Dynamics of Classical and Quantum Fields: An Introduction

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Fluids
Lecture - 21
Stokes' Drag - II

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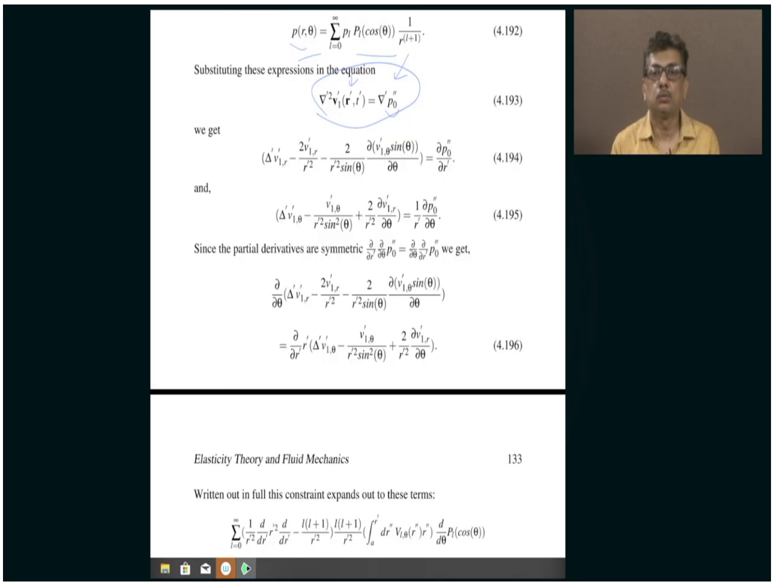

So, ok let us continue our discussion of Stokes' Drag on a sphere falling in a fluid. So, basically the goal of this exercise is to reach the point where we can derive the classical formula that we have encountered in our school days namely this one.

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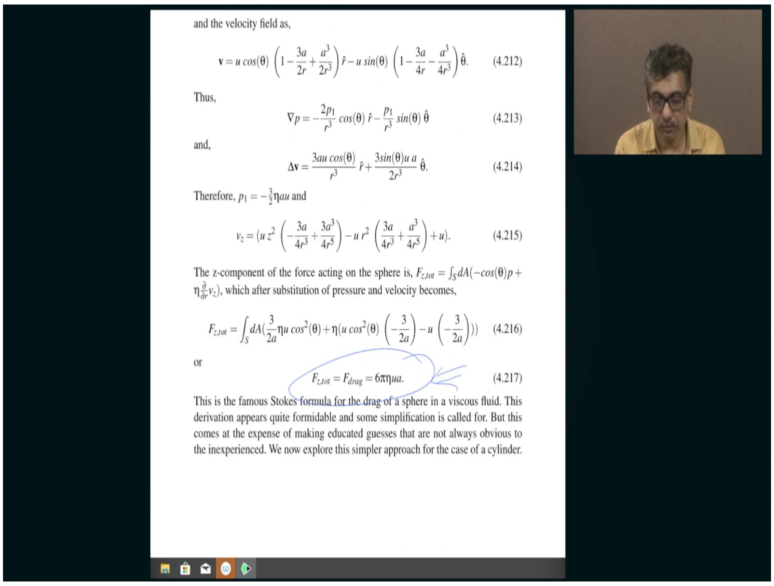

So, this equation 4.217, so that is $6\pi\eta ua$. So, eta is the coefficient of viscosity of the fluid u is the speed with which the ball is falling and that ball has radius a. So, basically that is the drag experienced by the ball when it is falling with that speed in the fluid.

So, that is what we want to derive.

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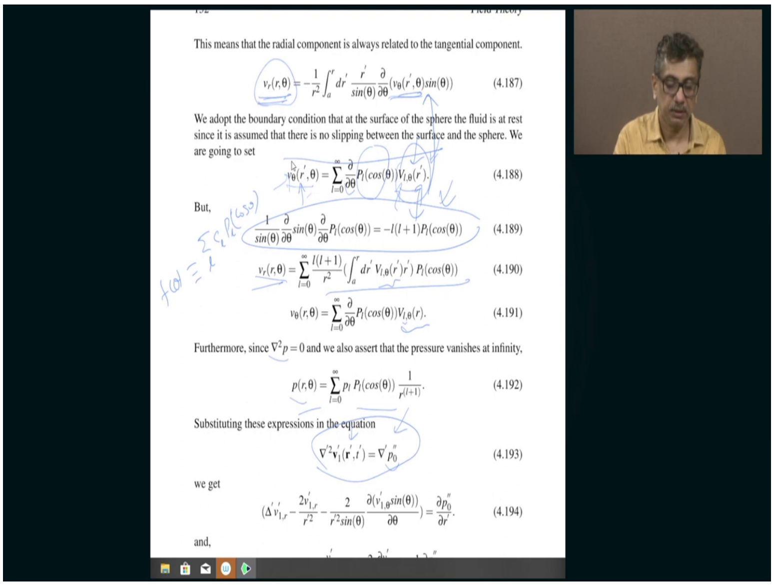

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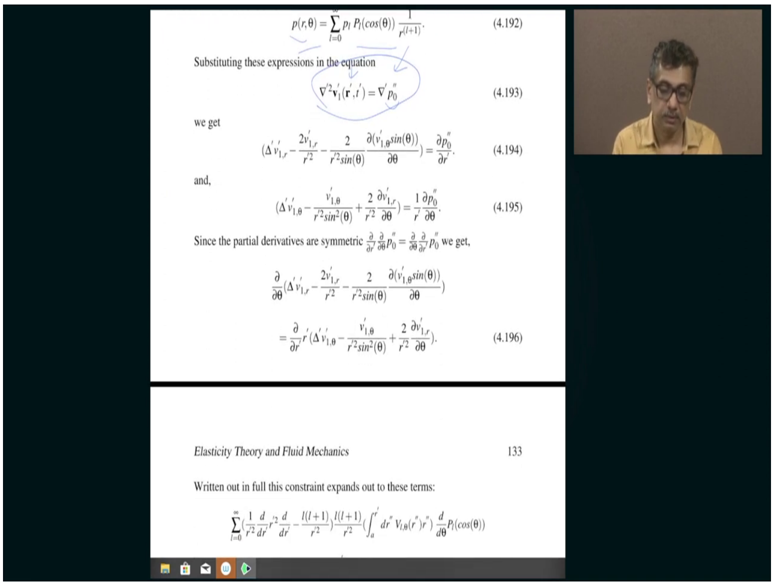

But then the derivation of that is not simple because it involves taking into account turbulence and the fact that there are these Reynolds numbers that are involved and you have to expand in powers of Reynolds number. So, I do not want to spend too much time on the technical details because it is very easy to get lost. So, I will just highlight the salient features.

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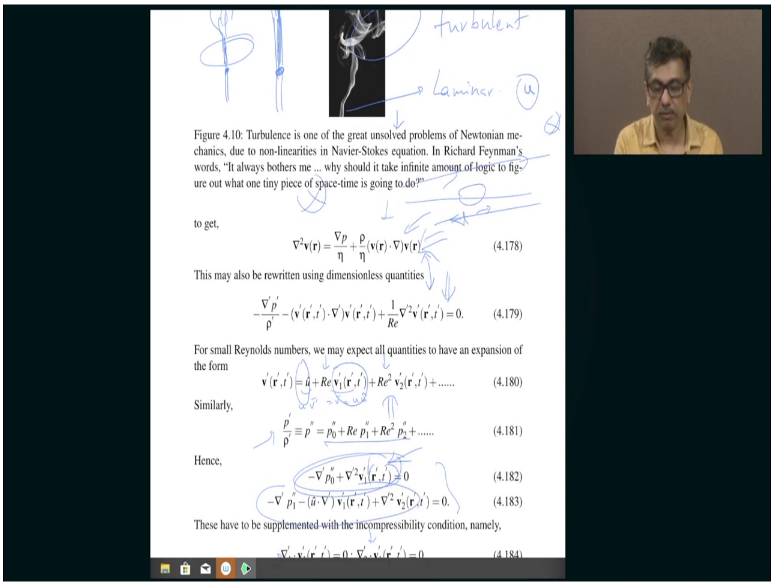

So, the first feature was that we have we expand this equation. That means, there is Navier-Stokes in the case of steady state there are no time dependences. So, first we render that dimensionless by rescaling the variables.

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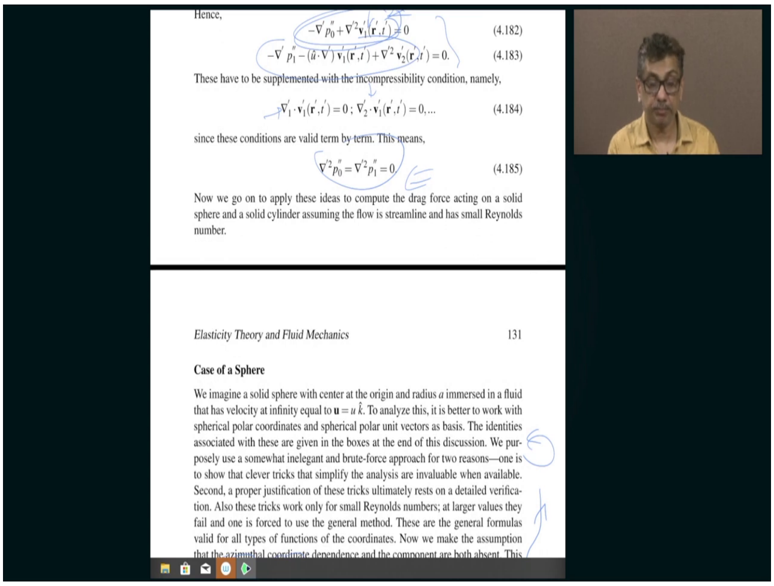

Then we expand all the unknowns whether its pressure or velocity and so on in powers of this Reynold's number. And you will see that the first order terms are related to the second order terms and so on. So, the first order in Reynolds number is related to the zeroth order and Reynolds number in this way.

So, 4.182 will tell you how V 1 dash which is the first order correction to the velocity of the fluid. So, I am assuming that the ball is at rest and the fluid is flowing around that ball which is the same as ball flowing in the fluid. So, whatever it is 4.182 is the one that tells you how the fluid velocity will change because of the Reynolds number.

So, V 1 dash is the first order correction to the velocity because of the obstacle ok. Point is that, so once we derive all this and then we take into account the fact that the we are talking about incompressible fluid. So, in steady state the you know velocities are divergence free.

So, that will mean that basically this pressure and velocity and all that which satisfies pressure especially satisfies the Laplace equation. Because if you take divergence of 4.193 for example, you will get del square p equals 0.

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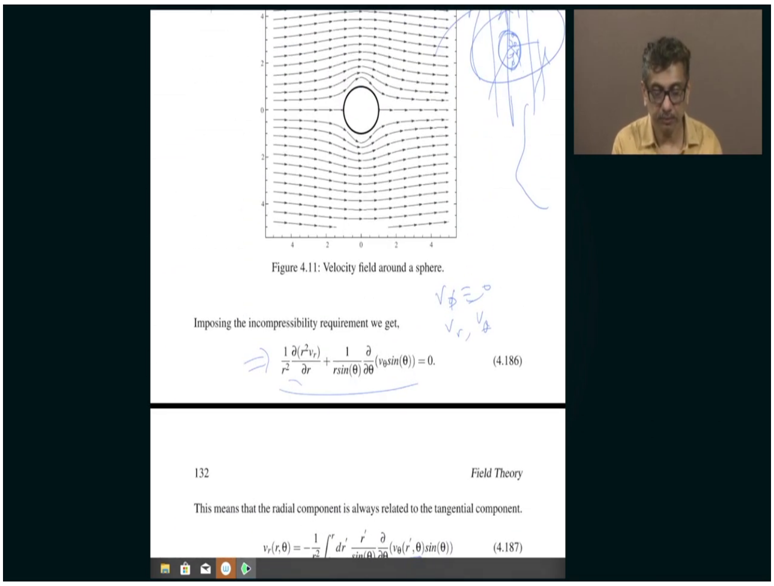

So, then you write down the del square V equal 0 and then you can express the radial component in terms of the tangential component. And the tangential component is expanded in linear combinations of these Legendre polynomials.

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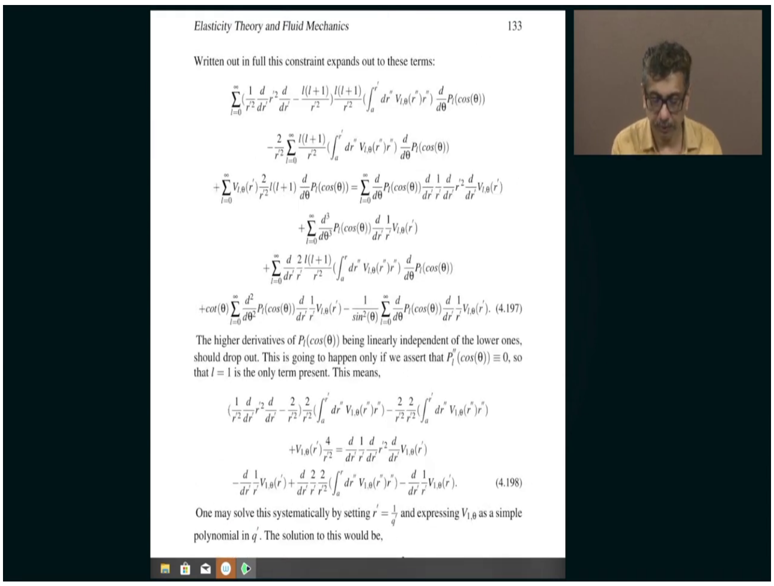

And then finally, we will be able to write down one equation purely for the coefficients of the that the coefficients that appear in this radial function.

So, you see this sorry the tangential component of the velocity is expressible in terms of these quantities. So, the basic point is that you have to calculate these quantities once you know what these quantities are, you know V theta and because you know V theta from this formula you will know V r.

So, if you know this quantity you will know V theta and V r. So, how do you calculate this quantity you insert that into these equations which are basically expanded out versions of 4.193. So, you take 4.193 and you expand it out. So, if you expand it out you will get.

So, where does this come from? This comes from matching the Reynolds number on both sides. So, you expand in powers of Reynolds number and you substitute into that Navier-Stokes in steady state situation and you compare the powers of Reynolds number and you get 4.193 from there. So now, that you have got this you go ahead and substitute your expanded out forms for V 1 dash in terms of the V r V thetas and therefore, the V ls.

So, from the from this you can also $\nabla^2 p_0$, so from this you can find out the pressure also, so it will have its own coefficients ok. So, when you insert all this you will see you will get one equation which will only involve the V l thetas ok.

So now, 4.197 is as one thing it looks very horribly complicated, but you will see that it is not complicated mainly because it looks complicated, but its finally not because most of the terms in that summation are actually 0. Because in fact, only l equal to 1 survives because all the see the thing is the P l cos thetas are basically linearly independent basis.

So, these two being equal means that all the higher order derivatives of P l cos theta, I have to vanish because you do not find you know things to cancel that out somewhere else. So, you will be able to convince yourself that because the higher derivatives of P l cos theta are linearly independent of the lower ones, the higher derivatives are going to drop on specifically you can convince yourself that P l dash dash is identically 0. So, therefore, all the higher derivatives identically 0.

So; that means. So, if P l dash dash is 0; that means, basically it is only up to l equal to 1. In fact, you will find that only l equal to 1 will survive.

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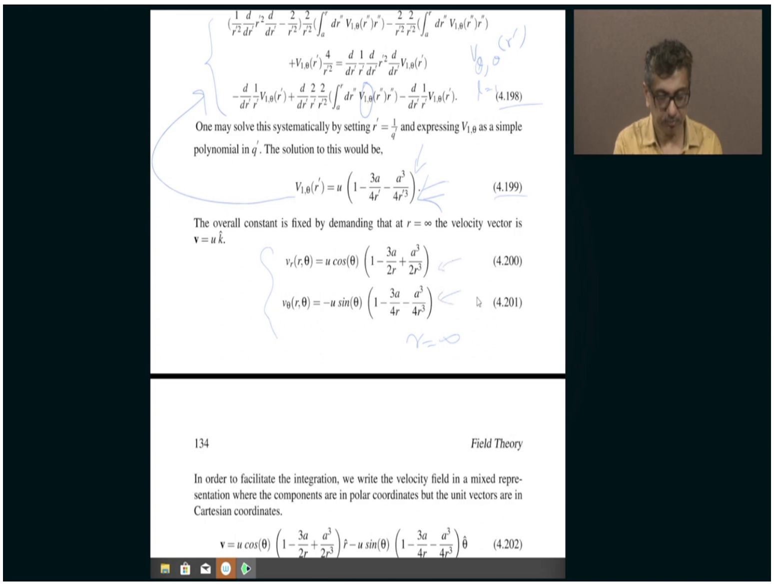

So, if you so what you do is you kind of first convince yourself that is in fact the case ok. So, this P 4.179 was obtained by inserting our answers which expands the velocity in terms of the complete basis which is Legendre polynomials. So, once you substitute into 4.1 you substitute that answers into 4.197 you get an answer for that the only relevant coefficient which is V 1 theta. So, it is only this is the only relevant one.

So that means, you remember that there was a V l in general it was like this, so where V l was something very general. So, it could have been anything. So, l could be anywhere from 0 to infinity. So, what this is saying is basically of all the possible V l's only the ones which is relevant is l equals 1 ok.

So, now you have to go ahead and solve this equation and you will see that this equation has a solution I mean the basically if you make a substitution of r dash is 1 by q dash. So, you think of 1 by r dash as your independent variable rather than r dash.

You will see that basically that this V 1 theta is a polynomial in 1 by r dash ok. And so you can find out what that polynomial is and you will be able to convince yourself that it is in fact this. So, I know that I am kind of you see discourse is somewhat unusual in the sense that it is parts of it especially the Stokes' drag calculation is fairly technical and it is not typical of the rest of the course.

So, do not want you to get intimidated by this discussion in the sense that not all parts of the course are going to be this technical and involved, it is only this calculation of strokes drags that is difficult. So, I am not going to necessarily insist that you appreciate all aspects of this calculation unless you really want to and I also do not want to burden you with asking these types of question in any examination.

So, this is only meant as a reference for you to you know go back to whenever if somebody if you yourself was wondering where that high school formula comes from. And you really were curious to know how to derive it I have just wanted to put it out there and. So, that you will know that there is such a derivation and you will feel satisfied that somebody has told you how to derive it.

So, it does not necessarily mean that you should kind of know it inside out unless you want to specialize in fluid mechanics. So, this is just merely meant to you know make you aware of the existence of this derivation and the salient step features and how the main procedure for deriving that formula ok.

So, having said that you see we can continue and say that look we wrote down the solution for that V l theta and we just convinced ourselves that only l equals 1 contributes and the answer to V 1 theta is basically a polynomial in 1 by r dash. So, having assigned that polynomial is going to be precisely this thing. In fact, you can rather than deriving this you can go ahead and substitute this in at this answer here and you will see it is an identity.

In fact, think of that as an exercise rather than going through all those steps in a very systematic way you could you know take. So, a lot of it on faith and randomly verify cross check whether some of these things make sense by you know back substitution like this. You substitute 4.199 into 4.198 and convince yourself that it is an identity.

So, the point is that you will see that this choice is consistent with the idea that at infinity if $r = \infty$ the velocity of the fluid should be what it was all along when that ball was not there which is u vector ok, so in the u in the z direction.

So, and in fact that is what you see here from this.

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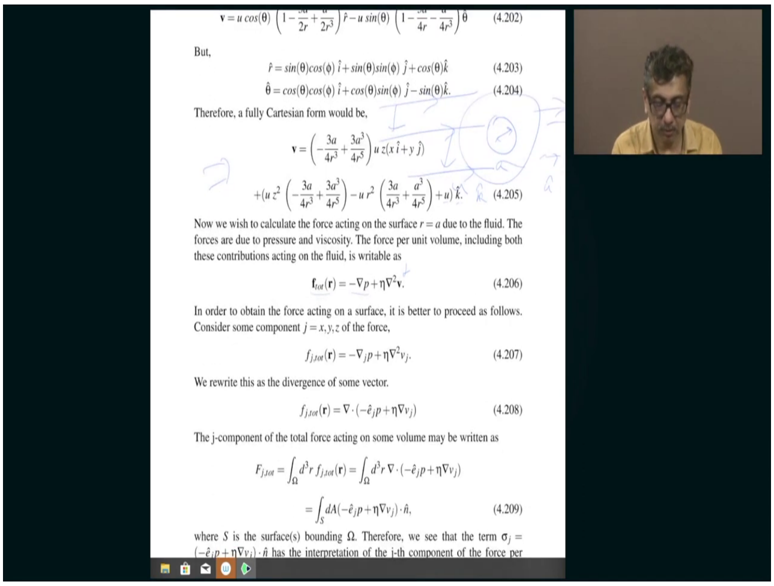

So now, you see this is in some kind of a polar form if you really want it back in Cartesian form you can go ahead and bring it back to the Cartesian form and the Cartesian form for the velocity would look like this ok. So that means, when r equals infinity all these terms drop out and you only get u k hat ok

So, that at far away situations far away from the sphere, so r equal to a is the sphere ok r much greater than a is far away from the sphere. So, if r r tends to infinity, so you can convince yourself that the velocity is u times k hat ok. So, this is going to tell you exactly what is the velocity field; that means the velocity of the fluid as it flows around the sphere.

So, this is. So, it is quite nice to know that you can explicitly write down such a formula. So, if there is a sphere sitting here. So, you can even if you do not follow that derivation fully or not at all you should certainly appreciate the final answer here. So, there is a sphere sitting here with radius a and there is a fluid flowing with velocity u.

So, it is it comes from infinity in the k direction with velocity with speed u and goes around the sphere and then again you know flows away to infinity with the same speed far away. So, far away on the left side the speed was u far away on the right side the speed was u. So, the question is what does the speed look like near the sphere and the answer is this.

It is really nice to know that you can write down such an answer ok, so the thing is that. So now, what we want to know is that the we also want to know what is the force acting on the sphere because of this fluid and the force acting clearly is due to two parts one is.

So, the force acting per unit volume at any point r in the fluid is due to two parts one is due to the pressure gradient. So, minus grad p and the other is basically the internal viscosity. So, one layer is rubbing against another layer. So, that also causes a force to act at a point in the fluid because layers are rubbing against each other. So, there is one contribution due to pressure gradient the other contribution is basically due to the viscosity.

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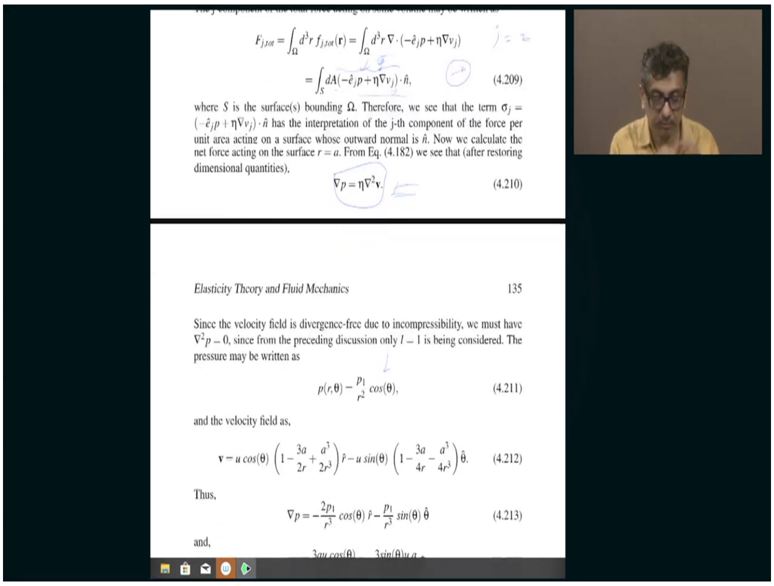

So now, what we really want to do is we want to calculate the z component of the net force acting on the sphere. So, what we have to do is we have to calculate. So, this force per unit volume. So, you integrate over the volume of that small sphere of radius r. So, that will tell you the net force acting on the sphere. So, it is force per unit volume then you integrate over the volume.

So, this will be the net force and then you can re-express this in terms of the surface integral on the surface of the sphere. And you will see that this has the familiar interpretation of a some kind of a stress tensor. So, you have a matrix here which is of that form and ok. So, this is a jth component of that.

So, this is itself a component and then there is a grad which has another component. So, in some sense that is a matrix. So, that matrix dotted with the normal component is still a vector. So, bottom line this tells you the jth component of the total force acting on the sphere and we expect only the z component of this force to survive. That means, j equal to z is the only one which survives because we expect the force to be along the z direction.

So, you see because of that Navier-Stokes Reynolds numbers expansion we have this relation. So, now what we want to do is we see we got v from v we want to get p. So, after we get p we substitute here because now p and v are related because we can restore the dimensional quantities in that Reynolds number expansion formula which tells you how p is related to v.

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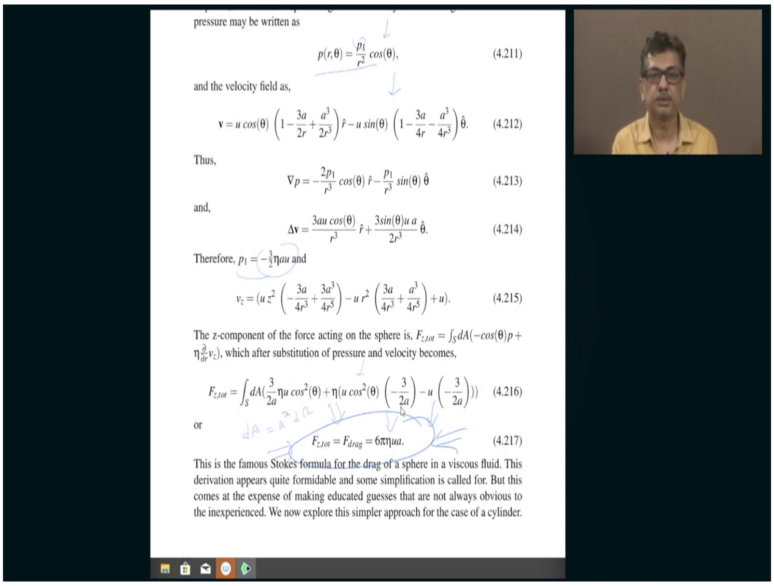

pressure may be written as

$$p(r,\theta) = \frac{p_1}{r^2} cos(\theta), \quad (4.211)$$

and the velocity field as,

$$\mathbf{v} = u\, cos(\theta) \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right) \hat{r} - u\, sin(\theta) \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) \hat{\theta}. \quad (4.212)$$

Thus,

$$\nabla p = -\frac{2p_1}{r^3} cos(\theta)\, \hat{r} - \frac{p_1}{r^3} sin(\theta)\, \hat{\theta} \quad (4.213)$$

and,

$$\Delta \mathbf{v} = \frac{3au\, cos(\theta)}{r^3} \hat{r} + \frac{3 sin(\theta) u\, a}{2r^3} \hat{\theta}. \quad (4.214)$$

Therefore, $p_1 = -\frac{3}{2}\eta a u$ and

$$v_z = (u z^2 \left(-\frac{3a}{4r^3} + \frac{3a^3}{4r^5}\right) - u r^2 \left(\frac{3a}{4r^3} + \frac{a^3}{4r^5}\right) + u). \quad (4.215)$$

The z-component of the force acting on the sphere is, $F_{z,tot} = \int_S dA(-cos(\theta) p + \eta \frac{\partial}{\partial r} v_z)$, which after substitution of pressure and velocity becomes,

$$F_{z,tot} = \int_S dA(\frac{3}{2a}\eta u\, cos^2(\theta) + \eta(u\, cos^2(\theta) \left(-\frac{3}{2a}\right) - u \left(-\frac{3}{2a}\right))) \quad (4.216)$$

or

$$F_{z,tot} = F_{drag} = 6\pi\eta u a. \quad (4.217)$$

This is the famous Stokes formula for the drag of a sphere in a viscous fluid. This derivation appears quite formidable and some simplification is called for. But this comes at the expense of making educated guesses that are not always obvious to the inexperienced. We now explore this simpler approach for the case of a cylinder.

So, from here you can conclude that p has to have this form ok and because v has this form and you compare both you will see that this constant is perfectly determined like this ok. So, it is explicitly determined.

So, now you have an explicit formula for the pressure acting in the fluid and also the velocity of the fluid ok. So now, you can go ahead and calculate the net force acting in the z direction and that is basically you just put j equal to z and that is what that is ok.

And then you go ahead and calculate that integral and you will find that this is nothing but, so this is dA would be what. So, it will be A square d omega ok. So, that is your surface area I mean there is a surface area element. A square d omega d omega is a solid angle sin theta d theta d phi, but phi integral is 2 pi and sin theta d theta is basically d cos theta and then you integrate and you will get this answer ok.

So, this is the Stokes' drag yeah. So, it is a lot of work it is a tremendous amount of work to get a formula that you already know from your high school days.

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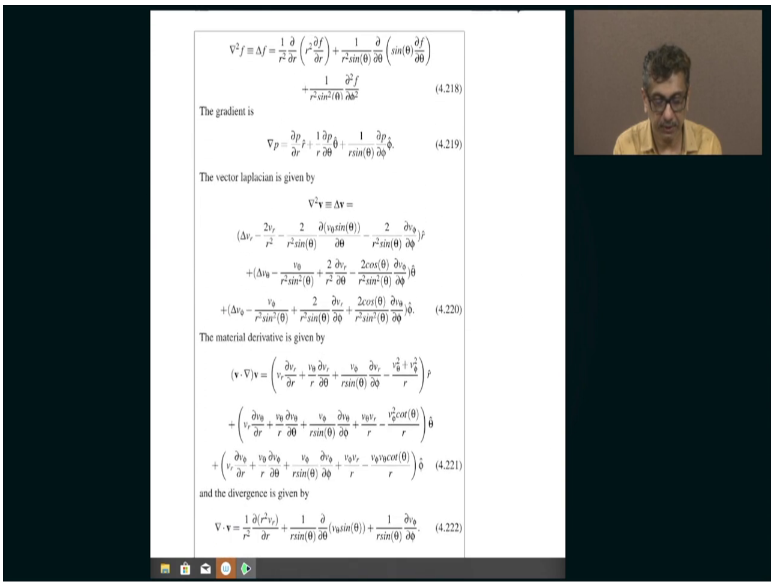

But I just wanted to point out that certain results which you are forced to memorize are actually very deep.

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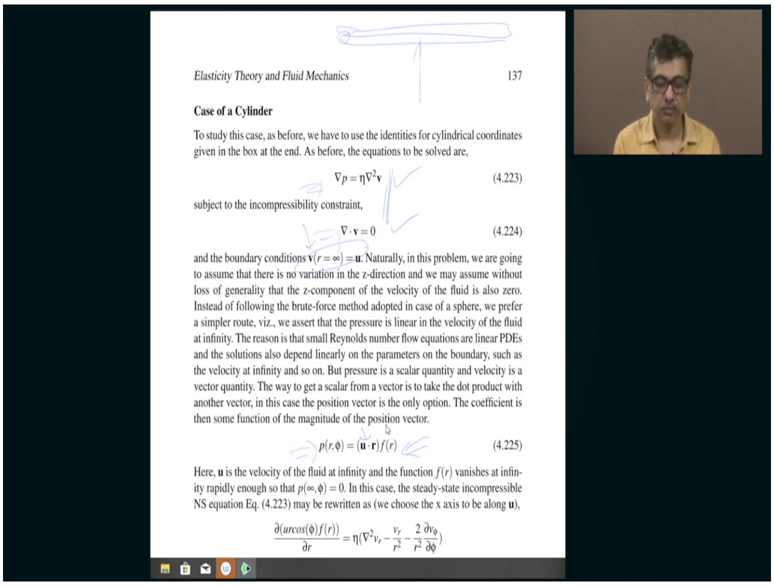
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Case of a Cylinder

To study this case, as before, we have to use the identities for cylindrical coordinates given in the box at the end. As before, the equations to be solved are,

$$\nabla p = \eta \nabla^2 \mathbf{v} \quad (4.223)$$

subject to the incompressibility constraint,

$$\nabla \cdot \mathbf{v} = 0 \quad (4.224)$$

and the boundary conditions $\mathbf{v}(r = \infty) = \mathbf{u}$. Naturally, in this problem, we are going to assume that there is no variation in the z-direction and we may assume without loss of generality that the z-component of the velocity of the fluid is also zero. Instead of following the brute-force method adopted in case of a sphere, we prefer a simpler route, viz., we assert that the pressure is linear in the velocity of the fluid at infinity. The reason is that small Reynolds number flow equations are linear PDEs and the solutions also depend linearly on the parameters on the boundary, such as the velocity at infinity and so on. But pressure is a scalar quantity and velocity is a vector quantity. The way to get a scalar from a vector is to take the dot product with another vector, in this case the position vector is the only option. The coefficient is then some function of the magnitude of the position vector.

$$p(r,\phi) = (\mathbf{u} \cdot \mathbf{r}) f(r) \quad (4.225)$$

Here, $\mathbf{u}$ is the velocity of the fluid at infinity and the function $f(r)$ vanishes at infinity rapidly enough so that $p(\infty,\phi) = 0$. In this case, the steady-state incompressible NS equation Eq. (4.223) may be rewritten as (we choose the x axis to be along $\mathbf{u}$),

$$\frac{\partial(urcos(\phi)f(r))}{\partial r} = \eta(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi})$$

And they come about for very deep reasons and this is one of them ok. So, you might be wondering you know this is too much and this is too much effort it is too technical is there a simpler way of getting this.

So, that is what I have said in this last paragraph you can, in fact many of the books that get this result through a derivation actually do not go through all these steps explicitly. They reduce the number of steps by making some assumptions which they finally, do not justify properly. So, it is just that the reader has to and of again believe some of those statements.

So, if you are going to believe some of those statements then you might as well believe this itself. So, why bother trying to go through some steps, where a lot of them are again things you have to memorize without proof. So, you might as well memorize this without proof..

So, that is the reason why I did not want to do that initially. So, I have spelled out all the steps that are involved in deriving Stokes' drag. But however, it is still worthwhile to see if now that we at least know we have some confidence that you can explicitly derive some formula if you wanted to. So, it is desirable to see if there is another version of this calculation which uses fewer steps.

So, in fact there is such a version and I am going to use that to calculate Stokes' drag not on a sphere, but on a cylinder. So, you have the same problem and you have this infinitely long cylinder ok and there is some kind of fluid that is flowing past the cylinder ok.

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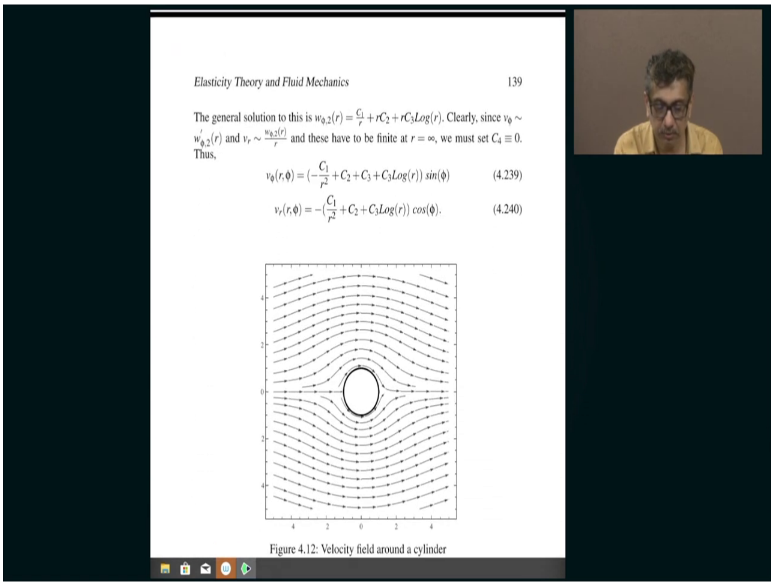

So, so you see in this situation. So, you see this is the cross section of the this is the cross section of the cylinder. And so there is a infinitely long cylinder and there is fluid flowing past this.

And clearly the fluid is going to drag this sphere around as it goes around this. So, so this is a cylindrical problem is cylindrical symmetry rather than spherical symmetry. So, the simplification. So, these are the standard starting equations this will this is because of the Reynolds number expansion with the units restored and this is the due to the incompressible fluid and there is this relation. So, we have to combine these two.

So, we have to solve these two with this additional as a assumptions. So, the question is you know. So, we did a very systematic for the case of the sphere we actually did a very systematic job of solving this by expanding in powers of you know the basis functions and all that. So, if you did not want to do that you have to simplify things further by making some answers.

So, this is an simplifying assumption that we say that this pressure is expressible in terms of the position vector in this way. So, that is exactly why I am saying that it is kind of not very convincing. So, most of the books actually start something you know they make some statements like this and then they proceed and that simplifies the equations a lot you get your final answers very quickly, but it is not at all clear why this should be the case.

But at the same time it is it is not that unreasonable because after all you see the pressure yeah. So, basically what this is saying is exactly what I actually derived. So, what this is saying is that it only involves cos theta it does not involve cos square theta or anything any higher power. So, what this is this is just cos theta its u dot r is basically u r cos theta.

So, it automatically assumes that p is proportional to p 1 cos theta p 1 cos theta is just cos theta. So, I in the case of the sphere I actually derived that I showed that all higher ls do npt contribute only the l equal to 1 contributes.

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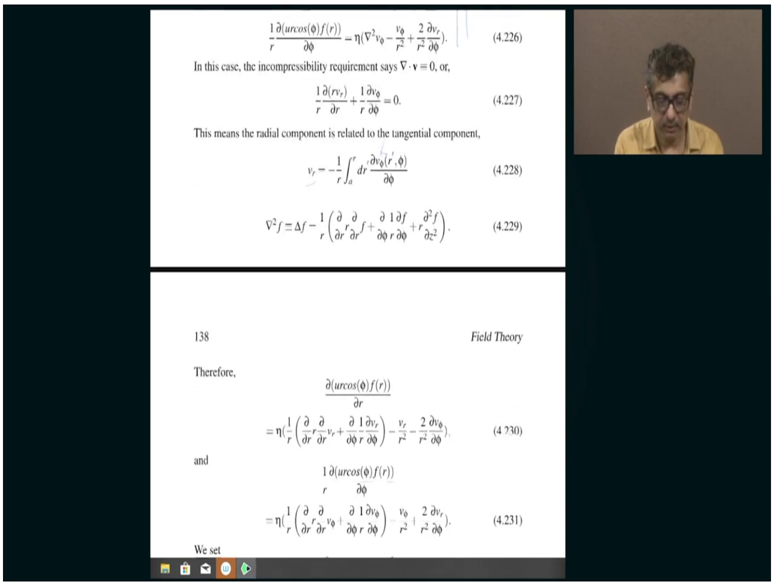

Whereas, here it is kind of assumed that that is the case and if you of course, assume that that is the case you can necessarily it is true that you can simplify this a lot and this thing gets simplified very quickly and as usual you express your radial in terms of the angular.

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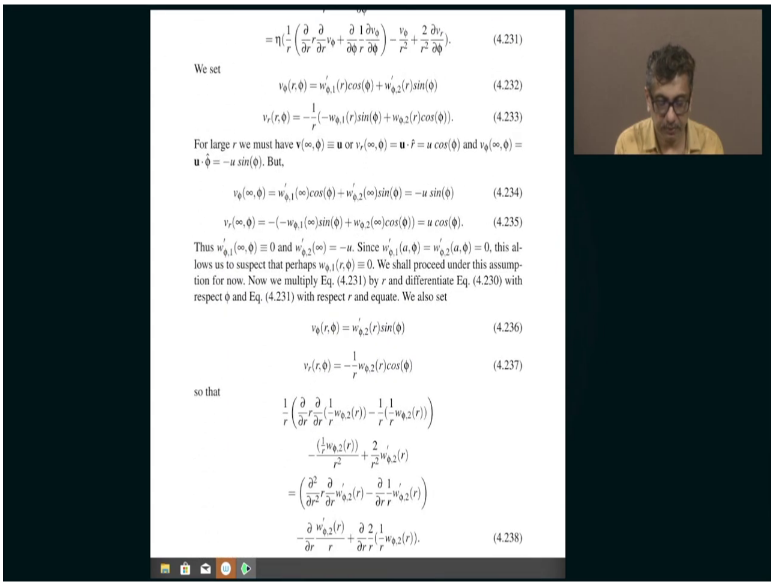

And then you go ahead and rewrite.

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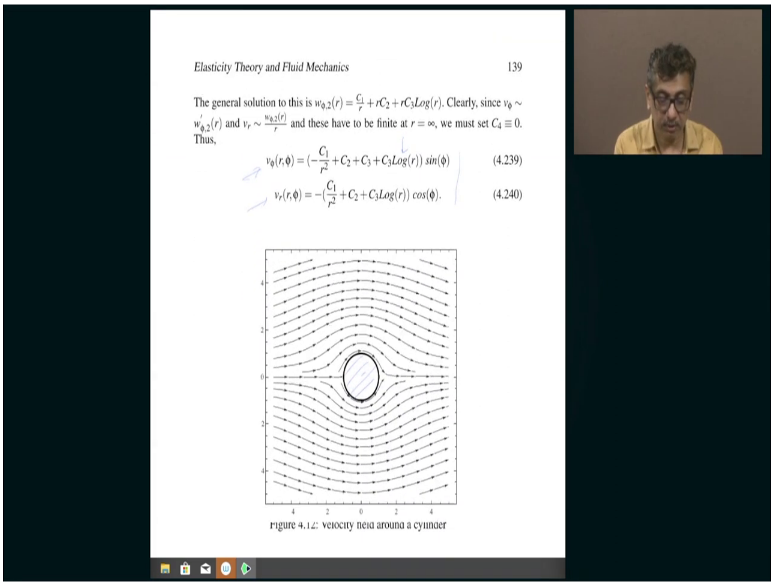

Yeah it is still long, but you can go ahead and in the case of cylindrical you get these two components in terms of the coefficients.

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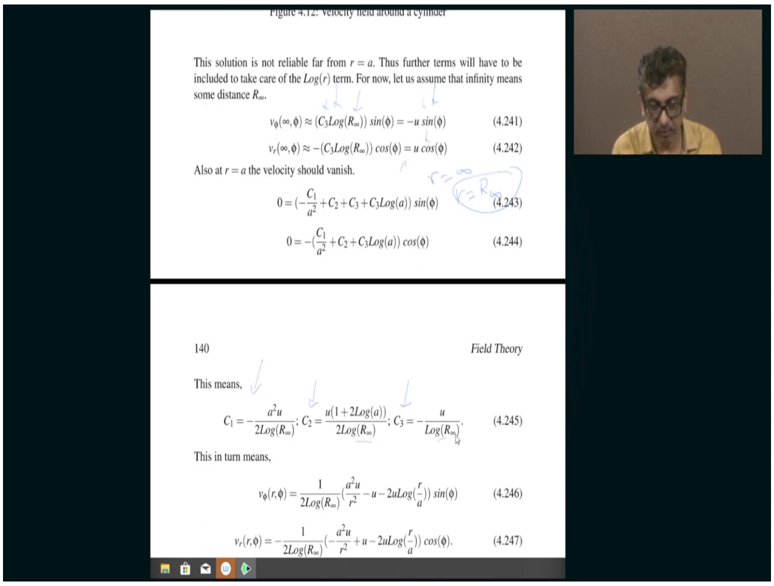

So, then you can go ahead and. So, in the case of the cylindrical term there are some a peculiarities which are not there for the sphere,

For example, you get this log term. So, this log term becomes unstable when r is very large. So, the basically the reason for this log term is because we have ignored the convective derivatives so. In fact, Oseen who is a researcher in this field has pointed out that if you include the convective derivative then you get some sensible terms even at r.

So, at R equal to infinity these equations do not converge to what we expect because of this log term. So, what we have to assume is that what Oseen has shown is that basically when you include the convective derivative R equals infinity effectively becomes some r equals R infinity which is a some very large term.

So, that I mean I am just being sloppy here. So, what basically it means is that you can get away by writing log r, but because you have ignored the convective derivative you should not you are not entitled to set us lowercase r equals infinity. You are only allowed to go up to R some large value of R which is R subscript infinity. So, that serves as a proxy for an small R being infinity.

So, the bottom line is that when this is the case you expect this to be you know u theta and so on. So, that is the way the C's are determined. So, the C's are determined by forcing v phi and v r to be these two known values, but not at R equal to infinity, but it is proxy value which is r equals R capital R subscript infinity which is large, but not actually infinite.

But then also of course, the velocity should vanish on the surface of the sphere because kind of there is no slippage condition. So, the cylinder is stationary. So, the fluid should be stationary along with the sphere when they are touching. So, from that you can fix the remaining coefficients.

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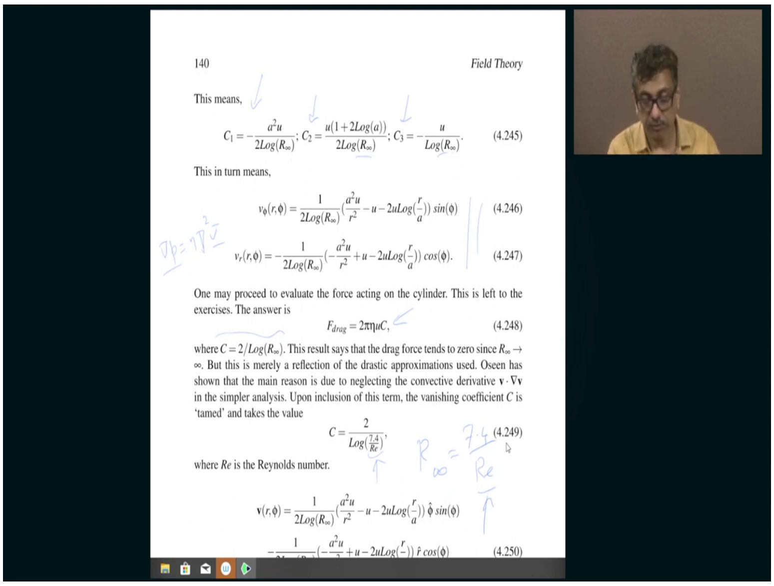

So, all these coefficients get fixed by these two requirements and the only thing is that you will have to deliver this peculiar proxy for infinite distance which is R subscript infinity ok. So but then you will see that finally, drops out of your calculation for the drag.

So, you will get an answer for the velocity and therefore, from the earlier result from the pressure also if you know the velocity you can get pressure because you have that grad p equals eta del squared v. So, if you know velocity you can solve this and get pressure, but all of them will involve this R infinity, but then later you will see that when you actually evaluate the drag that yeah. So, it will appear in the this form it will involve R infinity as a multiplicative constant ok.

So, it will come out as this and dimensionally also the viscosity in two dimensions is kind of different from what it is in case of three dimensions. So, there are two dimensional because its cylindrical symmetry the z does not play a role ok. So, it is kind of I mean the z is along the cylinder.

So, along the cylinder does not play a role. So, it is effectively a two dimensional problem, but bottom line is that this proxy for infinite distance Oseen has shown that if you properly study this by including the a convective derivative. This proxy for infinite distance of course, also in dimensionless units will be something which is inversely proportional to the Reynolds number.

So, remember that all this analysis is valid for low Reynolds number. So, we are expanding powers of the Reynolds number. So, therefore, this r infinity which is according to Oseen 7.4 by Reynolds number. So, bottom line is when r Reynolds number is small which is the regime in which this is valid this proxy for infinity actually is a divergent which is what we expect alright.

So, you get us less familiar formula for the Stokes' drag of a cylinder a placed in a moving fluid. So, this is not what you learned in high school because it has this peculiar thing that Reynolds number is involved, but for sphere that drops out to lowest order which is why you learn it in school ok.

So, I have come to the end of a fluid mechanics and elasticity theory.

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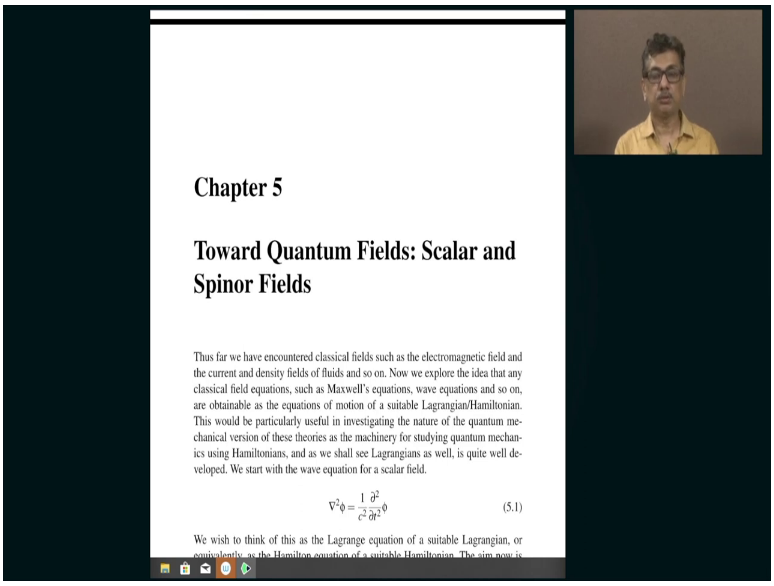

So, in the next class I am going to explain you know how to motivate the introduction of quantum fields. So, till now we have all we have studied there is no mention of quantum mechanics anywhere. So, it is all classical.

So, in from the next class onwards I will explain to you how it is that many of these concepts which involve a infinitely many continuous degrees of freedom, for of classical systems can now we studied quantum mechanically. So, if there are point particles you know how to study you know how to go from classical mechanics to quantum mechanics.

But if you have infinitely many particles a infinitely many classical degrees of freedom and if that infinity is of the continuous kind making it a field. So, it becomes important for us to you know what it is we are expected to do in order to study that quantum mechanically.

So, of the most important of these applications would be to study the Maxwell's study Maxwell's equations quantum mechanically. So that means, so if you look at empty space electromagnetic fields cause electromagnetic waves which are classical. But then if you study electromagnetic fields quantum mechanically you do not get electromagnetic waves you get quanta of energy.

So, basically you get discrete energies and these are called photons. So, this would be a first rigorous demonstration of the fact that radiation is actually made of quanta and this is first famously demonstrated or realized by Einstein in his theory of photoelectric effect. So, which we all learn in school.

So, a photoelectric effect simply cannot be explained if you posit that electromagnetic fields are classical waves. So, it can only be explained by invoking the quantum theory of the electromagnetic field or a quantum theory of radiation. So, the question is what is the logical justification for a quantum theory of radiation and that is basically involves studying the equations of electric and magnetic fields this Maxwell's equations not classically, but quantum mechanically.

So, that is going to be our a important goal in the next few lectures. So, I hope you will join me for that and.

Thank you for listening.