue where we left off. So, if you recall in the last class I was trying to explain to you that there is a concept called Green's function of any system.

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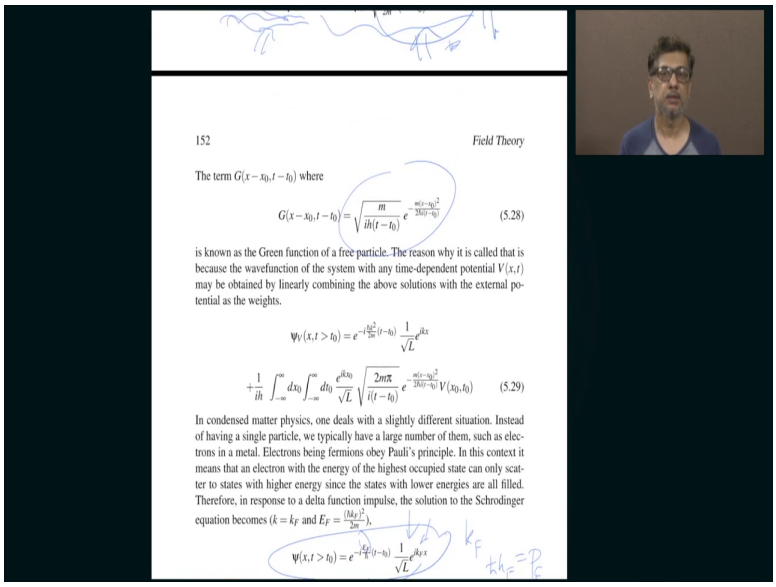

So, specifically I was thinking I had in mind a collection of fermions say in one dimension. So, where there are lots of electrons and because of Pauli Exclusion Principle you are forced to keep filling the higher states until you reach the maximum allowed state, corresponding to how many fermions you have with you and that is called the Fermi level.

So, now the thing is that if on such a system you apply a disturbance. So, the whole idea is that you have a system of fermion, so it is called Fermi gas. So, basically you have a collection of n fermions in a box of length from 0 to l and because of that you see you know that the energy levels are quantized. So, each I mean if there is just one quantum particle it is energy is h bar square pi squared n squared by 2 m l squared. So now, n is goes from one up to infinity. So, the point is that if you have say large number of fermions then you cannot put all of them in n equal to 1.

So, you have to put so if for example, no spin imagine that you have only up spin electrons or fermions there. So, in which case you can only put 1 fermion for 1 value of n, so if n equals one it can maximum accommodate 1 fermion. So, then after that you have to put in necessarily the next one in n equal to 2. So, like that if you have lots of electrons if you have 100 electrons you have to reach up to n equal to 100.

So, you have to reach up to however many electrons or fermions you have with you. So, that will lead to the concept of Fermi energy or the Fermi level, which means that basically it is represents the energy of the highest occupied state. So, now you have imagine you have such a system and you apply a disturbance. So, that disturbance is a localized; that means, that it is a delta function in space and delta function in time also. So, that is what I was trying to impress upon you.

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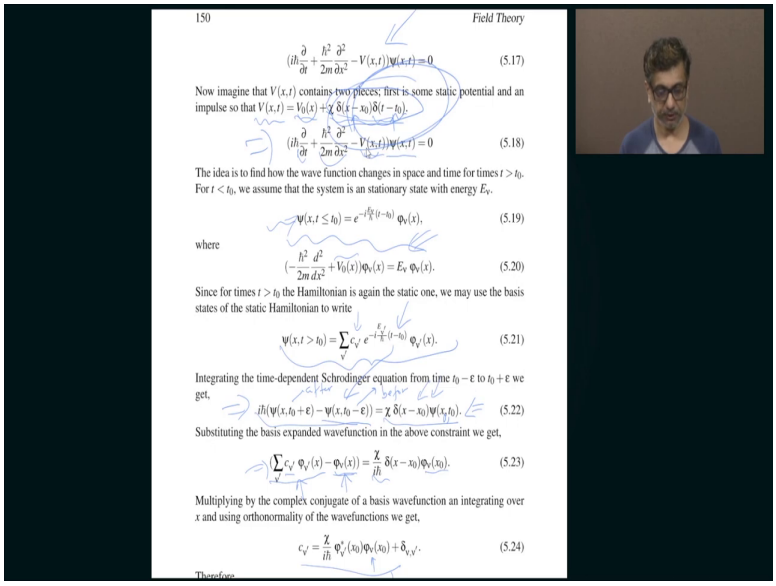

So, that is basically the problem we are studying. So, you have a Fermi gas and on top of that you apply a disturbance of this kind ok. So, that is delta function in space and delta function in time. So, it is a highly localized that means you apply it at x equal to x naught and then at time t equal to t naught.

So, then you ask yourself what will be the wave function of the system subsequent to this disturbance. So that means, it earlier it was a stationary state. So firstly, it was basically it is not really a wave function of one particle, now it is a wave function of many particles right. So now, the idea is that you are asking the easiest electron to disturb is basically the one whose energy is basically the Fermi energy.

Because it costs very little energy to excite that electron, because you see after all the states above that Fermi energy are unfilled. So, it is it takes infinitesimally small energy

to excite that electron. So, what we are trying to ask and answer in this earlier lecture was that what will be the suppose you focus only on that electron, which has energy close to the Fermi energy. So, what will how will the wave function of that look like as a function of time?

So, it is a simplistic analysis because it is not particularly accurate, because you see it I have assumed that the rest of the electrons are going to be completely inert which is not actually true. So, what I have done is I have just singled out that electron whose energy is the Fermi energy and pretended that only that will respond to the disturbance.

So, this is just to you know illustrate the concept of right mover and left mover I did not, there is no implication that it is exceedingly accurate or anything. So, if you allow me that latitude then you will see that basically the wave function. So now it makes sense to talk of the wave function of that particular single electron whose energy is the Fermi energy. Now when it is subjected to a disturbance of that kind then you see it is wave function will not will cease to be stationary.

So, in other words it is if you square the modulus of the square of the wave function is independent of time and it is stationary that is the case when there was no such disturbance. But now when there is such a disturbance we do not expect the wave function to be stationary, in other words the mod squared of the wave function of that particular electron that we have chosen to single out will now evolve with time. Now the whole purpose of the this exercise was to illustrate the concept of the Green's function of that electron.

So, basically now in this case the Green's function in this particular way of looking at things is synonymous with the wave function of that particular electron, after the disturbance is switched on. So, in other words subsequent to the disturbance. So, subsequent to the disturbance that particular electrons wave function will change in a very peculiar way and that wave function is basically called the Green's function of the system ok.

So, the wave calculate that is through of course utilizing. So, I told you the mathematical aspects of the mathematical steps in the earlier lecture. So, I am just compensating for

that by explaining the physical content a little bit better now. So, the whole point is that you solve for the time dependent Schrodinger equation.

So, you write down the time dependent Schrodinger equation, because you see disturbance is being switched on and switched off abruptly, so then it is a time dependent problem. So, before the disturbance was turned on it was in a stationary state. So, you end up solving the time dependent Schrodinger to find out how it looks like after it switched on.

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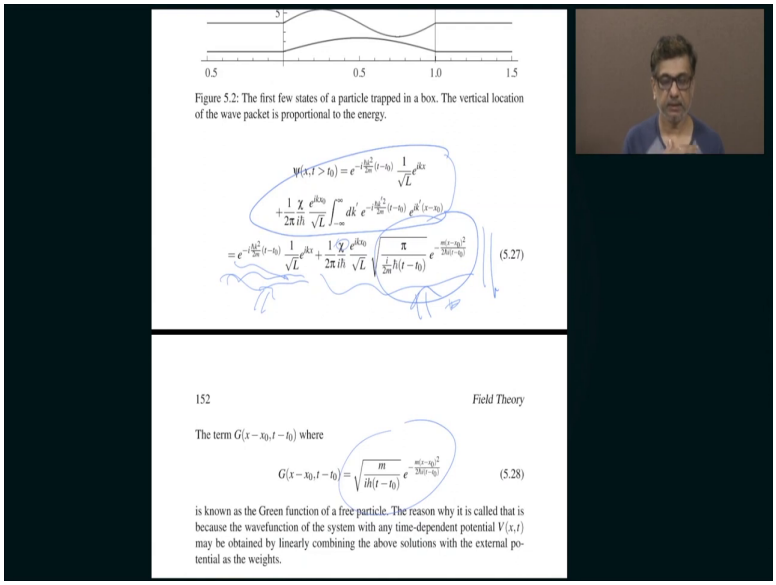

So, after putting in some effort you will find the answer to that question. So, assuming for example you are interested in.

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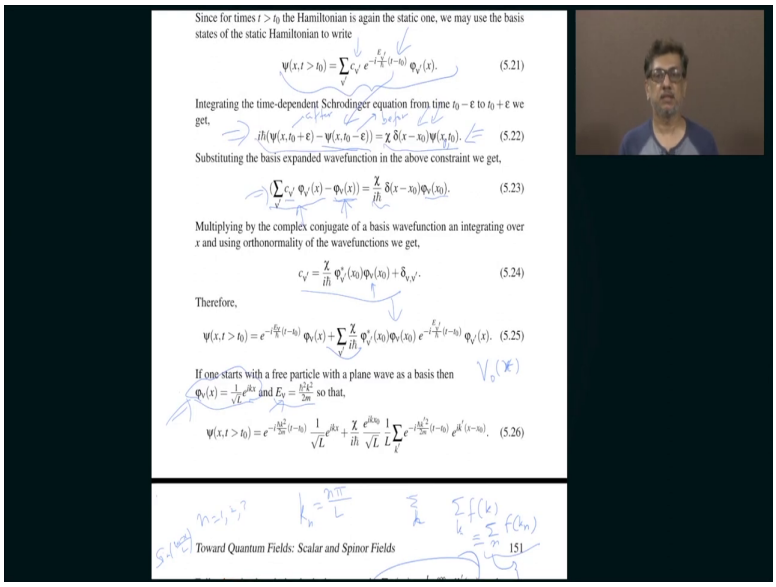

So, the ok one of the important ingredients in the solution is the assumption, that the electron that you know that responds to this disturbance, which is whose energy was initially close to the Fermi energy will now only scatter. Because now it is going to start scattering to other states, because now it is no longer in a stationary state will scatter to other states. So, the implication is that it will scatter to states that are not occupied. So, in other words it will ignore states that are less than the Fermi energy it will only scattered to states higher than the Fermi energy.

And then there is this further assumption that the scattering is sufficiently weak, in the sense that it will not scatter very far from it is initial energy. So, there is a bandwidth. So, there is a bandwidth whose in other words there is a sliver of energy close to the Fermi energy and greater than the Fermi energy within which it will scatter. So, if you make all those assumptions, you will find that the wave function of that particular electron subsequent to the disturbance is going to look like this.

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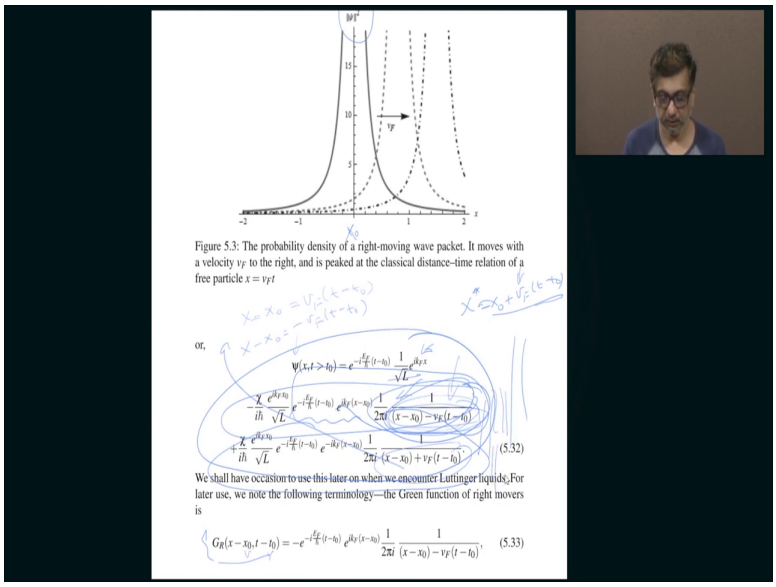

So, it is going to look like this, so I am sorry for the juvenile markings here, but that is from last lecture. So, point is that what this represents basically is that this is called a right mover and the reason why it is called a right mover is that you know this part of the wave function remains constant if this remains constant.

So, in other words specifically if this is close to 0, so if this denominator is close to 0; so, that is the dependence on x and t which maximizes the probability for finding the electron. So, what does that tell you? So, it basically tells you that the electron is more most likely to be found whenever x minus x naught.

So, it is likely to be found at x where x minus x naught is equal to t minus t naught times V F. So, what is this is basically the trajectory of a free classical particle moving with speed V F. So, you see so quantum mechanically, so this was a quantum mechanical problem because we had we are talking in terms of wave functions fermions and so on. So, there is nothing classical about that.

And yet if you ask a very specific question namely what is the most probable location of the electron after time t, the answer is exactly where you are likely to find it if the exactly where you are going to find it if the electron or if the fermion were actually

classical. So, the classical location of the particle is also the most probable location of the quantum particle.

So, if the particle bear quantum mechanically the most probable location is the location where you would classically find it for certain ok. So, that would correspond to a right mover because it corresponds to a particle moving with velocity or speed plus V F.

So, by contrast this corresponds to a left mover because when the denominator is 0, so this will the this will correspond to x minus x naught being equal to minus V F t minus t naught. So, by implication this means the particle is moving with speed minus V F and therefore, it is called the left over. And these parts of the wave function are basically called the Green's function of the right mover and this is called the Green's function of the left mover ok.

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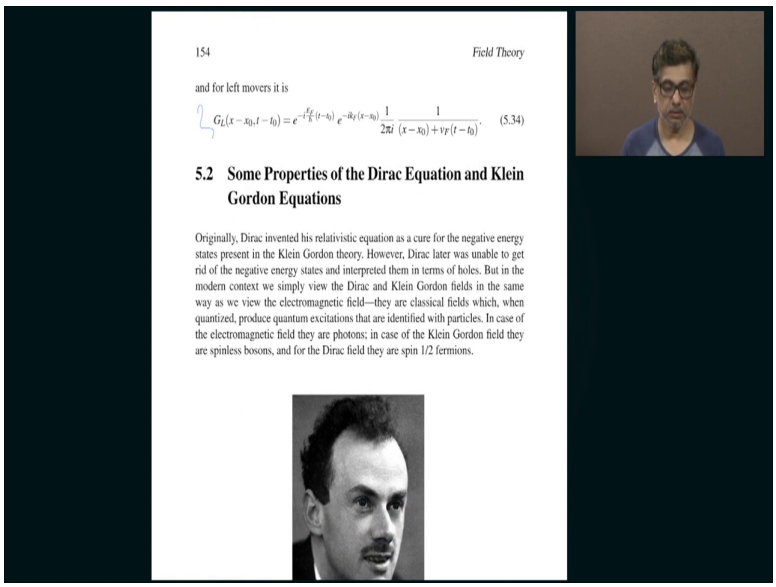

So, Green's function is basically the wave function subsequent to a localized disturbance in space and time ok. So, that pretty much completes my description of right movers and it basically it motivates the introduction of these concepts, which are going to be later on very important when we study a very interesting system called a Luttinger Liquid. So, we will come to that a little later alright.

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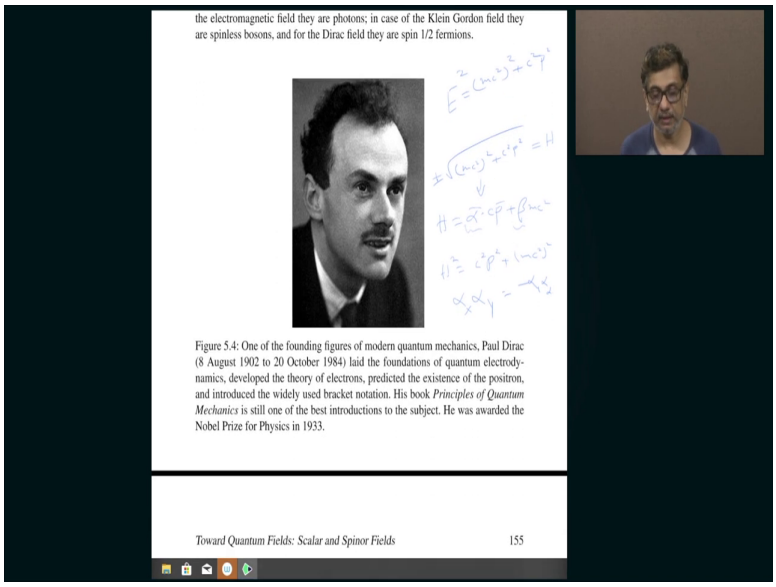

So now, let us let me switch gears and discuss some properties of the relativistic theory of quantum mechanics, so in other words relativistic quantum mechanics. So, that is something that traditionally quantum field theory was born as a result of you know the need to reinterpret certain aspects of relativistic quantum mechanics. So, you will see that if you follow the historical timeline where people tried to like Dirac and Klein and Gordon.

And they were trying to develop the relativistic version of. In fact, Schrodinger himself developed first before writing down his famous non relativistic Schrodinger equation, he had actually done the more ambitious thing of writing down the relativistic Schrodinger equation first. Because you know he said like let us start with the correct theory which is special relativity and then he found that when you. So, basically his idea was he wanted to explain the results that Somerfield had obtained by examining the fine structure of the energy levels of hydrogen.

So, he wanted to understand the fine structure of hydrogen the spectral lines. Many people knew by that time that it had something to do with special relativity. So, Schrodinger actually wrote down the relativistic version of the Schrodinger equation which we now call Klein Gordon equation and he introduced the coulomb potential between the electron and the proton and then he calculated the energy levels. And found

that it really does not match with the with what Somerfield had inferred from the observations. So, he was Schrodinger was very disappointed.

And there is this story that he retreated into a log cabin and he was very depressed and after a few weeks he consoled himself and said that he is going to write up the non relativistic version and then publish it. So, that is pretty much what he did. So, he wrote down the non-relativistic version and said that. So, this should suffice as a starting point. So, that is what we now call Schrodinger equation the non-relativistic one. But then you know of course, that did not deter the other physicists who were around at that time to pursue the relativistic angle more vigorously.

Among them Dirac was the most prominent one and in one of the conferences I think it was one of the Solvay conferences. So, somebody asked him what he was doing what was he thinking about research wise and he said he is trying to find the square root of something. So, he gave a very cryptic answer and that is that cryptic answer is something which we now call Dirac's theory of the electron.

So, let me spend some time so that was a historical preamble. So, let me spend some time trying to explain to you what all these things are and there is some understanding that these are parts of these are actually prerequisites for this course because, I simply will not be able to do full justice to Dirac's theory of the electron or anybody else's relativistic approaches. So, it is just I mean I am just going to breeze through all those concepts ok.

So, the bottom line is that what Dirac showed was that you see the energy of a relativistic particle was actually like this. Whereas, if you wanted to do quantum mechanics you needed a Hamiltonian, so if you take the Hamiltonian as the square root of this you will end up getting 2 signs so and that is something you do not find in non-relative.

So, this is a relativistic Hamiltonian of a free particle. So, if you have a free particle it looks like you get 2 signs. So, of course, you can decide let me take only the positive sign because, how can well if you choose the negative sign that implies that the vacuum is unstable. So, like most sensible people Dirac also initially ignored the negative sign and tried to proceed and he was not particularly successful.

And then he decided that there might be a better way and then he said that look I am going to see if I can get rid of the square root and simply find a way of writing this as alpha dot cp plus beta m c squared, where I will try to find alpha and beta which make sure that h squared at least is whatever I expected to be namely this. But the end result of that was that this alphas and betas are not numbers, but they are actually you know.

So, there are 3 alphas alpha x alpha y alpha z because it is alpha dot p when p itself has p x p y p z. So, you have 3 alphas and 1 beta, so the total of 4 objects which have to be determined and then what Dirac found that these are not really numbers. Because they have to anti commute in other words alpha x into alpha y should be equal to minus alpha y into alpha x etcetera. And alpha and beta should also anti commute and alpha squared and beta squared should be 1. So, he tried to find out. So, he then immediately guessed that candidate representation for such alphas are basically matrices.

So, you see if they are simple numbers they will not they will certainly commute, but if you want something that do not commute the simplest objects that do not commute which do not have any more details in inside them are basically matrices. So, he initially tried 2 by 2 it did not work 3 by 3 it did not work then 4 by 4 it worked, so which is why we call them Dirac matrices. So, alpha x alpha y alpha z and beta are called Dirac matrices and so I expect my listeners to know those things, I mean I am not going to spend time explaining Dirac's theory of the electron at all.

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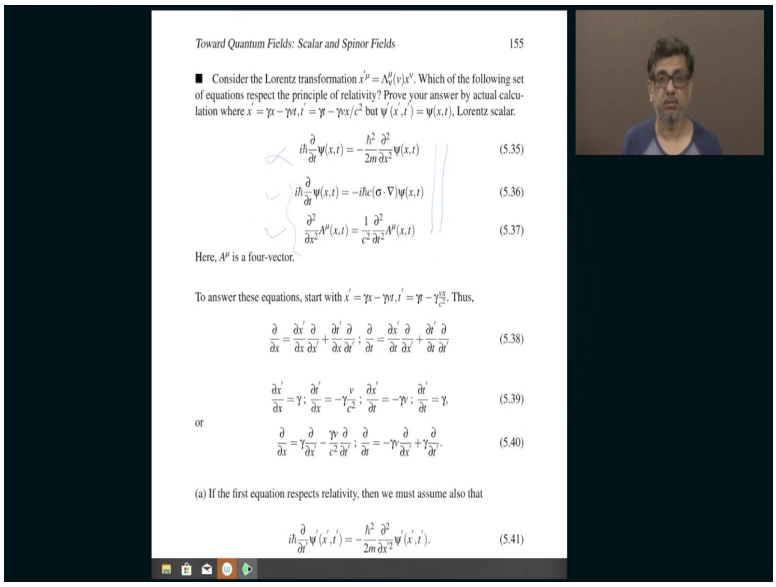

Toward Quantum Fields: Scalar and Spinor Fields 155

■ Consider the Lorentz transformation $x'^{\mu} = \Lambda^{\mu}_{\nu}(v)x^{\nu}$. Which of the following set of equations respect the principle of relativity? Prove your answer by actual calculation where $x' = \gamma x - \gamma v t, t' = \gamma t - \gamma v x/c^2$ but $\psi'(x', t') = \psi(x, t)$, Lorentz scalar.

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{5.35}$$

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -ihc(\sigma\cdot\nabla)\psi(x,t) \tag{5.36}$$

$$\frac{\partial^2}{\partial x^2}A^{\mu}(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A^{\mu}(x,t) \tag{5.37}$$

Here, A^{μ} is a four-vector.

To answer these equations, start with $x' = \gamma x - \gamma v t, t' = \gamma t - \gamma\frac{vx}{c^2}$. Thus,

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}; \frac{\partial}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} \tag{5.38}$$

$$\frac{\partial x'}{\partial x} = \gamma; \frac{\partial t'}{\partial x} = -\gamma\frac{v}{c^2}; \frac{\partial x'}{\partial t} = -\gamma v; \frac{\partial t'}{\partial t} = \gamma, \tag{5.39}$$

or

$$\frac{\partial}{\partial x} = \gamma\frac{\partial}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial}{\partial t'}; \frac{\partial}{\partial t} = -\gamma v\frac{\partial}{\partial x'} + \gamma\frac{\partial}{\partial t'}. \tag{5.40}$$

(a) If the first equation respects relativity, then we must assume also that

$$i\hbar\frac{\partial}{\partial t'}\psi'(x',t') = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}\psi'(x',t'). \tag{5.41}$$

But so now so with that exceedingly short preamble let me go to some examples ok. So, what I have tried to do here is. I am going to try to see if so the this is the question. So, which of these are consistent with Lorentz transformation? So that means, that if imagine that each of this is valid in a certain reference frame. So, if I decide to do a Lorentz boost.

So, say in the x direction which of these equations will look the same with you know like by simply replacing the psi with psi prime x with x prime t t prime and is there a simple way in which you can relate psi prime and psi. So, you will find that the answer to that very predictably is only these 2 and this will not be consistent with special relativity.

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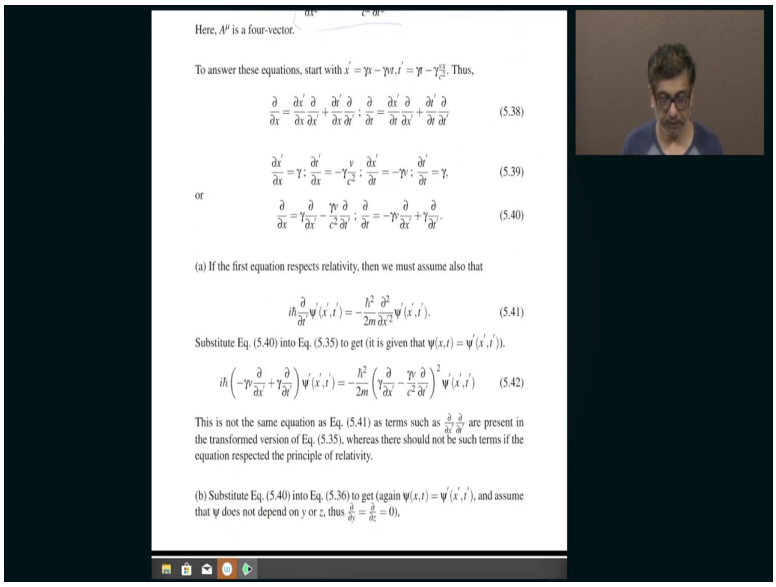

And I have spent some effort trying to prove that you and I think you can go through these steps yourself, because again I do not want to spend too much time discussing algebraic manipulations. Because that is part of the practices that I want yours want my listeners to themselves perform. So that means that I want them to sit down and work out all these details and if they get stuck they can always go back to this book and so speaking of the book.

So, if you recall in the beginning I advertise that my book is available on Amazon, but till now it was just the international edition which was forbidding expensive and it was a hard bound. But now my publishers tell me that very soon end of this month that is the end of June 2022 there is going to be a Indian edition out and whenever I get the details of where you can purchase it from I will let you know until then you can watch out for it. So, it is going to be released end of June 2022, which is the this is exactly the same book, but at a fraction of the original cost ok.

Alright, so now what I showed in the what I have been able to show is basically through this by working out these problems in detail is that this is not consistent the first one is not consistent with special relativity. After all what is the first one basically the non-relativistic time dependent Schrodinger equation that is 5.35 that is not consistent with special relativity the second one is and you might be wondering what is the second one.

So, it is basically it represents it is of current or meaning it is of interest since maybe 2004 when a material called graphene was stumbled upon. And that is basically a 2 dimensional material and the electrons the quasi particles the charge carriers in that material obey this sort of equation ok. So, that is the reason why there is of interest because these are of 2 by 2 pole matrices. So, that is that angle and now the third one is of course, the familiar electromagnetic wave equation for the vector potential the 4 vector potential.

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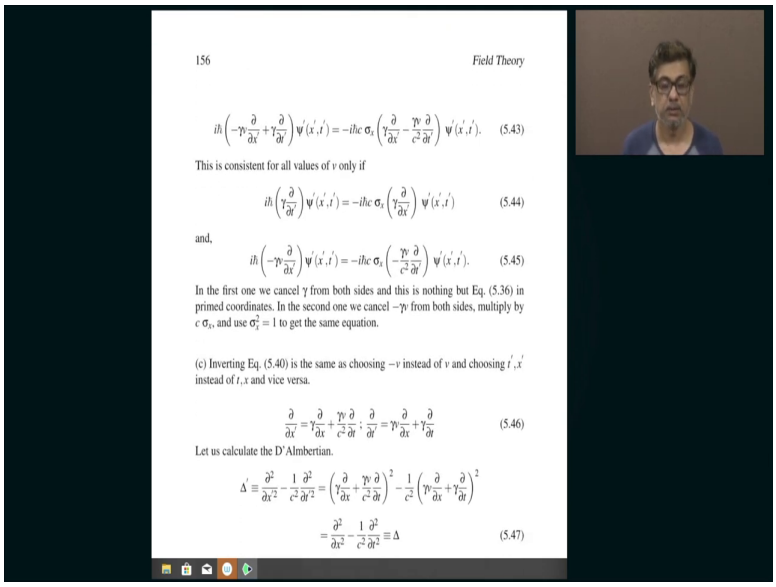

So, that is what I have done here in great detail.

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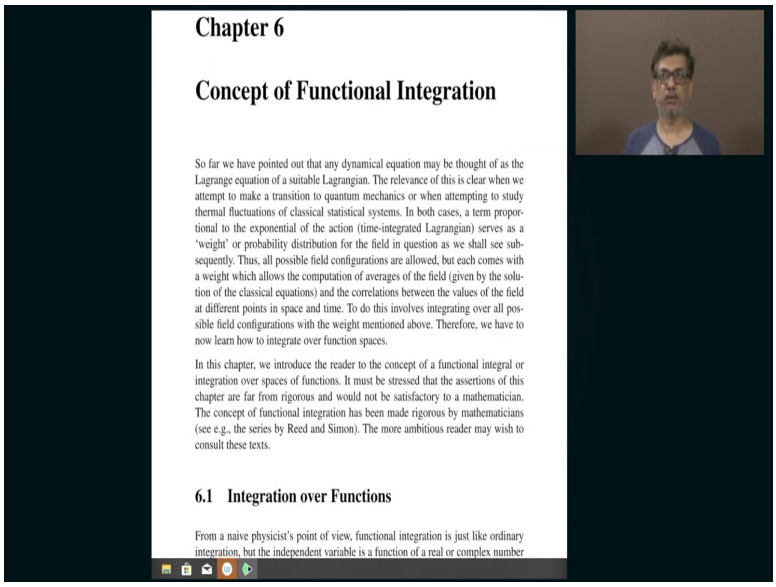

And so what I am going to do is that I am going to now switch gears and try and introduce a different topic. So, I do not have much time. So, I am going to spend some time trying to explain why I am jumping to this next topic. Because till now this earlier chapter whatever I discussed was a kind of a bridge, it was like a buffer between the first half of what I was discussing namely Classical Field Theory.

Because, if you remember I discussed you know Navier stokes, elasticity whole bunch of things. So, they are all fully classical, but then what comes later is basically the theory of Quantum fields. So now, I needed a buffer by which I kind of motivate fields by just pointing out you know phenomena in quantum mechanics that lend themselves to a description in terms of fields.

So, one of them is basically the idea of the right mover and left mover. So, that is basically the Green's function of a system of electrons which you can think of as a kind of a say gas a Fermi gas. So, that has a flavour or you can think of that in terms of a field because after all it is a kind of a continuum, but the underlying system is still quantum mechanical. And how that system responds to external disturbances could be a legitimate line of inquiry.

And that would come under quantum field theory. So, whatever it is so that the earlier chapter was a kind of buffer to help you understand how to go to the next set of topics which is basically Quantum Field Theory. So, the legitimate first entry into this new idea of quantum field theory is the basically this concept of functional integration.

So, I have to explain to you what functional integration is before you can really appreciate all aspects of quantum field theory. Of course, you can appreciate a lot of it without knowing what functional integration is but it helps a great deal if you do know it ok. So, the idea is that you see the point is that in your undergraduate courses you are usually taught quantum mechanics from the point of view of Hamiltonian.

So, in other words if you remember how you are taught I mean how you are taught quantum mechanics, it starting with some classical system and then you make analogies with waves and then you stumble upon this correspondence that the momentum should be writeable as a operator which is minus ih bar d by dx and then you go ahead and write down the Schrodinger equation. So, there was a huge number of conceptual leaps, but nevertheless there was this systematic procedure.

So, for example replacing poison brackets with commutators and so on, see but those are all you know in some sense sort of conceptually a little unsatisfactory even though the procedures are very rigid and concrete and easy to implement. But it also gives the wrong impression that somehow quantum mechanics can only be studied using Hamiltonian's.

See, but that is of course, not true because you can also study quantum mechanics starting from a Lagrangian because it is not necessary that you start with a Hamiltonian and then promote all the objects in that Hamiltonian to operators and then you talk of Hilbert spaces and so on. So, that is what people typically do. You do not necessarily have to do that.

So, the whole purpose of this particular chapter is to impress upon you that you can also do quantum mechanics using Lagrangian and that is something that is not very frequently discussed in courses at least in India. So, people will kind of gloss over that the so called path integral approach to quantum mechanics, so which is what I am going

to discuss now. So, path integral approach to quantum mechanics simply means you are discussing or you are trying to derive all the things you can derive using Hamiltonian's, but using Lagrangian instead ok.

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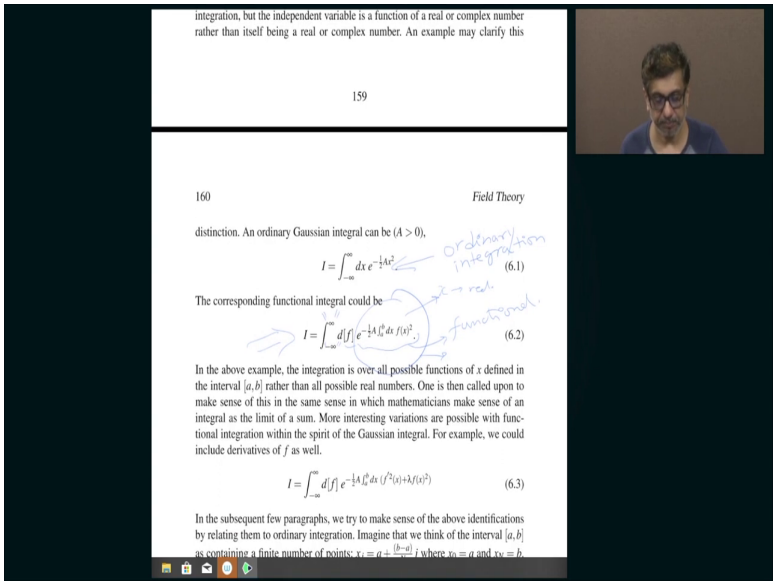

So, but then so the reason why it is not very frequently encountered in coursework or part of syllabi in Indian universities or most of the other universities is because, it entails knowing how to integrate over function spaces which is a rather technical and even now mathematically rather controversial idea. So, the bottom line is that because it involves integrating over spaces of functions doing quantum mechanics using Lagrangian is not it is technically more difficult, so because it involves this rather unfamiliar concept of integrating over functions.

So, but then still it is worth it, in other words I want to spend some time explaining to you how to make sense out of integrating over function spaces ok. So, let me tell you what I mean by integrating over function spaces? So, imagine you have an ordinary integration like this. So, what does this mean? So, you have say imagine A is some positive number and x is a real number and you are trying to integrate over this. So, you get some answer. So, you will get some square root of pi by A as your answer, by then what I mean? So, this is an ordinary integration ordinary.

So, whereas so what I mean by functional integration is that it is actually instead of x being a real number. So, in this case x is a real number which you are integrating from minus to plus infinity. So, instead of that what a functional integration is basically you look at not the space of real numbers, but you look at the space of functions. So that means, you look at all possible functions of a real number and then you integrate over all possible functions. So, this is actually in quotation mark I mean. So, like what I mean by this is basically a space over integrate over all possible functions.

So, here there is a I mean with x there is a well-defined you know starting and ending minus to plus infinity. So, if you go systematically from minus infinity to plus infinity you exhaust all possible real numbers. But that is of course, not the case in here when you are talking about the space of all functions, I mean this is just a very symbolic way of writing integrate over all possible functions.

So, this does not minus infinity or plus infinity has no particular significance, it just as a reminder and mnemonic that reminds you that you have to integrate over all possible functions. So, f is your function from a space of real numbers to real numbers. So, now so and this is called a functional ok. So, functional is basically a device that takes a function as an input and spits out a real number.

So, this is basically it takes input f is a function which exists between A and B for example, it takes a function like that as the input and the end product is a real number. So, this after doing all this you ends up getting a real number. So, the input is a function where the output is a real number, so that is called a functional and once you get a real number it makes sense to add up a whole bunch of them, which is what an integration is.

So, now what you are doing is you are taking this output and you are changing the function that you are inputting and you are constantly changing them and finding newer and newer answers for this integration and then you are adding them all up. So, that is what integration over functions mean ok.

So, basically so this is an example of a functional integration, where you are summing over all possible functions. So, you will you are probably kind of feeling a little dazed

and because that is understandable because you would probably have never seen this before.

And that is the reason, why many of the courses do not discuss the Lagrangian approach to quantum mechanics? Because in order to understand the Lagrangian approach to quantum mechanics you are forced to learn how to make sense out of these type of things which are of course, very unfamiliar.

And to be honest that they are also mathematically not very rigorous, in the sense at least firstly the way physicists use them in any case physicists generically are very blasé about mathematical rigger they very casual about being they in fact not rigorous at all. So that is one criticism, the other one is that even the mathematicians have not been able to make full sense out of this sort of thing as far as I know except in very limited cases where, which is not of much use to physicists ok.

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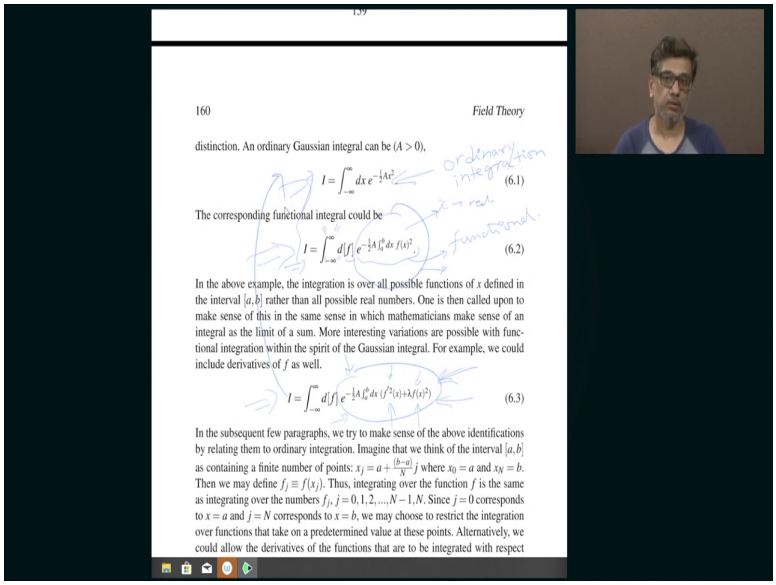

So, the other example which is more interesting is that. So, if you have a functional which only depends on f that is fairly easy to follow, but then you can have more interesting functional where the input is a function the output as usual is a real number so this is what that is. So, it takes a function as an input output is a real number, but then

you see what goes inside the black box is not only the function, but also it is first derivative.

So, that is what makes it interesting and that is what like you know differs it from this ordinary completely, see so it so this has no resemblance now anymore to the ordinary integration. So, you might think that these look rather similar because all you are doing is just integrating over functions. This is a function inside and function outside is simply integrating there is an x inside x outside integrating.

There is a much richer set of possibilities when you decide to do functional integration. So, your functional can no longer need not necessarily only have functions inside you know when you open the hood as it were. You not only find functions inside it, but you might also end up finding derivatives of that function inside it.

And that is the reason why this functional integration idea is a lot richer than the usual integration that you are familiar with ok. I think now is a good time to stop. So, what I am going to do in the next class is I am going to tell you how to this is fairly easy and uninteresting and also not that irrelevant to physics. So, what is of immense relevance to physics is this sort of thing. So, I am going to try and explain to you how to make sense out of how to evaluate such integrals.

So these are called functional integrals as I told you earlier. So, in the next class I am going to tell you how to evaluate this by converting basically this sort of thing to something more familiar. So, I am going to convert this very peculiar type of integration to a sequence of ordinary integrations that we are all familiar with. So, I am going to stop now and I hope you will join me for the next class.

Thank you.