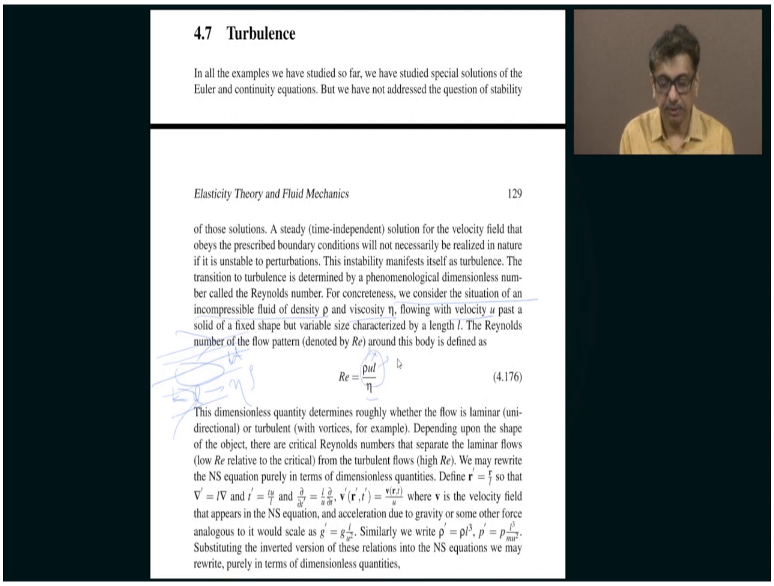
Dynamics of Classical and Quantum Fields: An Introduction

Prof. Girish S. Setlur
Department of Physics
Indian Institute of Technology, Guwahati

Fluids
Lecture - 20
Stokes' Drag - I

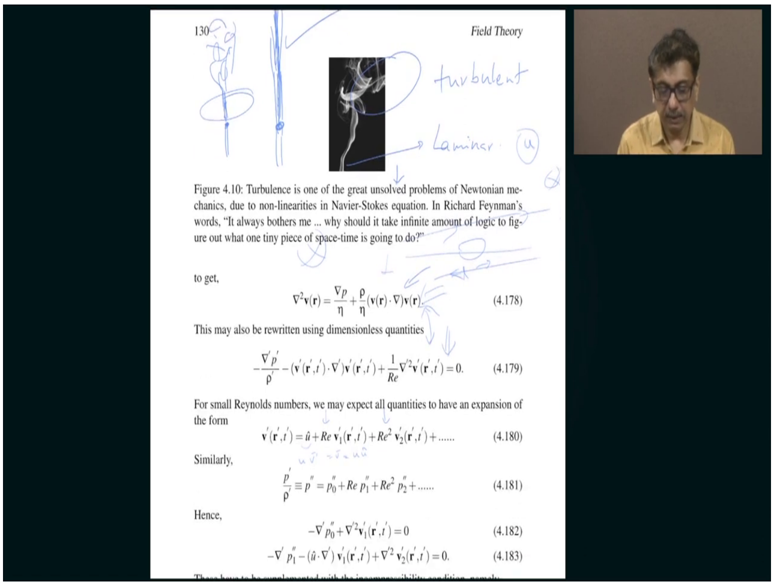
(Refer Slide Time: 00:31)



Today we will discuss a new topic and this is likely the last subtopic in the subject of Fluid Dynamics and this topic is basically Turbulence. So, a turbulence is not possible without viscosity ok. So, we have to necessarily incorporate that the idea that fluid is not ideal; that means, that layers of the fluid drop against each other and there are dissipative phenomena in the fluid. So, that is one of the main causes of turbulence.

So, let us try to understand, what is turbulence and why it is important and also extremely difficult to handle. In fact, before I proceed with the technical description it is worthwhile pointing out certain observations that is prominent physicists have made about turbulence.

(Refer Slide Time: 01:21)



For example, you know Richard Feynman in his very famous and much loud you know series of books on Physics. So, he talks about Feynman Lectures of on Physics he says that, "it always bothers me why should it take an infinite amount of logic to figure out what one tiny piece of space time is going to do?" Ok.

So, I mean what he meant by that is that it appears that turbulence is one phenomenon which is which kind of transcends length scales and time scales so; that means, what happens at the shortest length scales influences what happens at larger length scales that is somewhat unusual in physics and, but it is quite standard in the phenomenon of turbulence. And that is what makes it hard for people to study turbulence rigorously.

So, let us try to understand mathematically what is turbulence and how best we can go about studying it. So, the idea is that you see these Navier-Stokes equations and the continuity equations put together presumably form a complete description of the fluid that is assuming you know our continuum description of the fluid is valid at all length scales, but that may not be necessarily correct, but let us go along with that.

So, if that is the case then you see the point is that these two equations admit solutions that correspond to steady state; that means, that you can have a situation where the velocity distribution is independent of time and the density distribution is independent of time. So, the density of the fluid changes from point to point, but not from time to time, similarly the velocity of the fluid changes from point to point, but it is independent of time.

So, such distributions are called steady state distributions and you can always find examples with appropriate initial conditions and boundary conditions, you can find Navier-Stokes and the continuity equations obeying these kinds of expectations. But now the question is that just because some solutions exist for these equations it is not clear that such solutions are seen in nature. So, in other words so, we should really be seriously asking ourselves that do we really see velocity distributions of a fluid that change from point to point, but are strictly independent of time.

So; that means, if you just sit at one point the velocity of the fluid is strictly the same at all times. So, the other thing is a same with the density. So, you will see that is unlikely to be true. So, let me give you examples of the sort of things I am talking about. So, in Indian households it is very common to light these [FL] so, which are incense sticks and you can see that its a very common occurrence in all Indian households that when [FL] are lit you see a narrow column of smoke ascends from the a stick upward.

But if you are near the stick that column of smoke appears completely straight and vertical, but as it ascends it kind of dissipates and the smoke kind of stops going straight it is starts to going in a haphazard way ok. So, that is what turbulence is basically its a so, the straight smoke that emanates from the [FL] near the place where its lit that is called laminar flow and then later on when it ascends it is called turbulent flow.

So, in fact, I have a picture here so that is what this is. So, this is when it has already reached turbulence. So, below this there is this [FL] that is lit here. So, you can see that so, this is turbulent and this is laminar or even below this is laminar ok. So, in other words the so, it ascends like this laminarly and then finally, it kind of does that ok.

So, the point is that you see your the. So, the point is that these Navier-Stokes and the continuity equations admit solutions which correspond to this situations, this situation namely where the laminar flow extends to infinity ok. So, in fact, your Navier-Stokes and continuity equation will allow this as a possibility, but this is never seen in nature and the question is we have to understand why it is never seen in nature and that is because of turbulence.

So, the bottom line is that just because some distribution is a solution of the Navier-Stokes equation does not mean that solution is stable to perturbations. See, if a solution of Navier-Stokes equation has to be seen in nature not just in mathematical calculation if it has to be seen in nature, it has to be stable to perturbations; that means, if you change something slightly this the solutions also should only change proportionately slightly.

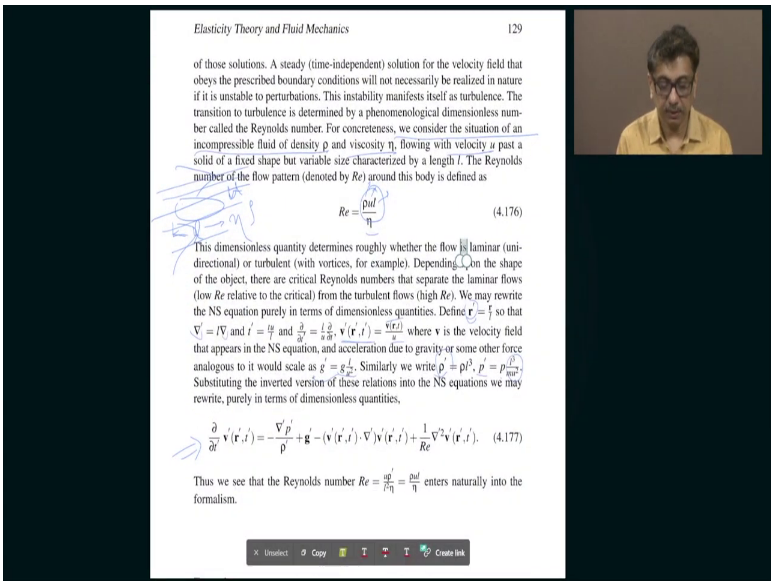
So, if the solutions change drastically when you make some small changes; say for example, you just lightly blown to the that streamline flow just very lightly blown to it and see the streamline flow should if it changes only slightly then it is called stable. So, the moment you blow slightly no matter how slight it is if your solutions change drastically then its called unstable.

So, you will see that in many examples the solutions are unstable and when solutions are unstable we say that we have encountered turbulence ok. So, now, how do we understand turbulence mathematically? So, to this end we have to introduce certain an important notion called Reynolds number ok. So, let us read the sentence. So, imagine a situation where there is an incompressible fluid of density rho and viscosity eta, flowing with a velocity u past some solid of a fixed shape, but variable size.

So, imagine, some shape like this and this is your size of that and there is your fluid that is flowing like that across this ok. So, the speed of this fluid is u and viscosity of this fluid is eta and density is rho and this is your length of those that. So, now, you can ask yourself see what is the. So, one can define an dimensionless quantity called Reynolds number and that dimensionless quantity is defined as you multiply rho with the speed of the fluid, rho is the density of the fluid and l is the length of that obstacle, the basically it is the linear dimensions of the obstacle and eta is your viscosity.

So, dimensionally you can verify that this is in fact, a dimensionless quantity and this is called Reynolds number.

(Refer Slide Time: 10:07)



So, you might be wondering why did I introduce this peculiar concept, because it seems out of the blue I mean like it has to be motivated and you will see very soon that they it has a very good reason why such a concept will appear in your equations; namely, your Navier-Stokes equation when you recast them in a certain way you will encounter this number naturally. So, that is what I want to convince you today ok. So, how do you do that?

So, first let us. So, the first step is to realize that once you invoke a length scale like l which corresponds to the linear dimensions of an obstacle then you have sufficient number of dimensional parameters in your formalism to recast all the independent variables in terms of these dimensional quantities, so that you can render them dimensionless.

So, what I mean by that is you know suppose you have you know typically in when you go to a flower market, see the especially in South India where you have these flower sellers was sitting on this on the footpath and have all these flowers to sell and you ask them you know how much is it they will say you know 10 rupees for one [FL] that is what we say in Kannada. So, that corresponds to the length from the tip of your finger middle finger to your elbow.

So, basically what they have done is they have used some length of some physical object like the forearm, forearm length is the unit and they are expressing all other lengths in terms of the length of the forearm. So, that is why as a result they have rendered the length concept dimensionless.

So, if they say 5 [FL] so; that means, it is 5 times the length of the forearm. So, you that way you can render pretty much any dimensional quantity dimensionless by writing it as the multiples of you know appropriate quantities which are which appear in the physical world. So, if its length it can be the length of some physical object. So, density would be therefore, mass per unit length cubed.

So, length we have already defined in terms of the physical object. So, then we can define mass also indirectly if you know the density. So, that way you can density of any object can be multiples of the density of the fluid. So, like that you can define a pretty much every. So, there are three independent dimensional quantities length, mass and time and so, you have you know sufficient number of dimensional quantities to render everything else dimensionless.

So, given that observation we can go ahead and define a dimensionless quantity called r dash. So, if r is the position vector of some point in space. So, that position vector will have some direction and length, I mean it will have a size magnitude. So, you divide by the characteristic length l of the obstacle that you are considering. So, you will get a dimensionless vector which is called r dash.

So, similarly gradient will also the, because gradient is inversely related to length it will have this property that it is it will be. So, its kind of length times something inversely related to length is dimensionless. So, that is what I have called as grad dash. So, that is basically grad with respect to the r dash.

So, similarly with time also you can do the same thing. So, if so, with time you see you now have another dimensional quantity which is the speed of the fluid flow which is u and u by l has dimensions of inverse time. So, therefore, t into u by l is dimensionless. So, that I have called it as t dash. So, therefore, d by d t dash will be similarly related to so, d by d t dash will also be dimensionless because t dash is dimensionless.

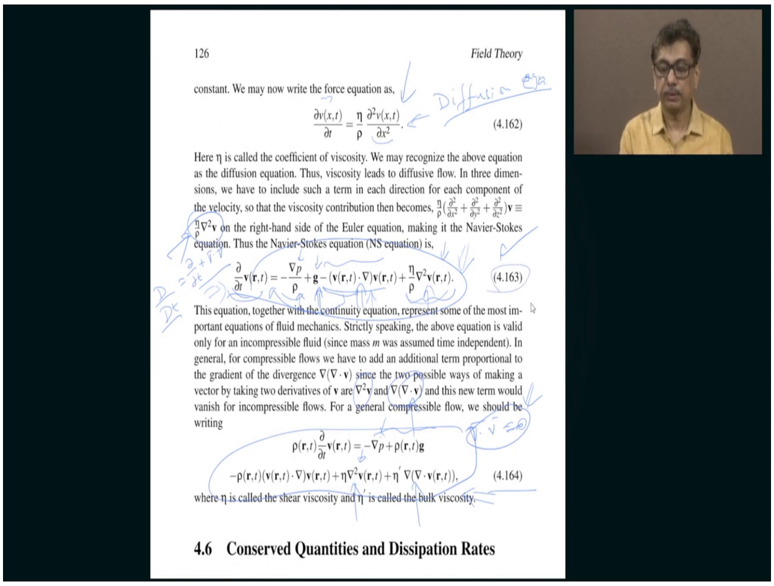
So, clearly, the velocity of the fluid if it is v, I divide by u and I get v dash which is the dimensionless version of the velocity of the fluid. So, now, there are other dimensional quantities in the Navier-Stokes. So, if the body forces like the weight of the fluid. So, then the acceleration of due to gravity is also a dimensional quantity that appears in your equation.

So, we can also render that also dimensionless by multiplying by something which has dimensions of inverse acceleration. So, that is basically l divided by u squared ok. So, g times l divided by u squared is dimensionless and that we have called it as g dash. So, similarly, we can define dimensionless version of the density which we define as rho times l cubed.

So, because rho is a mass per in this case it is number density not mass density it is number of you know whatever particles per unit volume. So, then you multiply by volume you just get a number. So, that is basically what rho dashes is dimensionless. So, lastly, you can redefine pressure also to be dimensionless. So, you define p dash as p times something which has dimensions of pressure and then you will see that in terms of these quantities it is defined like that ok. So, having done that you see this m appeared in your equations also ok it is m.

So, bottom line is that you can go ahead and so, recast your Navier-Stokes equation like this. So, remember what that was. So, that I should go all the way back here because I think we have come a long way. So, we have forgotten where that was and it is here, this one.

(Refer Slide Time: 16:42)



126 *Field Theory*

constant. We may now write the force equation as,

$$\frac{\partial v(x,t)}{\partial t} = \frac{\eta}{\rho}\frac{\partial^2 v(x,t)}{\partial x^2}. \quad (4.162)$$

Here η is called the coefficient of viscosity. We may recognize the above equation as the diffusion equation. Thus, viscosity leads to diffusive flow. In three dimensions, we have to include such a term in each direction for each component of the velocity, so that the viscosity contribution then becomes, $\frac{\eta}{\rho}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})\mathbf{v} \equiv \frac{\eta}{\rho}\nabla^2\mathbf{v}$ on the right-hand side of the Euler equation, making it the Navier-Stokes equation. Thus the Navier-Stokes equation (NS equation) is,

$$\frac{\partial}{\partial t}\mathbf{v}(\mathbf{r},t) = -\frac{\nabla p}{\rho} + \mathbf{g} - (\mathbf{v}(\mathbf{r},t)\cdot\nabla)\mathbf{v}(\mathbf{r},t) + \frac{\eta}{\rho}\nabla^2\mathbf{v}(\mathbf{r},t). \quad (4.163)$$

This equation, together with the continuity equation, represent some of the most important equations of fluid mechanics. Strictly speaking, the above equation is valid only for an incompressible fluid (since mass m was assumed time independent). In general, for compressible flows we have to add an additional term proportional to the gradient of the divergence $\nabla(\nabla\cdot\mathbf{v})$ since the two possible ways of making a vector by taking two derivatives of $\mathbf{v}$ are $\nabla^2\mathbf{v}$ and $\nabla(\nabla\cdot\mathbf{v})$ and this new term would vanish for incompressible flows. For a general compressible flow, we should be writing

$$\rho(\mathbf{r},t)\frac{\partial}{\partial t}\mathbf{v}(\mathbf{r},t) = -\nabla p + \rho(\mathbf{r},t)\mathbf{g}$$
$$-\rho(\mathbf{r},t)(\mathbf{v}(\mathbf{r},t)\cdot\nabla)\mathbf{v}(\mathbf{r},t) + \eta\nabla^2\mathbf{v}(\mathbf{r},t) + \eta'\nabla(\nabla\cdot\mathbf{v}(\mathbf{r},t)), \quad (4.164)$$

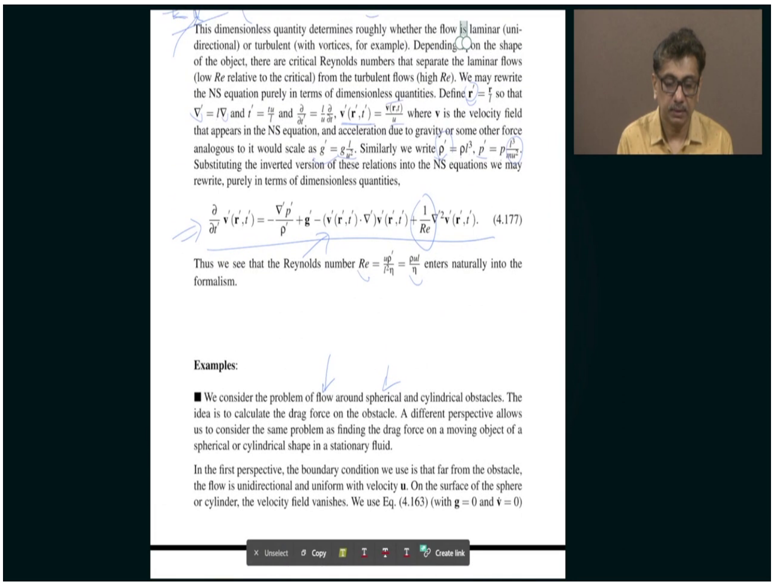
where η is called the shear viscosity and η' is called the bulk viscosity.

4.6 Conserved Quantities and Dissipation Rates

So, this is Navier-Stokes because you see this is tells you the rate of change of the velocity and it is related to all bunch of forces like body force, pressure gradients, this is the convective derivative and this is the viscosity here. So, now what we are doing is basically we are rewriting this in such a way that all the quantities whether it is dependent variable or independent variable they are all rendered dimensionless.

So, that is what we are trying to achieve here, we are trying to convert this equation 4.136 which is Navier-Stokes equation into a form where all the variables whether it is independent or dependent variable they are all made dimensionless.

(Refer Slide Time: 17:24)



So, now how do you do that you just rewrite all your rho's and p's and g's and in terms of the corresponding primed values and the primed values are defined like this. So, the gradients are also primed expressed in terms of the primed values. So, when you do that you get a version of Navier-Stokes equation which involves only the primes, both the independent variables are prime and also the dependent variables are prime.

But then now you will see that this when you do it this way you will get an. So, clearly every term here is dimensionless including this one, but then this one is the only one which will involve viscosity and that term will actually appear as 1 by R e where R e is what we have been calling Reynolds number. So, it will appear this way ok. So, $R_e = \frac{\rho u l}{\eta}$.

So, you can see that is the reason why I invoked Reynolds number in the beginning you would have rightly suspected or you know wondered why I should introduce such a arbitrary concept without any motivation, but now it is you can see the motivation now once you rewrite Navier-Stokes equation dimensionlessly it will appear naturally. So, Reynolds number appears naturally. So, and this is defined as Reynolds number ok.

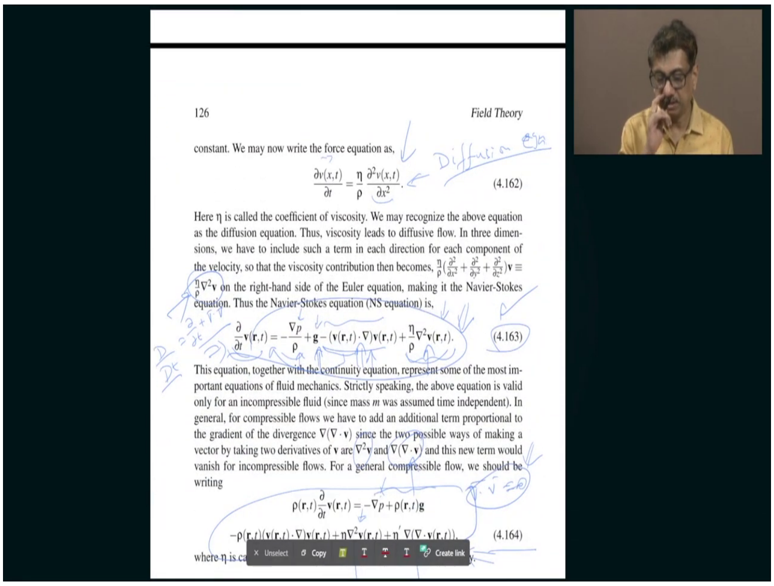
So, now let us go ahead and try to see how to study some. So, in other words now we want to analyze this equation under certain limiting cases. So, we want to consider situations where the Reynolds number is small ok. So, that is corresponds to laminar flow and when Reynolds number is large which will correspond to turbulent flow. So, that is something we want to analyze.

So, we will be doing expansions in powers of Reynolds number and we will try to see what that tells us. So, there will be leading term which corresponds to laminar flow and then successive terms will correspond to turbulent flow. So, now, so let us focus on examples. So, the example is that let us consider a problem of a flow around a spherical and cylindrical obstacles.

So, what we want to do is that you see, suppose you have a sphere and you imagine that there is fluid flowing or air flowing across that sphere. So, what is going to happen is that the air experiences drag because of the sphere that is sitting in the middle, alternatively you can imagine that the air is still, but the sphere is falling in air. So, in which case the sphere will then exhibit or experience drag, because it is falling through some air which has some viscosity. So, that is what we want to study ok.

So, we want to study this. So, we want to find the drag force on obstacles because of turbulent or basically because of viscous fluids flowing across it. So, to do that, so let us first consider this equation where we will consider steady state ok. So, let us consider steady state. So, if you consider steady state you will see that the. So, if you look at 4.163.

(Refer Slide Time: 21:28)



So, let us go back to 4.163 ok. So, what is 4.163 basically the Navier-Stokes. So, if you look at Navier-Stokes and you try to look at steady state so; that means, you ignore this explicit time derivative then you will see that that equation can be rewritten in this way ok. So, it can be rewritten in this way. So, we I am ignoring body forces for now. So, there is pressure gradient there is viscosity ok. So, that is the whole idea. So, you have pressure gradients and viscosity. So, you can write it like this.

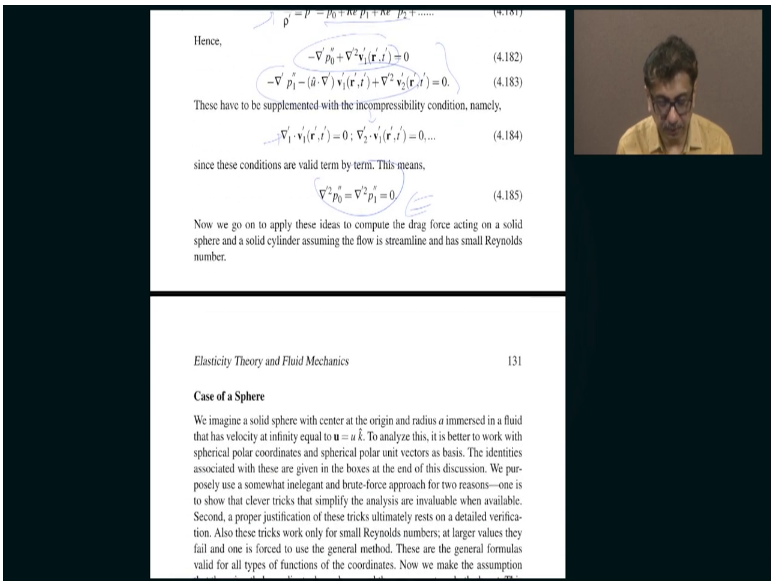
So, now what I am going to do is I am going to try and see if I can recast this equation in terms of dimensionless quantities. So, if I do that I can rewrite this equation in this way. So, I told you earlier how to recast equation in terms of dimensionless quantities. So, you replace t with t dash time some appropriate dimensional quantity and so on. So, then you end up getting this equation.

So, keep in mind that what we are going to assert is the following that. So, there is this obstacle here ok and there is fluid flowing across this. The assumption is that far away from here and here far away from this obstacle the flow is perfectly laminar and the speed is unidirectional with the velocity is unidirectional with speed u ok.

So, in other words if I expand v dash in powers of Reynolds number. So, remember what v dash is, v dash is velocity measured in units of u it is basically the. So, it is small u times v dash is the velocity of the actual velocity of the fluid. So, specifically if there is no turbulence. So, if there is only laminar flow so; that means, the fluid is basically ignoring the presence of the obstacle it is just pretending there is no obstacle it is continuing along that straight line with that constant speed u.

So, if Reynolds number is 0; that means, basically the fluid is ignoring the obstacle. So, in which case the 0th order term will clearly be u hat because v dash is the dimensionless. So, v dash into u is velocity. So, u into v dash is v. So, if a Reynolds number is 0 this is basically u into u cap. So, that is your velocity.

(Refer Slide Time: 24:16)



So, Reynolds number is 0 means the fluid as ignoring the obstacle, but in general you can expect an expansion like this; that means, that you can expand v dash in powers of Reynolds number ok. So, you have the 0th order term proportional to Reynolds proportional to square Reynolds like that. So, similarly with this also I can expand. So, the ratio of pressure and density when it is rendered dimensionless will also have such an expansion.

So, when you go ahead and substitute all that you will get all these equations ok. So, successively you will start. So, if you compare the you know coefficients of the Reynolds number on both sides you end up getting these equations ok. So, I am I have stopped after the first term here I mean. So, now, of course, the we also have to invoke the continuity equation because this is the other one the this Navier-Stokes, but continuity is also important, but then continuity in the case of time independent situations when you have steady state.

So, that is basically a statement that and we are also going to assume incompressibility; that means, density is strictly constant. So, if it is strictly constant then clearly the divergence of v 0 we have said this many times already so, because divergence of v is 0. So, we can go ahead and take divergence of this equation and divergence of this equation and you will conclude that del squared of this p dash dash 0 is 0 and del squared of p dash dash 1 is also 0 ok.

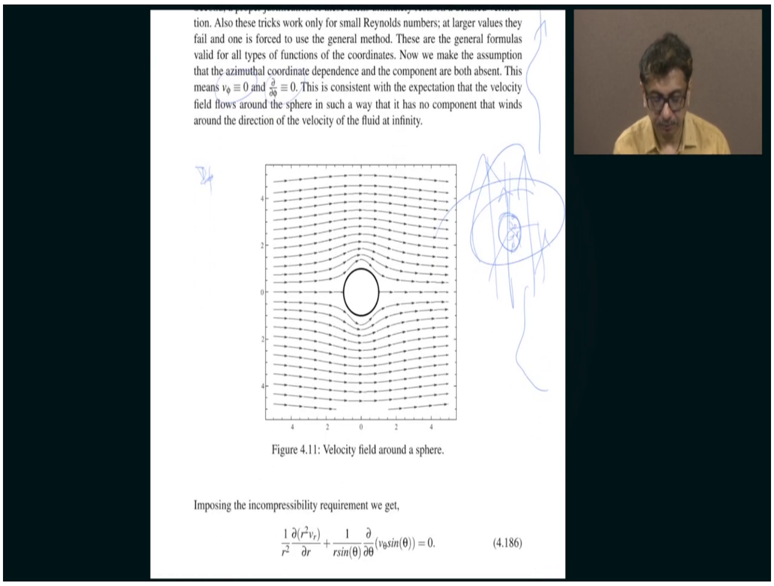
So, I mean these are all intermediate steps in a long calculation that I am going to display now. Basically you will see that the end result that we are going to obtain namely one should not lose sight of what we are trying to achieve and that is we are trying to find the drag force. Suppose, you take a spherical object and drop it in a fluid for example, it will experience some drag.

So, that is called Stoke’s drag and the formula for that is well known to every high school students that is 6 pi eta u where u is the speed of that object. So, basically it will experience a drag and it has that formula which we kind of memorize in our school days and just reproduce in any examination that asks that question, but it is the derivation of that is quite technical and that is why of course, at that level nobody explains that to you, you simply ask to memorize it and accept it as given.

So, this particular course kind of is meant to open your eyes to the fact that those formulas that you thought were very familiar to you actually are quite technical and deep and its derivation is not that easy. So, I think that is the reason why it is worthwhile going through that derivation once, so that you appreciate the depth of this subject. So, that formulas that are seemingly very familiar to you have a very deep reasons for why they are that way ok.

So, now that we have reached this far we can now go ahead and apply. So, this was for any shape in particular we did not specifically assumed sphere or anything till now, although I kept saying sphere, but at if the level of the equations I am not assumed as yet ok.

(Refer Slide Time: 28:07)



So, now, I am going to assume sphere ok. So, now, you imagine that there is a sphere with radius a immersed in a fluid that has some velocity u at infinity ok. So, now, clearly we the best coordinate system is basically the spherical polar coordinates ok. So; that means, there are three types of independent variables r theta and phi.

So, now if you think of you know what r theta and phi is it is like this. So, this is your r and this is your theta and this is your phi ok. So, but if fluid is flowing like this for example, you know across this spherical obstacle, clearly things are independent of phi ok and the fluid does not you know spin around like that it will go around like this I mean it will it does this.

So, it does that or it does not do that ok so; that means, that v phi can be we can confidently set v phi to be 0. So, that is the phi component of the velocity, it looks like nu it is there is a font.

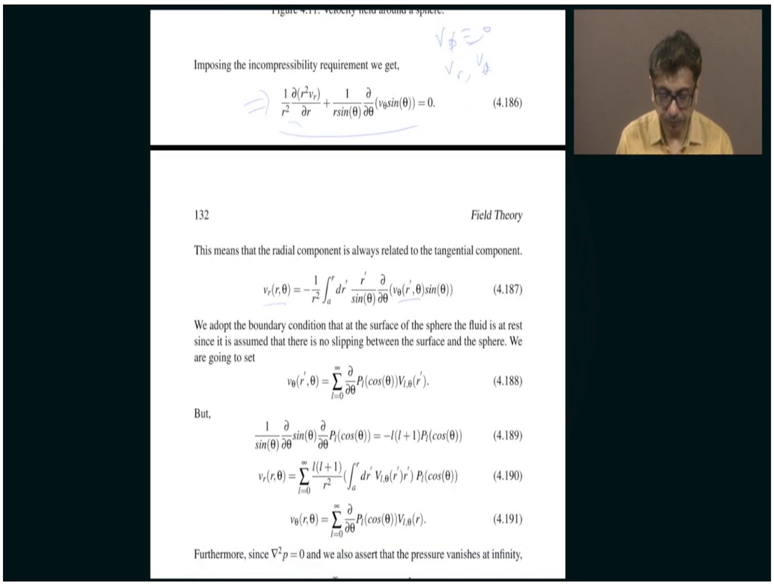
Student: Ok.

Problem there. So, this is actually supposed to be v phi I mean this is there throughout the book. So, please bear with me. So, it is v phi and there is no change relative to this azimuthal angle there ok. So, that is one point to just keep it at the back of your head. So, basically I am just pointing out some interesting useful facts that we have to keep in mind as we proceed.

So, now let us get to some more substantial points and that is that we have to actually impose the assumptions we have made; namely incompressibility. So, incompressibility clearly means that the divergence of the velocity is 0 ok. So, the velocity field does not kind of originate from some more divergence is 0 means that velocity does not converge to a point and diverge from a point because see if velocity converges to a point it means that density at that point keeps increasing with time.

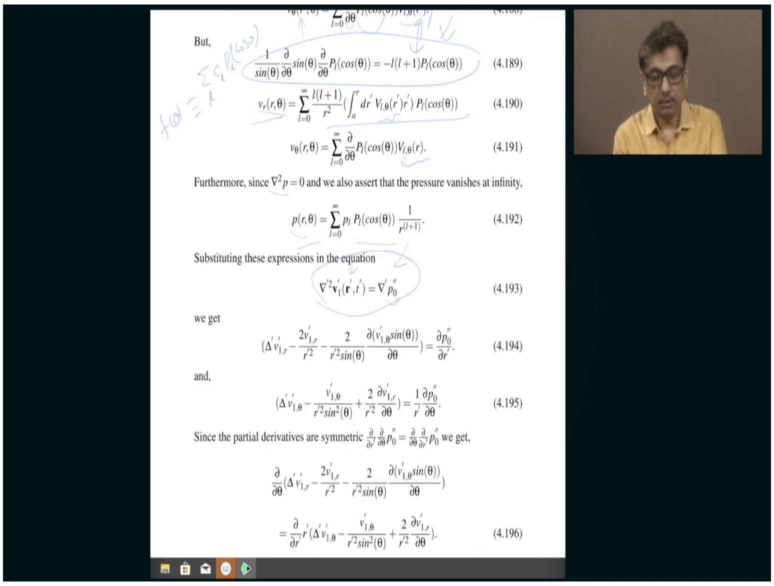
So, if velocity you know diverges from point that there is some kind of depletion at that point density changes at that point. So, the only way you can maintain constant densities by disallowing velocity to have divergence. So, in spherical polar coordinates divergence is 0 means imposing this condition this is divergence is spherical polar coordinates. Keep in mind that v phi was 0.

(Refer Slide Time: 31:23)



So, that the only the other two are there v r and v theta ok. So, its possible to now simply integrate this equation and write down the answer for v of r in terms of v of theta because you see it is just it involves first derivative of v of r with respect to the radial coordinate and clearly you can simply integrate and rewrite this ok. So, now what we are going to do is the following.

(Refer Slide Time: 32:11)



So, we are going to use our idea that the angle dependence ok. So, the angle dependence of any function can be written in terms of basically a basis and the basis we are going to select is the familiar Legendre polynomials. So, because we know that is a P of l cos theta is basically a basis for any function. So, if any function f of theta can always be written as sigma l C of l P l cos theta ok. So, you can always rewrite this like this.

So, it will have all the necessary properties that you expect from something which depends on theta. So, but then I have written it peculiarly like this as derivative of P l cos theta because I will require it later for some other reason ok. So, it is always possible because the derivative of P l cos theta are again linear combinations of other P l s so, which I can always rewrite that as.

So, the reason why I am writing that is because I am going to exploit this identity. So, I have basically what I have done is I have look first I have written v of r in terms of v of theta, but now I am going to expand v of theta in terms of some basis functions which is P l cos theta in this particular case I have chosen some peculiar version of that which involves the derivative with respect to P l with respect to theta, but the reason for that is because I can readily identify the. So, in other words I can exploit this identity ok.

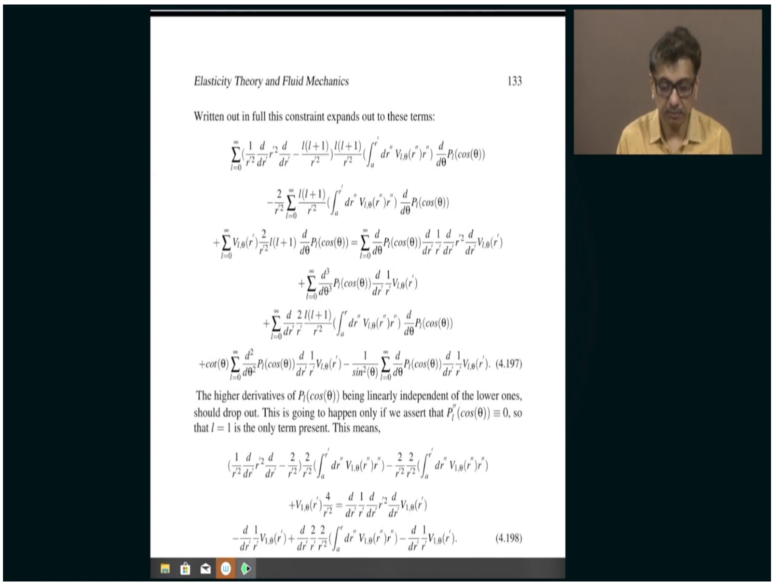
So, when I do that you will see that when I insert these two equations here I can rewrite v of r theta like this. So, it will immediately be rewriteable like this ok, it will be rewriteable in terms of these coefficients, but then keep in mind these coefficients will have to necessarily be a function of r dash because that was what was remaining. So, the it is only the function of theta that is expressed in terms of basis, this we do not know how it looks like as of now. So, it continues to be something which we do not know what it is which we will finally, determine. So, bottom line is you can rewrite the velocity components in terms of.

So, the velocity components are function of two things r and theta, but now you have successfully reduced that into a function of only theta, but also function of this discrete l. So, now, we can go ahead and also do the same for the pressure because since del squared p 0 I can always rewrite the pressure also like this ok. So, now, we can we are equipped to study what will happen to the next order so; that means, that so, once you have written down all this expansion you can go ahead and back substitute in these equations.

So, you see that there was one equation which told you how the first order so; that means, that so, what is v dash 1 it is the turbulent part of the velocity. So, that R e 0 is the laminar part of the. So, this is the laminar part of the velocity. So, this is the lowest order turbulent part of the velocity. So, what this equation tells you is that how does the lowest turbulent part of the velocity depend upon the pressure gradients in the system ok. So, that is what it says.

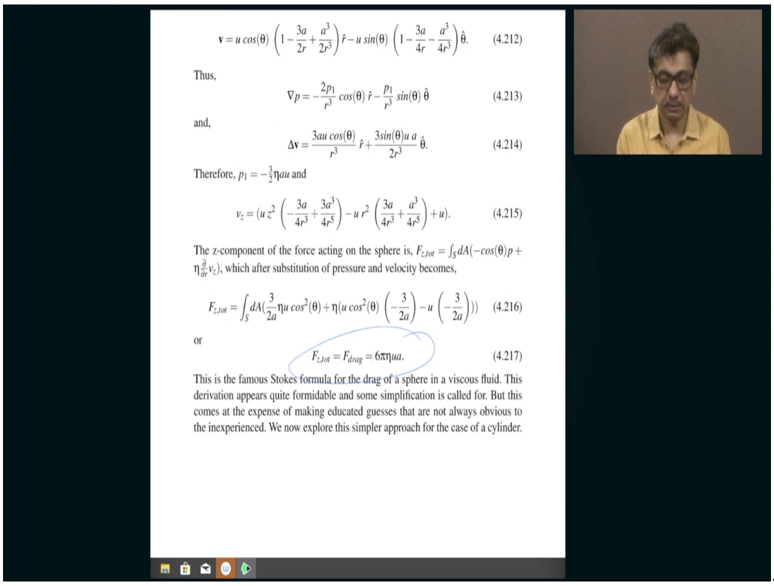
So, now, we can go ahead and try to find the turbulent part of the velocity by solving this equation, because now we have an expansion for p 0 0 also ok, because it is basically similar to that ok. So, I will I think now is a good time to stop because the rest of the details is just the lot of extremely tedious algebra, but unfortunately necessary because it is only when you go through all these steps.

(Refer Slide Time: 36:43)



Then, you get the final answer that you are familiar with from your school days and that is something I am going to display right there ok. It takes all this while to reach here.

(Refer Slide Time: 36:50)



So, eventually you are going to find that the drag is what you learned in your school days which is $6\pi\eta u a$, a is the radius of the sphere and u is the speed of the fluid in which that sphere is immersed. So, reaching this familiar result from the equation of fluid dynamics is an extraordinarily tedious process and it is something that you have to go through in order to appreciate the depth of the subject, because it is only when you do this then you understand you know that many of the things that you thought were simple are actually not that simple ok.

I am going to stop now and in the next class I will try to highlight some salient features of the remaining steps in the process and then I will arrive at the Stoke's drag for the motion of a sphere in a fluid. So, you can actually do something similar for the motion of a cylinder and a fluid which is less familiar and I will quickly mention that and then move on to some other topic ok.

So, thank you and hope to see you in the next class which will be the final class on fluid dynamics.