Dynamics of Classical and Quantum Fields: An Introduction

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Quantum Systems
Lecture - 23
Right and Left Movers

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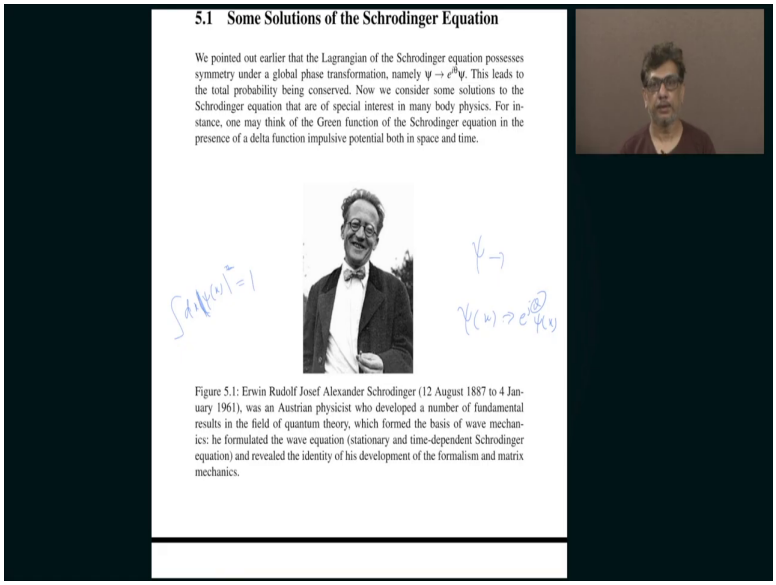

So, today I am going to discuss this topic that I had briefly started, namely I just want to find a bridge between the topics we have been discussing till now. Namely, classical field theory and the topics that are going to come subsequently which is quantum field theory. So, this particular lecture is going to be an a bridge between these two subjects.

And I am trying to lead you gently into quantum field theory by giving you examples that you are familiar with or you know at least in in certain contexts you are familiar with them. But I am going to utilize them in a new context which there will be some new ingredients, ok. So, in order to achieve this I am going to introduce this familiar Schrodinger equation which is the time dependent Schrodinger equation.

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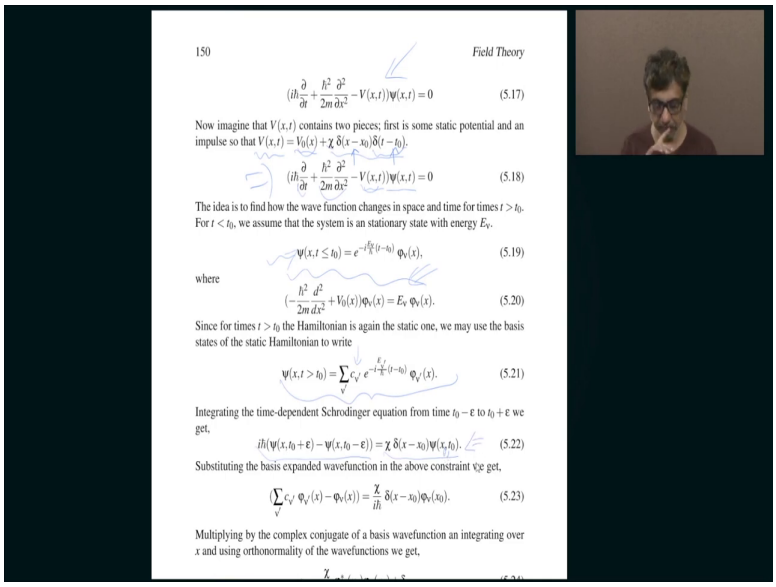

And we all know that this is what it is in one-dimension. And so there is nothing surprising or new about this idea. The only thing the new ingredient is going to come when I make reinterpretation of this potential energy function. So, I am going to say that this potential energy function has two parts to it, one is the time independent part and the time dependent part is going to be an localized impulse. That means, that you know what an impulse is.

Basically its a disturbance that exists for an infinitesimal duration. That means, it is an extremely strong disturbance that exists for a very short while. So, that is what an impulse is. So, you see that is the reason why there is a Dirac delta function and an associated amplitude, but not only do we have to understand or explain what the potential is as a function of time.

So, we have decided to say that it is basically a time independent part plus the time dependent part which is an impulse. But then we also have to explain what how that impulse changes from point to point in space. So, I am going to assert that the impulse is also impulsive in the spatial part also. That means, it is a localized impulse. That means, that impulse only exists at a position called x 0 and then it exists for a very short while at time t 0.

So, given these two conditions. So, you can imagine, so it is like you know just taking a you know a dagger and tabbing that line at some point momentarily you know. So, the electrons are minding their own business or these particles these quantum particles are minding their own business, and then you just take a dagger and stab that at x 0 at time t 0 and then you release.

So, then you see what happens to that system. So, obviously, it is going to get disturbed. So, the Green's function of the system is basically the difference between the solution of the Schrodinger equation, before it was disturbed and after it was disturbed. So, that is what the Green's function of the system is because its called Green's function because you see the disturbances and impulse both spatially and temporally.

So, now, we are going to assert that we know how the wave function looks like before the impulse. So, we are going to assert that it was in some stationary state determined by the solution of the time independent Schrodinger equation. So, that is given by this the stationary state eigen functions.

So, now, after impulse it is clear that it is because these stationary state eigen functions forms a basis, I can use that basis to represent the wave function after the impulse as a linear combination of that basis. So, which is precisely what I have done here, ok. I have used the psi as basis, phi nu as the basis, and I have expressed the wave function after the impulse in terms of the wave function before the impulse. So, the wave function before the impulse was stationary states, ok.

So, now, how do I go about determining these coefficients which we will be necessary in order for me to express this or basically write down the solution after the impulse explicitly? So, the answer to that is the following that, you first of all realize that this potential function has a part which is an impulse which is a Dirac delta function.

So, if you integrate with respect to time, so if you integrate both sides of this equation with respect to time, from t equal to just before t 0 and just after t 0. So, you integrate from t 0 minus epsilon to t 0 plus epsilon. Then, you will see that clearly the terms which do not involve the delta functions drop out.

So, that is the reason why these things do not contribute, but the one, so even this does not contribute. So, what contributes is the fact that the integral of the derivative of with respect to time is clearly the wave function. So, what is that going to say that basically the wave function is discontinuous, and the strength of that the jump, the value of the discontinuity is determined by the strength of the delta function there.

So, it multiplies by the wave function by then the wave function is evaluated at the time at which the impulse takes place, ok. So, it is actually x 0, t 0, ok. So, so this is what that is. So, now, you can go ahead and you can use your answers, namely this answers, you can use it here, ok.

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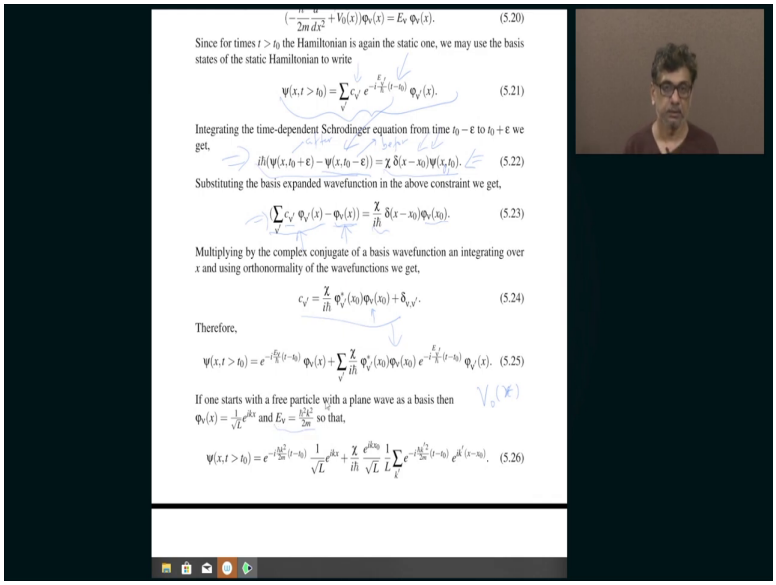

So, if you use it there, so what this says is basically that before the impulse it was just this and after the impulse it became this, right. So, this is before, this is after. So, before the wave function was this, after it is now this, because it is after it is a combination of stationary states. It is not stationary anymore after the impulse.

So, this difference therefore, is the jump or the strength of the or the basically is the measure of discontinuity, how much how discontinuous the wave function is and that is given by the coefficient of the impulse. Basically, it is given by this expression. So, you might be wondering that that seems to be at odds with what you might have learnt in

your quantum mechanics, courses that wave function always has to be continuous. But I am seeming to imply here that the wave function is discontinuous and the discontinuity is determined by this.

So, I think what the instructors of your traditional courses forgot to tell you or they did not have in mind is potentials which are very very singular like this. So, they implicitly assume that real life does not produce potentials that are extremely singular like this. That means, stabbing kind of potential, that is it is impulsive both spatially and also temporally.

So, those are unusual type of potentials and certainly if you include them in your discussion, wave function is not continuous, ok. Its continuous whenever you say that other than that any other potential is allowed. So, if any other potential is allowed then wave function and usually its first derivative is also continuous, ok.

But in this somewhat pathological example, the wave function itself is discontinuous, ok. And the discontinuity is determined by the strength of the Dirac delta function, basically the strength of the impulse. So, now, how do I determine these coefficients? So, clearly I use the orthogonality properties of the basis and then I go ahead and multiply by the complex conjugate and then I integrate. So, if when I do that I get this result, ok.

So, this is going to be my; so, keep in mind that nu is the starting you know the system was in that particular stationary state called nu before the impulse was turned on. So, that is that always exists in your final answer because it is a new prime that is getting summed over, ok. So, the nu always exists it was there earlier, it is still there now. So, nu is what we started off with the system was in a stationary state labelled by nu, ok.

So, now, you go ahead and substitute the answer we got for these coefficients. And so, suppose now you specialize to the case of a free particle so; that means, you know that V 0 x. So, I told you that well I am talking about stationary. So, before the impulse the system was in a stationary state determined by the solution of the time independent Schrodinger equation where the potential function was V 0 x.

But then you can make a further simplification, and ask yourself what do these solutions explicitly look like when the system does not have a inverse no forces acting on the particle. No force acting on the particle then clearly the energies are just, energy of a free particle which is p square by 2 m and keep in mind that p is h bar k. So, its $\frac{\hbar k^2}{2m}$ and the wave functions are plane waves and that is what you get, ok.

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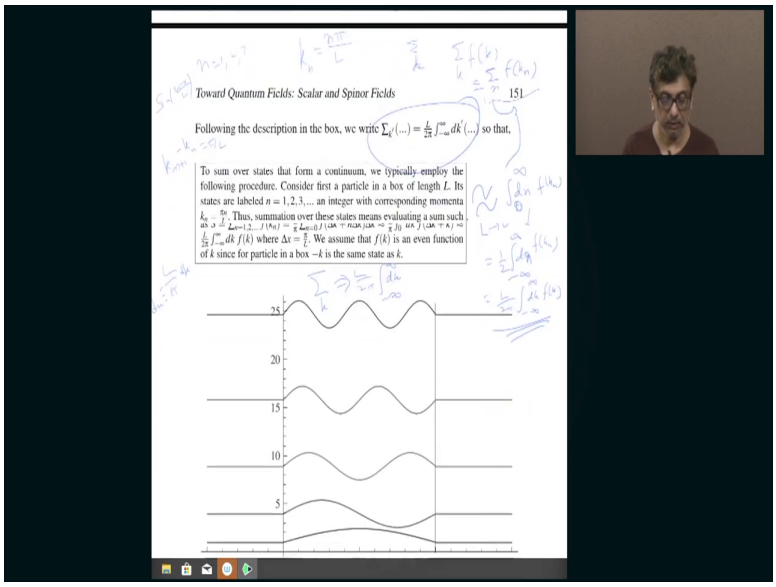

So, the bottom line is the following. So, this is the answer basically. So, this is how it works. And then you have to convert the summation to an integration. So, keep in mind that you know the way you will be usually deal with free particle is not really to, because if you if it is genuinely free then it would violate some basic postulates of quantum mechanics.

Namely, the genuinely free particle has a plane wave eigen states, and plane waves are not normalizable. The mod square of the wave function cannot be; if you integrate it over all space you get infinity instead of 1, which is what you are supposed to get, ok. So, there is no way you can normalize a wave function like that. So, typically what people do is that we assume that there is a finite extent to the spatial locations of the particle. So, we first confine the particle to a box, namely starting from x equal to 0, all the way up to x equal to L. We say that the particle is confined to this box.

Then, we ask ourselves what would be the properties of the system. So, then once you confine it to the box, then clearly the wave function is normalizable. Then, we ask ourselves what would the wave functions look like and not only the wave function, but observable properties look like as you make the box larger and larger.

And so, in the thermodynamic limit you get back your original what you wanted namely that you did not want the box or you did not want the boundaries of the box to be present. So, you could achieve that by first placing the system in a box, then calculate your usual useful physical quantities, and then you make the size of the box very large.

And certainly you know ratios of quantities that are proportional to the size of the box are going to be finite as you make the box larger and larger. So, which is what we are typically interested in. We are not interested in the whole quantity itself with the quantity per unit length, so that sort of thing. So, those are typically finite.

So, another question is that you know if you confine a particle to a box, so the summation over k has a very specific meaning. That, see once you confine a particle to a box that k is going to become discrete n pi by L. So, if you recall n was equal to 1, 2 you know 0 was excluded because the wave function was sin n pi x by L, and you would make the wave function becomes 0. So, if the wave function is 0, then its integral of its mod square will never be 1, it will still be 0. So, it will not be normalizable even then.

So, 0 is excluded for that reason. And negatives are excluded because you see the negative values of n differ from the positive, the wave functions are the same apart from a sign. And you know that wave functions that differ by a sign or even a phase, there is a complex number of unit modulus. They refer to us, they correspond to the same state. So, therefore, we only consider positive integers for n and k is written as n pi by L, ok.

So, now, how do I convert summation to an integration? So, the way I do that is that first of all I am going to say that look if I look at the values of k, ok. So, if I look at, so this is k n. So, if I look at k n plus 1 minus k n. So, these two are going to be incredibly close to each other, ok. So, in other words, the difference between the k's, ok are going to be very small.

So, rather than summing over n, I could equally well integrate over k. But then in order to do that what I am going to do is that I am going to say that look summation over n is same as integrating over n because n is dimensionless now. And summing over n without loss of generality when L is very large, I can replace this by an integration over. So, this was basically by definition this, ok.

But then I am going to say that this is approximately, when L is very large this is approximately integral d n, f k n, but then n goes from 1 to infinity. But now you see the point is that if f of k n is even; and so, one more point that you see the bottom line is that when you sum over all the ends, the result that you get are typically of the size of the system. They are of the order of the size of the system capital L which goes to infinity.

So, now, so even though n equal to 0 was excluded. So, there is no loss of generality in just including it. So, I am going to just write that knowing fully well that the mistake I am making is negligible as the size of the system becomes larger and larger. So, now, if f f is even, so this can clearly be written as one-half of integral d n from minus to plus, ok d n. So, now, keep in mind what n is. n is L by pi into k and d, d n is therefore, L by pi into dk. So, this can be written as L over 2 pi into integral dk f of k.

So, this is the reason why you know hear of people say that in the case of systems in the thermodynamic limit rather than summing over discrete states, you integrate over the wave number. And the way to convert the summation to an integration is through this prescription. Namely, you write sigma k it maps to an integral over k, ok.

So, that is a hand waving justification for converting a summation to an integration. So, that is what I have written here. So, when you do that, so you get this result. So, this is what you get, ok. So, you convert this there was a summation earlier and that got converted to an integration.

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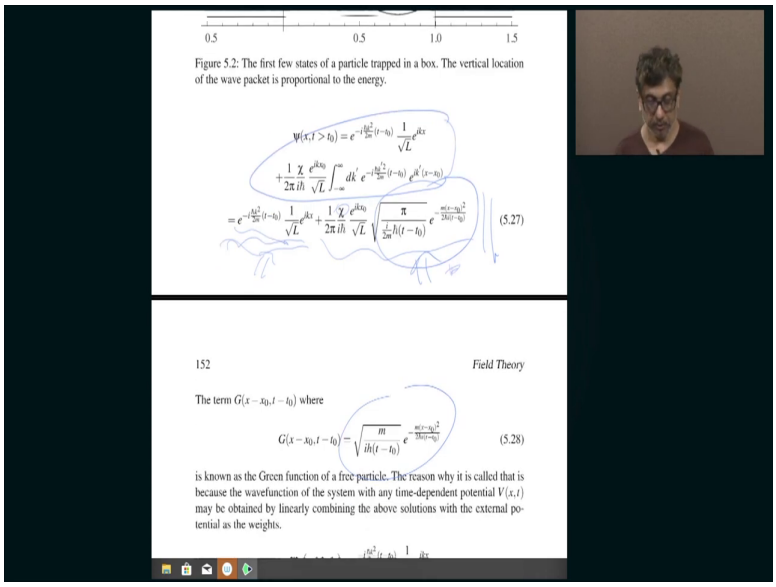

So, when you evaluate that you get this result, ok. So, there is this particle, I mean this is this, so the wave function after the impulse would have been the same as it was before the impulse namely, this is the wave function before the impulse. So, after the impulse it would have still remain this, had it not been for the fact that now we have an impulse with strength chi. Now, because of the impulse the wave function suffers a modification, namely it becomes this, ok.

So, now this particular quantity is called the Green's function of the system, ok. So, its called that because the wave function with respect to, so this particular solution was for an impulsive you know Dirac delta space and time. But if you wanted to study the same problem for another different completely different V x comma t you could first evaluate the Green's function which we have done now. And then, multiply by the appropriate V of x naught t naught and integrate over your x naught t naught to basically get whatever you want.

So, so basically the solution of the Green's function enables you to write down the solution of the time dependent Schrodinger equation in a general way. So, that means, you will find the most general solution to the time dependent Schrodinger equation, ok. So, that is the utility of Green's function.

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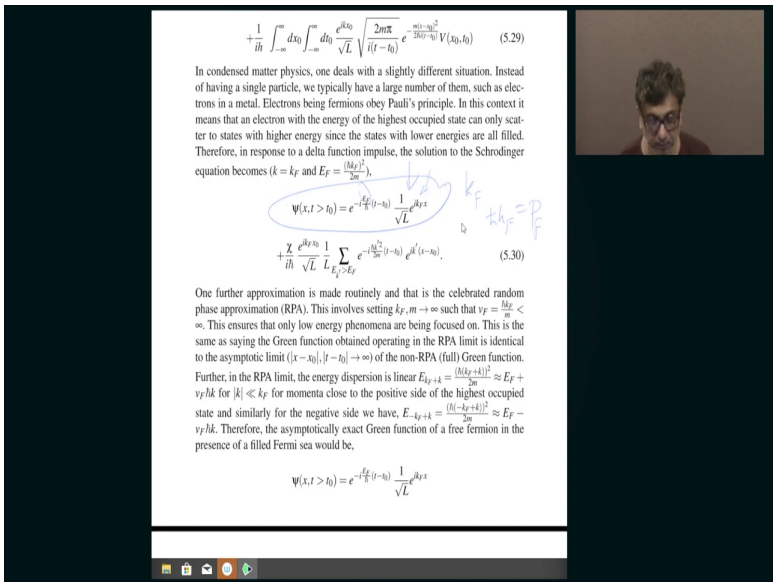

So, now I am going to ask myself look, I am going to now shift my interpretation to something interesting. So, this is you know if there was one free particle in a box and it suffered an impulsive disturbance. And then, its wave function changes according to this manner. That is what we have found.

But now instead of one single solitary quantum particle in a box imagine a whole bunch of them. Specifically, imagine a large number of them and imagine they are all Fermions. So, if they are Fermions, then clearly you cannot really populate more than 2 of them at a given quantum number n.

So, the bottom line is that we all know that for Fermions the case get quickly filled up, and it fills up to a maximum value determined by how many particles you have in your system, the total number of them. So, the point is the following that, so imagine such a system, ok. So, you have such a system and now you focus on the electron which has the highest possible energy and that is called the Fermi energy.

See, if all the particles in your box were Fermions, they would necessarily occupy the only a few or 2 of them would occupy the lowest energy, the next lowest energy would occupied be occupied by another 2 and so on and so forth. Very rapidly you will reach a

very high energy because you have a huge number of electrons that you are attempting to fill into the system.

So, the point is that now that you have filled your system up to that high value of an energy, you ask yourself what is the wave function of that electron, ok. Suppose, there was no disturbance of any kind. So, this is the system, this is the stationary state that you are in to begin with. So, now, the question is what would be the wave function of an electron with that particular energy.

And keep in mind its a free particle. If its a free particle then its energy is called the Fermi energy which is denoted by E F, and its wave number is called the Fermi wave number and h bar k F is called the Fermi momentum which is P F, ok. So, this is how the wave function of the particle with the highest possible energy look like. So, assuming that the box that you have in front of you contains a large number of Fermionic particles, ok. So, and the one the Fermionic particle with the largest amount of energy is what we are interested in.

And we have written down the wave function of that particular particle when there are no further disturbances. So, in other words, when it is a stationary state. So, when it is in a stationary state this is what its wave function is. So, now, we ask the question we have been asking earlier, namely that how would this wave function change if there was a impulsive disturbance. So, the disturbance is impulsive as usual not only temporarily, but also spatially.

So, I am going to as usual impose or disturb the system in a particular peculiar way. And the way I do that is by switching on an impulsive potential that is also highly localized in space. So, when I do that I want to ask myself how does the wave function look like subsequent to that impulse. Then, I will allow you to work out the details, but the answer comes out to be this.

So, it is comes out to be this because see the point is that now we have made this physical assumption that after the disturbance, the particle which had the highest possible energy cannot all of a sudden occupy any lower energy because it only occupies states that are available to it. Namely, now the ones that are not already occupied.

So, the energy states that are not already occupied are clearly greater than the Fermi energy because less than the Fermi energy, there is this Fermi Sea. That means, you have all the Fermions have kind of decided to occupy all the states below the Fermi energy in order to minimize the energy overall energy of the system. So, they have obeyed poly exclusion principle and occupied all states below the Fermi energy.

So, now, once this impulse is applied, clearly the electron with the highest or the Fermion with the highest possible energy can do nothing other than excite itself to as state with energy greater than the Fermi energy. So, which is precisely what I have said here. So, normally it would have been sum over all k, k dash.

So, its k dash here. So, normally, it would have been sum over all k, but now because k value is below the Fermi wave number are not available. So, in other words, they are not available because other Fermions have already occupied those states. So, therefore, the available ones are the only ones that this Fermions can scatter into. And so that is what it does here, ok.

So, now, I am going to tell you how to evaluate this. So, you see this description that I am giving you that the impulsive, so I am implying by this description that somehow the impulsive disturbance only excites the Fermion with the highest possible energy. That is clearly not true. Because there is no reason why other electron should not also find or should not also feel the impulse.

But of course, here the implication is that impulse also has a kind of bandwidth; that means, it is it picks out electrons close to the highest possible energy and only allows them to get excited. So, its only then all this interpretation makes sense. So, given that interpretation of the impulse, namely that it has a bandwidth which enables it to preferentially you know attack electrons with in the vicinity of the highest possible energy.

So, electrons with the highest possible energy have momentum either equal to h bar k F or minus h bar k F because in both cases the energy is h bar squared k square by k F squared by 2 m. So, now, the energy of the electron when it is close to plus k F, its wave number is close to plus k F, may be written as follows. So, this square term clearly has 3

pieces, one is k F square which is the original Fermi energy, and there is a cross I mean there is a cross term.

So, the cross term tells you that it is basically, so this small k is basically the deviation from, so this is the actual wave number k F plus k itself. So, k therefore, is the deviation of the wave number of the particle with the highest possible energy and k F. So, it is the deviation from the actual wave number and the Fermi wave number. So, it tells you how much those two differ.

So, now, you see the energy is linear in that difference. So, its proportional to k which is that the difference between these two. And the proportionality is basically related to what is called Fermi velocity. Fermi velocity is nothing, but the Fermi momentum divided by mass, ok. So, similarly for if the particle is close to the other end, namely it is momentum is minus h bar k F then its energy is going to look like this. So, it is going to be less than E F. Well, that depends on the sign of k. So, it is certainly has a different sign compared to when it was near k F, ok.

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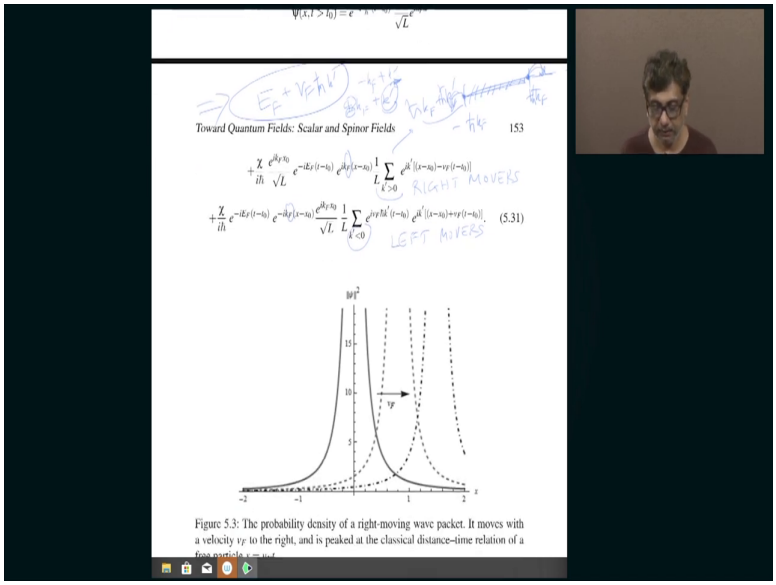

So, now there are two possibilities. So, you see when you are summing over all the, so you want to see what the energy of the what the electron with the highest possible energy does when an impulse is switched on. So, the thing is that when the impulse is switched

on the system, then the electron with the energy close to the momentum close to h bar k F is going to try to increase its momentum to h bar k F plus k.

The reason why it does that is because it wants to scatter into an place where there are no electrons. So, see for any momentum greater than h bar k F there are no electrons because those are the unoccupied states. So, the electrons would like to switch to a situation where the wave number or the momentum is basically greater than h bar k F, ok. So, this is your h bar minus h bar k F and plus h bar k F.

So, the electron which is here wants to scatter there, because here everything is filled up. So, similarly an electron here would want to scatter there. So, you see, so that is the point. So, the energy is going to look like this. So, if you just in general write like this, right. So, for the right movers, so the these are called right movers and left movers.

So, the electrons whose wave number or their Fermi momentum is close to h bar k F plus h bar k F, they are called right movers. They are called right movers because basically their momentum is positive which suggest movement to the right, at least in the classical way of thinking. See whereas, minus h bar k F are called left movers because that momentum suggests movement to the left.

Now, generically the energy of either right movers or left movers may be written as E F times or E F plus Fermi velocity times h bar k dash. So, now, this k dash you see, if you are a right mover you want k dash to be positive, so that your energy is greater than k F. So, you are scattered. So, this is the energy that the electron has after it scatters because of the impulse.

So, now, because of the impulse the electron scatters to an energy which is greater than E F if it was a right mover. But if it was a left mover it will actually try to go here, ok. So, it is new momentum is minus k F plus, ok its minus k F minus, it is even smaller, ok. So, it is minus k F minus k dash, ok. So, it is k dash less than 0. So, in other words, it is like; it is like this.

In general, it is always like this, plus or minus k F plus k dash. Now, if you choose the plus sign you should make this also positive, but if you choose minus sign you should

make this negative. Because if you choose minus k F as your starting point then you want to scatter to the left, right, because then you will be out of that Fermi Sea. So, this Fermi Sea is within this segment.

So, if you are at the point minus k F which is the left extreme, so after scattering you want to continue moving left because that is where all the empty states are. But if it is plus k F you want to continuing right because that is where again all the empty states are. Because if you are at the right and you after scattering you move left, then you will encounter all these filled electrons there. And then you will be violating Pauli principle and you do not want to do that.

So, the point is that these states are now going to correspond to right movers, ok when its k dash is greater than 0. And they correspond to left movers, so they correspond to left movers. So, they correspond to left movers when? The Fermi momentum is negative, ok. So, now, you see, you can see that, so in this case the Fermi momentum or the Fermi wave number is plus k F here is minus k F, ok.

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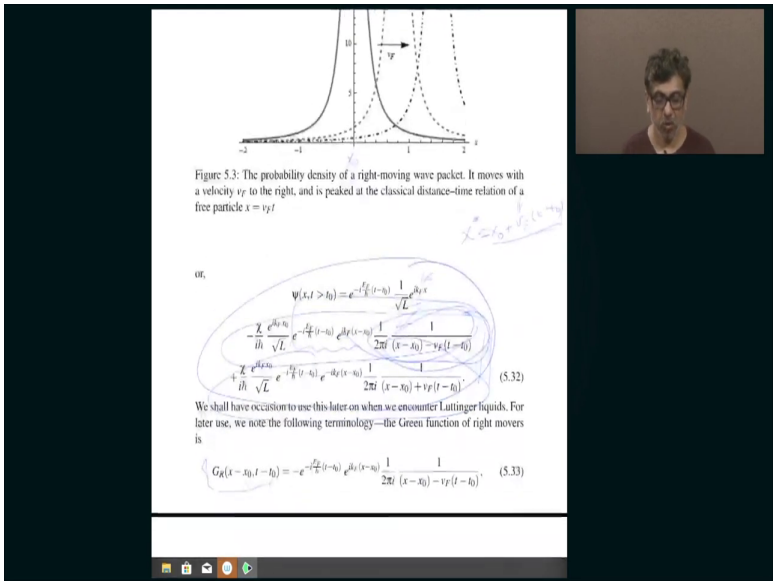

So, now you can see that this allows you to do the following. So, it allows you to write down the wave function of that particle with the highest possible energy after scattering as follows. So, suppose if it, so in other words if it was like this to begin with, ok, so that

means, if it had plus scattering, so it would scatter this way. So, it would after scattering it would look like this, ok. So, if it was minus k F, it would look like that, ok. So, if it was plus k F, there would be the original piece plus the scattering piece.

But then it would only involve the scattering piece if it was minus k F, ok. So, bottom line is that the because this is summed over both the regardless of whether it is right or left to begin with you are summing over k dash. So, there you have to sum both right and left, ok.

So, now, this if you just you know highlight in your highlighter, if you highlight this part, this is going to correspond to the Green's function of the right movers. It is called that as usual because if you multiply that by your strength of your potential, if you multiply that with your potential you get the actual wave function.

And then, if the potential is not impulsive, either spatially or temporarily, then you will have to integrate over the location and the time at which the impulse is. So, in other words, you have to integrate over all those x 0s, t 0s, in order to get a more realistic potential. So, in other in other words, in order to solve the Schrodinger equation for a more realistic potential.

So, this would correspond to the Green's function of the right mover, this would correspond to the Green's function of the left mover. So, I hope you have understood this concept. This is an exceedingly important concept, the concept of right movers and left movers. So, basically right movers are parts of the Fermionic wave functions that in a many body system.

So, if you have a many Fermion system, and you will look at the wave function of a particle of a Fermion with the highest possible energy. And you ask yourself what is the wave function of that particle with the highest possible energy look like, if its momentum is also positive. So, the wave function that corresponds to all those descriptions or qualifications, those are called right movers, ok.

And conversely if you are interested in the wave function of a Fermion with the highest possible energy in a sea of you know Fermions that are filled up to that particular energy.

Then, and you further demand that the wave the momentum or the wave number of that Fermion is negative, then you would be looking at the part of the wave function that corresponds to what we call left movers.

Now, I am just giving you a graphical example and if you plot the probability density of only the right movers, so what you are going to effectively get is this. So, you are going to get this. So, what this says is the square of that that is. So, you see that particular function if you plot it as a function of x minus x 0. So, you see that if initially if it is at x 0, later on it will not remain at x 0. So, it will actually peak. So, this this will peak always at x 0 plus v f t minus t 0.

So, basically what this says is that this the probability of finding this particle keeps shifting to this new value of x. And that new value of x is the original value plus the distance it has travelled, if it were some disturbance travelling with some speed v F. So, that is the interpretation you see.

So, right mover is basically a disturbance that travels with a speed plus v F. So, that the wave function that you are looking at always peaks at the original starting value plus the distance the wave function or the peak of the wave function has travelled in that particular time t minus t dash. So, the distance travelled is v F times t minus t dash. So, this is a description of right movers and left movers.

So, I am going to stop here. And I hope this is an exceedingly important concept because later on you will see that I will be invoking this idea to describe what are called Luttinger liquids. And you will find that they are very useful, and they are very interesting, they have counter intuitive properties and so on. So, they were learning.

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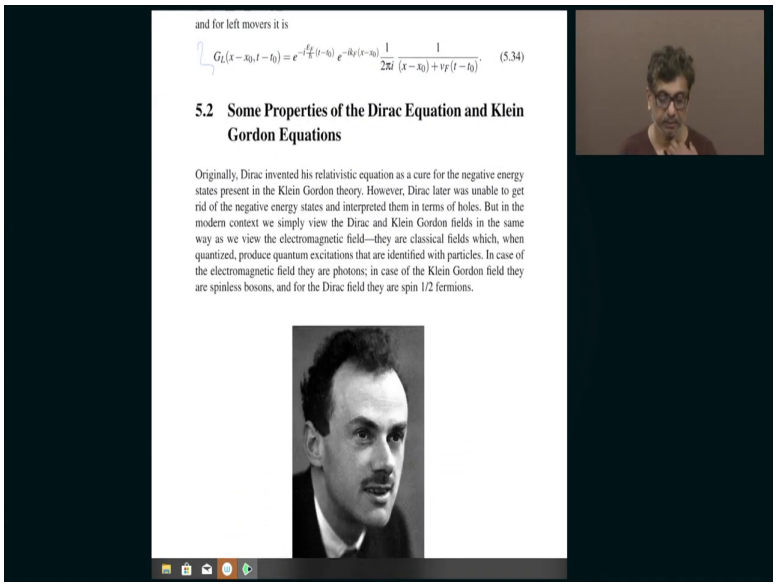

But before I get to all that I am going to again shift gears and in the next class I am going to describe more conventional field theoretic introductory topics namely the Dirac equation and the Klein-Gordon equations, and why field theory is needed to properly interpret them. Because in the absence of field theory none of these equations make sense, ok.

So, in the next class, I am going to touch upon these ideas and try to explain to you why you know without field theory, the relativistic theory of the electron makes very little sense, ok. So, I am going to stop here. And I hope you will join me for the next class.

Thank you.