Dynamics of Classical and Quantum Fields: An Introduction
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Review of point particle mechanics
Lecture - 02
Lagrangian Formalism

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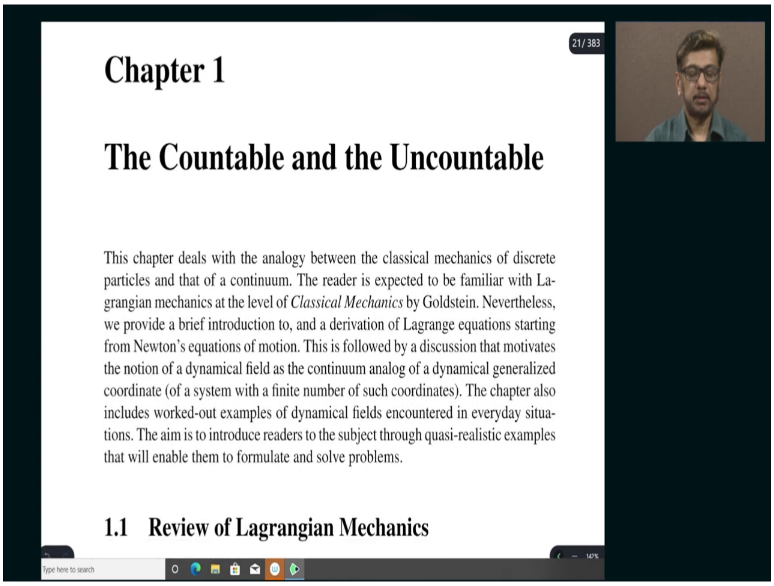

Hello, welcome to this first lecture on the course Dynamics of Classical and Quantum Fields. So, I want to start this lecture with a description of point particle classical mechanics that you are all familiar with, so namely Lagrangian mechanics. I want to specifically remind you how the Lagrange equations are derived starting from Newton's second law of motion. It is true that I expect these topics to be prerequisites for this course, but I am just refreshing your memory.

So, let me begin by pointing out that the fundamental feature of this particular course is the notion of a dynamical system with infinitely many degrees of freedom. So, specifically that infinity is of the continuous kind; in other words it is not discretely infinite, it is actually continuously infinite.

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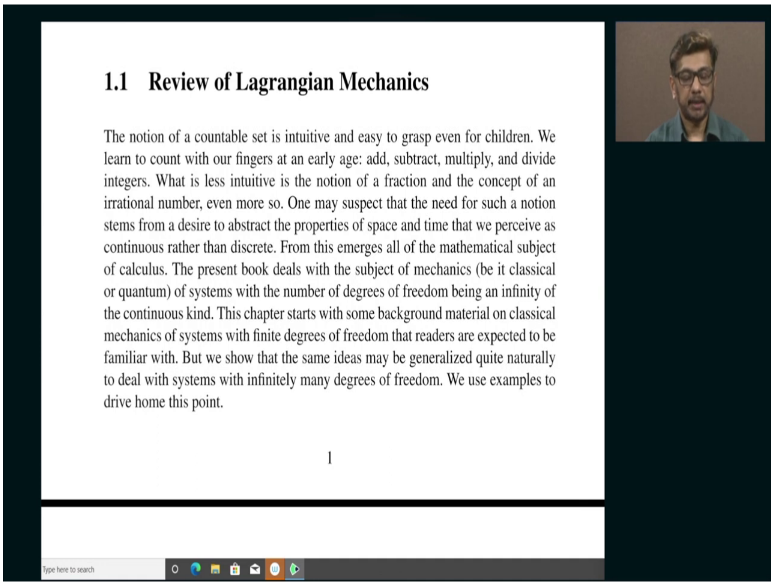

So, that is the reason why I have titled the first chapter as countable and uncountable. So, this chapter like I am just going to intermittently read off the chapter sentences themselves, because they are quite well written. So, this chapter deals with the analogy between classical mechanics of discrete particles and that of a continuum ok.

So, this chapter also includes worked out examples of dynamical fields that you will see later on and the aim of this chapter is to introduce the readers to the subject through semi realistic examples that will enable you to formulate and solve problems on your own, ok. Let me get to the brief reminder of what Lagrangian mechanics is.

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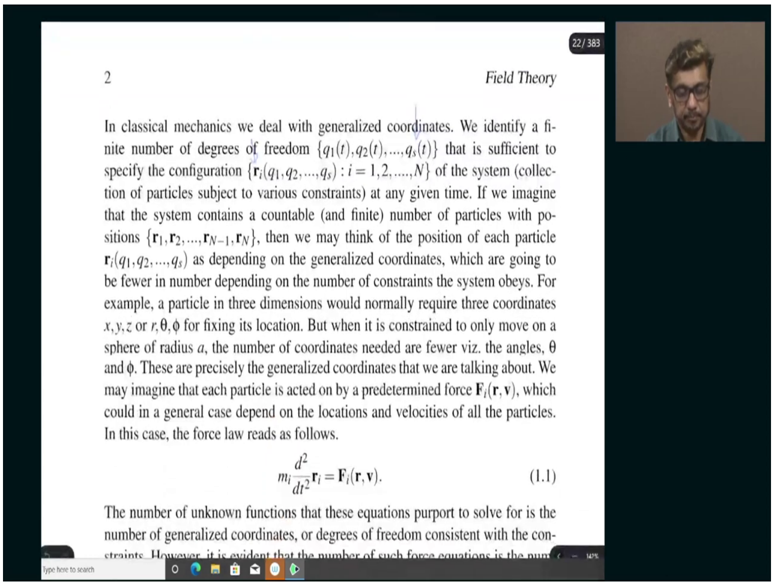

So, you all know that in Lagrangian mechanics, the basic feature of Lagrangian mechanics is that it is able to handle constraints. So, in other words you can normally what happens in Newton's second law is that, in order to utilize Newton's second law, you have to specify the forces explicitly including forces of constraint.

So, if you do not know what I am talking about, let me remind you that forces of constraints could be; for example, you can have a mass that is sliding along on a ring. The point is the forces that constrain the mass to the ring are quite complicated and usually not known. What Lagrangian mechanics allows you to do is that, it allows you to work, it is a workaround which enables you to solve the interesting dynamical questions without having to know what exactly the forces are that constrain the particle to slide along the ring.

So, those are the forces of constraints which Newton's second law will require you to know; but Lagrangian mechanics allows you to bypass and allows you to not solve the problem without knowing what they are. So, the way this is accomplished is the by the introduction of what are known as generalized coordinates. So, there is a distinction between the actual coordinates of a particle, which is basically the position of the particle in three dimensions.

So, a suppose a particle is we all know that in general every particle is located in three dimensions. So, strictly speaking you always require three coordinates to describe the location of the particle; whether it is the Cartesian x, y, z coordinates or the polar r, θ, ϕ coordinates or whatever else.

So, but then the point is that when you have constraints, it is obvious that you do not require 3; you require fewer than 3, that is because you already know the particle is say constrained to move on a circle or is constrained to move on the surface of a sphere or it is constrained to follow a certain path or live on a certain surface.

So, in that case you need fewer and the idea is that the fewer generalized coordinates are described by basically q1, q2, q3 up to qs so, the small letter s that I am describing here in. So, let me write something and tell you what I am talking about. So, this s here, so this s refers to the number of generalized coordinates. So, in other words that could be 2, if I am talking about a particle that is otherwise in three dimension, but lives on a surface.

So, it could be just 1; if there is a particle that normally lives in 3 dimension, but forced to slide along a circle along the circumference of a ring. So, the bottom line is you need fewer than the actual number of dimensions in the problem. So, now the point is that if you have large number of such particles, the position of each of those particles ri is described by specifying the configuration of generalized coordinates.

So, in other words by specifying the generalized coordinates, the magnitude of those generalized coordinates; so in other words their numerical values, you will be successful in locating each of those particles, their physical locations are therefore specified. So, that is what we mean by saying that particles obey constraints.

So, in other words they are described by fewer set of coordinates, which we call generalized coordinates. So, that is the beauty of the Lagrangian formalism that it allows you to take into account constraints in this very transparent way without having to indirectly deduce those conditions through the specification of forces which we usually do not know anyway, ok.

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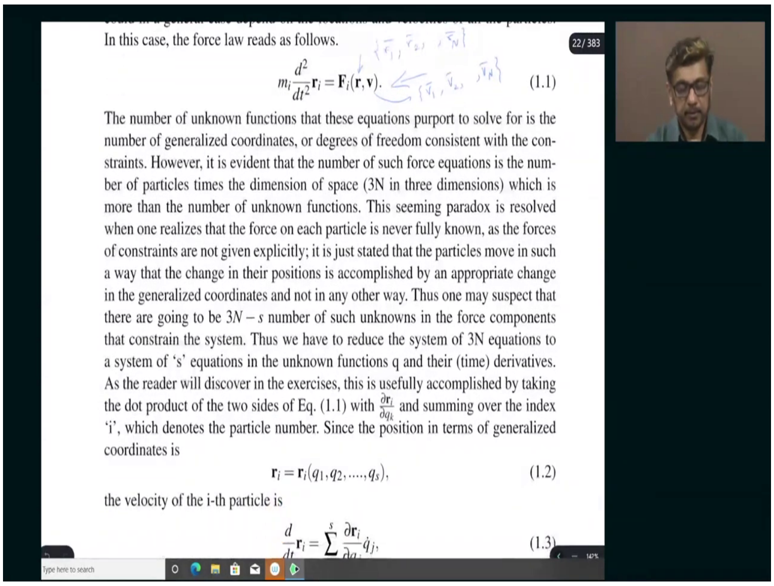

So, now, as we all know the Newton second law talks about the mass times acceleration being the force. So, it is written in this form mass times acceleration is force, where r is the position vector of the ith particle and v i. So, I should have written possibly well it this r actually now is a collection, it could be the collection of all the particles.

So, you see the point is that I have deliberately written r because it is quite possible that the forces acting on the ith particle depend on the location of all the other particles as well. So, so I think it is not an oversight or a typing error it is specifically meant to say this.

So, less usually it could also depend upon the velocities of all the other particles including its own. So, this would be the ultra general way of writing Newton's second law. So, the question is now; of course in most examples these forces are never fully specified, usually we only know the forces that we are explicitly applying to the system.

But the forces of constraints that are also present that force those particles to move in a certain way that constrain them to move along surfaces or rings and so on. They are not specified and they are not of much interest either. So, it would be very desirable to have a technique, which enables us to not consider them at all. So, that technique is precisely the Lagrangian approach to classical mechanics.

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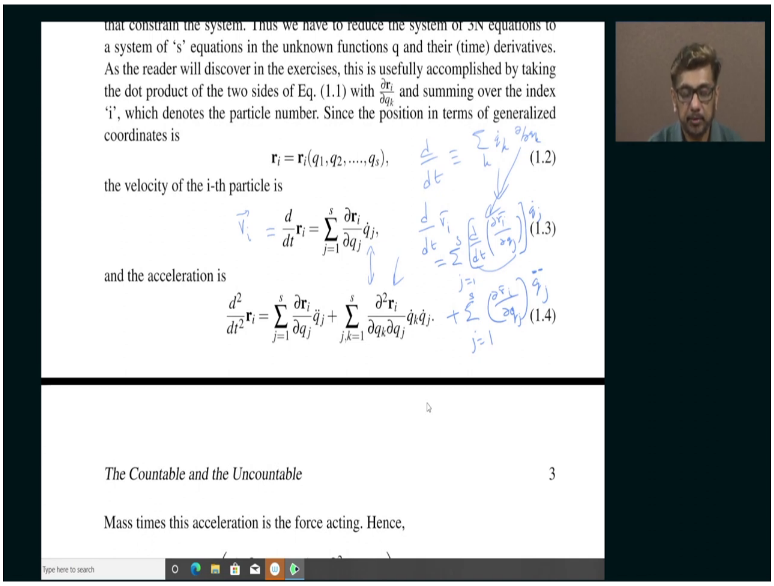

So, having said all this, let us start by trying to find out the acceleration of the ith particle in terms of the rates of change of the generalized coordinates themselves. So, now, the ith particles position is described by the specification of these s generalized coordinates. Now, if I find the velocity vectors. So, remember what this is, this is the rate of change of position which is the velocity.

So, the velocity vector is basically determined by chain rule by partially differentiating the position with respect to the each of the generalized coordinates and then multiplying it by the rate of change of the that particular generalized coordinates and then you have to make sure you take into account all of them, which basically means that you sum over all the generalized coordinates.

So, now having found the velocity we can go ahead and find the acceleration by simply differentiating once more. So, if I differentiate once more. So, those of you who are following along should put in some effort to fill in the steps. If you feel that it is not obvious how I got from this step to the next step; so you should not switch off and pretend that you have understood, but you should pause the video and then make sure you actually understand how I went from that step to the next.

So, here in particular the way I reached 1.4 from 1.3 was that I differentiated 1.3 once again and so, that meant that I have to do this and also that. So, I differentiate this first and then multiply by that or alternatively differentiate this first and multiply by that ok. So, when I do that the, this is pretty much what I get. But then the question is what is this? This is again you have to redo this chain rule all over again.

So, basically if you want to differentiate with respect to time, you first differentiate with respect to generalized coordinate and then differentiate the generalized coordinate with respect to time. So, that is precisely what you get here. So, $\frac{d}{dt} \equiv \sum_k \dot{q}_k \frac{\partial}{\partial q_k}$. So, then I am going to insert that here and then this is what I get, ok. So, I hope this is clear. So, there are some steps missing when I go from 1.3 to 1.4 ok. So, let us proceed.

So, now this is the acceleration and why did I want acceleration, because Newton's second law tells me that the left hand side of Newton's second law involves acceleration. So, mass time's acceleration is force, so which is why I needed acceleration. Now, I have successfully expressed the acceleration in terms of the derivatives, time derivatives of the generalized coordinates.

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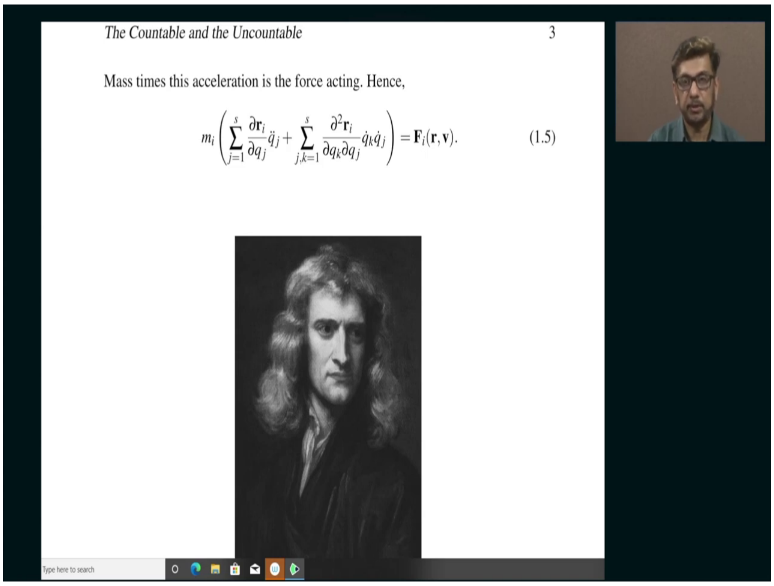

So, now let us go ahead and substitute equation 1.4 into equation 1.1. So, that will allow us to write mass times acceleration in purely in terms of the time derivatives of the generalized coordinates equals force, ok. So, that force is something very complicated, it could be very complicated and usually we do not even know all the components beforehand.

Because like I told you usually the forces, the components of forces that are known are the forces that you yourself have decided to apply on the system. There are other forces which the circumstances that the system finds itself in those circumstances apply those forces which Newton's law also requires you to know; but practically there is no way of knowing what they are, like I told you the forces of constraints.

So, the forces that a ring might be applying on a mass that is sliding on it in order to force that mass to remain on the ring, so that is of that is of no interest to anyone except that it is important in ensuring that the mass slides on the ring and you have no way of knowing what that force is.

So, it is really desirable to have a technique which does not involve that force at all and that is what Lagrangian mechanics allows you to do, ok. So, let us proceed and see how it allows us to do this, because at this stage 1.5 involves all components of the force including forces of constraints.

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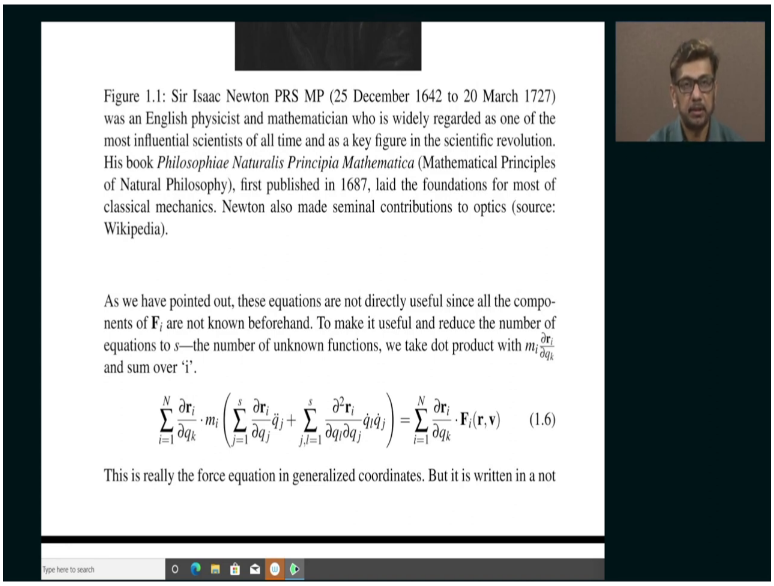

So, we have to somehow get rid of, we have to cleverly get rid of the constraint forces, ok. So, the way this is gotten rid of is that; obviously there are more components of forces than there are components that you already know about.

So, in order to get rid of the components that we do not care about; what we do is that, we multiply by we take the dot product of this equation 1.5, we take the dot product with respect to the rate of change of the position with respect to the kth generalized coordinate. So, we take the rate of change of the position of the ith particle in terms of the kth generalized coordinate and we sum over all the particles. So, you might be wondering this is a very arbitrary operation and why did we do it, what is the motivation.

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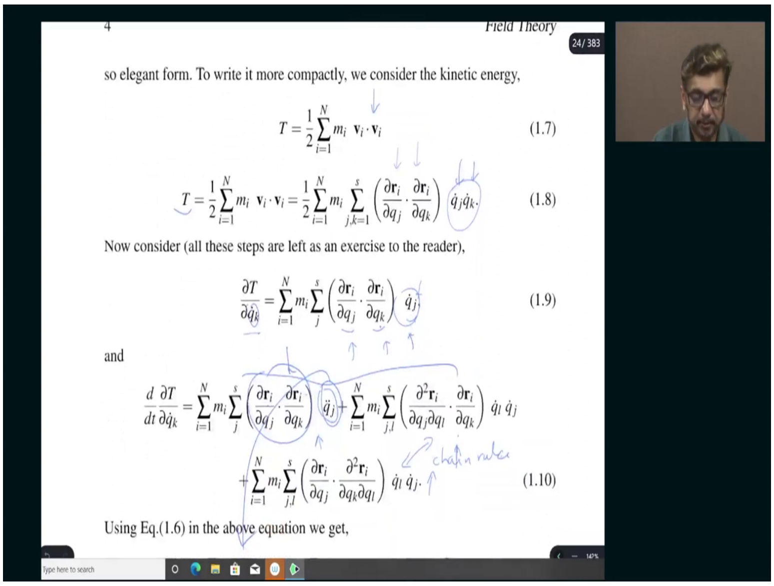

So, you will soon see that there is a very good reason for doing this and so, let us proceed further and then somewhere down the road I will tell you what the motivation is, at this stage it seems very ad hoc. So, now, let us at least proceed algebraically and see if we can simplify these equations until we can understand what is going on and why we did all this. So, keep in mind that the kinetic energy of the system of particles is basically given by mass times square of velocity divided by 2 summed over all the particles, which is what this is.

So, now like we said earlier the velocity of the ith particle is expressible in terms of the rates of change of the generalized coordinates of the of those corresponding particles. So, if that is the case, then the kinetic energy is expressible in terms of the rates of change of the generalized coordinates of the various particles. So, having written the kinetic energy of the system in terms of the rates of change of generalized coordinates; now let me tell you what it is I am trying to do.

So, basically you see in 1.6 there is this rather complicated looking set of terms. So, what we want to do is basically express some combinations of these in terms of derivatives of quantities, which are of physical interest like kinetic energy and so on. That we can easily identify with; because at this stage these quantities seem very arbitrary and they are not familiar to us and so it is. So, what we are doing now is this exercise is to render 1.6 into a form that is more familiar to us.

So, in other words express it in terms of quantities that are more familiar. So, you have to admit certainly that the kinetic energy of the system of particles is a familiar quantity and it could be really nice if we could express some of those terms in terms of some appropriate derivatives of the kinetic energy. So, that is what we are doing here.

So, if you take the derivative of the kinetic energy with respect to the rate of change of the generalized coordinate and then you further differentiate with respect to time, ok. So, again I could be skipping a lot of steps, but then it is really important for you to not only here; but throughout the course it is important for you to make sure that you do not that you understand each and every step and if you do not understand, you please pause the video and work out the intermediate steps and only then proceed.

Because there is you cannot really follow physics, if you just take the word of the instructor as if it is something everybody knows. It is for you to understand, everybody may know; but you are the student or the listener who was trying to follow and you have to understand by putting in that effort ok. So, the question is how do you proceed from here? So, it is obvious that the derivative of the kinetic energy with respect to the rate of change of generalized coordinate is this.

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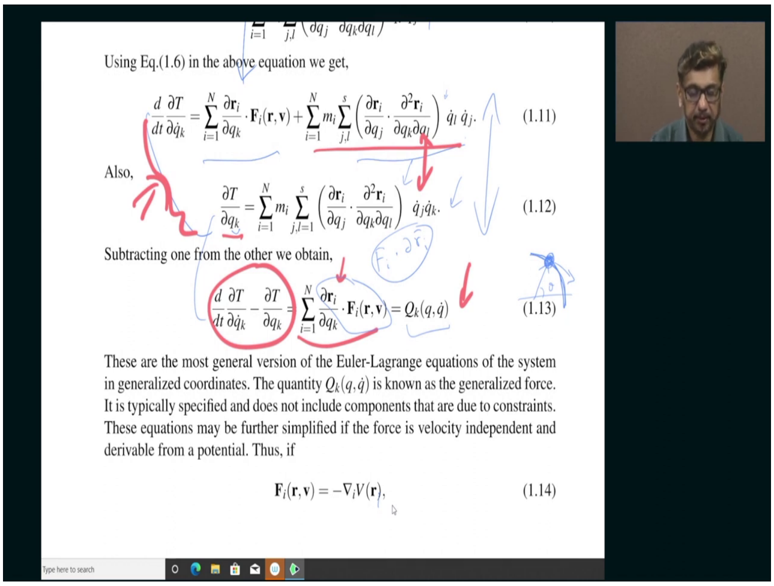

Now, if you differentiate with respect to time, you get you get all these terms. So, it is obvious where this comes from. So, like I told you differentiate with respect to time. So, there are three things which are potentially time dependent these three, right. So, if you want to differentiate the left hand side with respect to time, what you do is; you first differentiate this with respect to time multiplied by whatever was there already, then you differentiate this with respect to time.

But then once you differentiate something with respect to time which implicitly depends on time through the generalized coordinates, you have to of course also invoke the chain rule all over again. So, when you invoke the chain rule you get these terms. So, I am not going to fill in absolutely all the steps, but I am just pointing out to you where they come from. So, this come these two terms come from invoking chain rule ok; whereas this is obvious where it comes from, it is just differentiating this once more, ok.

So, now what you do is that, you notice that this 1.6. So, let us remember what that is look there is a term involving second derivative with respect to the generalized coordinate and that is what this is, right. So, that times notice that these two get multiplied, right. So, that is what this is. So, the derivative of the position with respect to generalized coordinate times something very similar times the second derivative of the generalized coordinate and that is what that is.

So, now plus this term and together gives you this term, but then there is this term that that gets left out, ok. So, these two put together becomes this ok, this gets left out. So, let this be as it is. But now go back and differentiate this kinetic energy not with respect to $\dot{q}$, but with respect to q. So, in other words see here notice that I have differentiated with respect to $\dot{q}$; that is of course a different thing altogether.

So, you are aware that $\dot{q}$ in Lagrangian; well that is the I mean that is the implication. So, if I say $\partial\dot{q}$, the implication is $\dot{q}$ is independent of q. So, you might be wondering how can that be, how can $\dot{q}$ be independent of q; it obviously is if you do not know if the trajectory of the particle is not specified, if the dependence of q on time is completely arbitrary, there is absolutely no earthly relation between q and $\dot{q}$.

The particle can be anywhere it wants and have any velocity it wants, these two are unrelated; position and velocity can be completely unrelated unless you already know what that trajectory is. But of course we do not know what the trajectory is, because we are that is what we are trying to determine right now. So, we do not a priori we do not know what the trajectory is.

So, at this stage q and q dot are completely independent. So, if that is the case, then the rate of change of the kinetic energy with respect to q is given by this formula is not it. So, where does this come from? So, notice that these two are by definition independent of q; so but then the q dependence is only contained here and it is very symmetrical. So, if you differentiate both once, you will get a factor of 2 and you get this result, ok.

So, now you go ahead and subtract these two formulas, ok. So, point is that I am going to subtract this and this; I am going to subtract these two. So, if I subtract these two, what I get is exactly this, ok. So, I get that. So, I subtract these two and I get this. So, now, when I subtract them, these two cancel out. So, that is the reason why I did this; of course this is from hindsight, so I know that this term is basically the same as this.

So, as a result I subtract that out and I get this really nice looking compact formula. So, now, I have succeeded in recasting what was earlier a Vectorial equation, namely Newton's second law is a vector equation. So, what do I mean by that? I mean by that exactly this that, the left hand side of this equation is a vector right hand side is a vector; but then remember somewhere down the road in equation 1.6, I took the dot product. So, that once I take dot product, I am converting an equation that is a vector equation into a scalar equation.

So, obviously the implication is that the information contained is going to get diluted; because earlier I had more components, taking dot product involves projecting out some of those components along some directions. So, obviously the information contained is fewer, but that is just as well; because I really do not need the information which forces me to know what the forces of constraint are. So, I purposely did this, so that I do not have to you know care about the forces of constraint which I anyway do not know.

So, indirectly by the knowledge of the fact that the position of each of those particles depend on a predetermined set of generalized coordinate that enables me to indirectly recover the remaining components of the force equation which I have somehow eliminated by taking this dot product. So, I know that it is little bit difficult to put this in words, but I am sure you understand intuitively what I am talking about.

So, bottom line is that, Lagrange equations basically involve scalar quantities, whereas Newton's second law involve vector quantities and there is no loss of information at all; both are equal equivalent, there is same information contained, because the information that is lost by making this transition from a vector form to a scalar form is recovered through the knowledge that the particles specifically depend upon certain generalized coordinates that are pre assigned, ok.

So, now the equation that takes the place of Newton's second law is now this, equation 1.13 and the right hand side involves the forces that are acting not all the components; but only the components that are parallel to the rates of change of the position with respect to the generalized coordinates, ok.

So, that is important because what it says is that, if there are components of forces which are perpendicular to the rate at which the positions of the particle change as you change the generalized coordinates, those forces do not contribute to this equation. So, to give you a specific example; so this has a physical meaning which is worth, which is worth understanding.

So, imagine you have a particle that is sliding along on a ring, ok. So, now, you see what is the position of the particle it is this and what is the generalized coordinate, it is this and the rate at which the position changes with respect to a generalized coordinate is basically along this, this direction is not it. So, this is the rate of change of the position with respect to generalized coordinates.

So, the δr is tangential. So, now, what this says is that, that you have to specify the forces that are acting which are parallel to δr. So, in other words those forces have to do work. So, this is in some sense the work done. So, it is this is like F dr. So, the right hand side only depends upon forces that actually do work, ok.

So, the forces that do not do work, do not contribute to the scalar version of Newton's second law ok; but they are important in ensuring that the particle slides along on this curve so, which is presupposed and assumed. So, as a result it indirectly amounts to specifying the constraint forces, even though we do not know what they are, ok. So, bottom line is that it this is how it is and then the scalar version of those original Newton's second law can be written like this.

So, now this these are purely in terms of the generalized coordinates. So, like you; so, the kinetic energy is expressible purely in terms of it is, expressible in terms of the velocity vector of the individual particles, which are in turn expressible in terms of the time derivatives of the position and the position depends on time only through the generalized coordinates, which happen to depend on time and the rate of change of the position of the particles depend on the rate at which the position changes with generalized coordinate times the rate at which generalized coordinates changes with time. So, that is chain rule.

So, basically the left hand side of this depends upon the way in which generalized coordinates change with time and the right hand side depends upon the forces that do, actually do work on the system and not the forces of constraint, ok. So, that is these are called generalized forces or well you can call it whatever you want; they are typically called generalized forces, so the forces that actually do work.

So, this is as far as you can proceed if you do not know anything further about the nature of the forces that are acting; but you can do a lot more if you know beforehand that the forces that are acting are only dependent on the positions of the particle and specifically they only depend upon the, so, imagine that they only depend upon the. So, each particle is acted upon by some external force and they do not interact with each other; suppose we assume that that is the case, it need not be that way.

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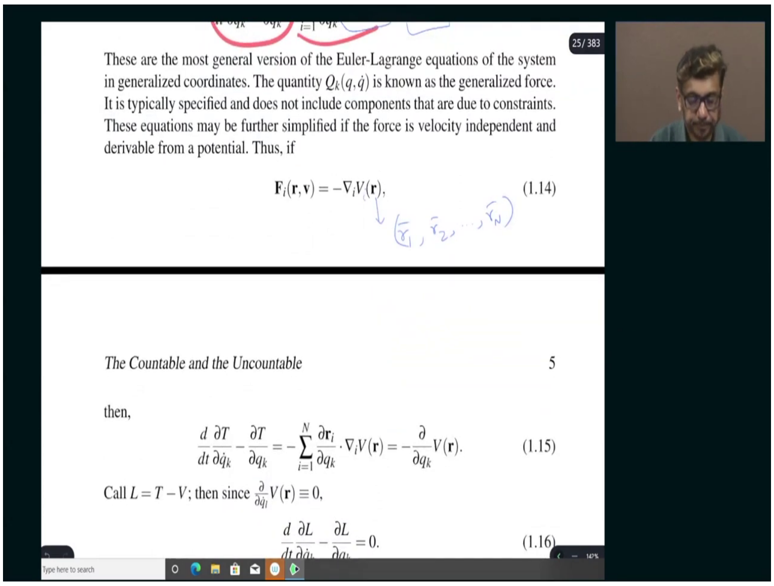

So, in fact I think I have allowed that to be more general, right. So, yeah so this could be more general. So, this $V(r)$; like I told you earlier this r is actually not just, this r actually means all of them, it could mean all of them, because the potential energy of a part of the ith particle ok could depend upon the, so it could yeah this is better.

So, there is the potential energy of the ith particle could depend upon the positions of all the other particles including the position of that particle itself; because there can be an external force acting on all of them, but on top of that each particle can be experiencing forces due to the remaining particles one by one. So, this allows for that possibility, ok.

So, bottom line is that if you do this. So, here I think I have made this simple assumption that, that there is a single potential energy. So, you could do that more general thing or you can do what I have done in the book, which is assume that there is a single potential energy, ok. So, in other words there is an external force and the same force is acting upon all the particles and the force simply depends only on where that particle, each of those particles are located.

So, if the ith particle is located at some r, there is a certain force acting. Now, if you instead of the ith particle being located there; if some i + 1 particle is now located at the same point the same force acts, basically $V(r)$ is the potential energy of any particle that happens to find itself at position r ok. So, if that is the case, then the force acting on the ith particle would be simply the derivative of that.

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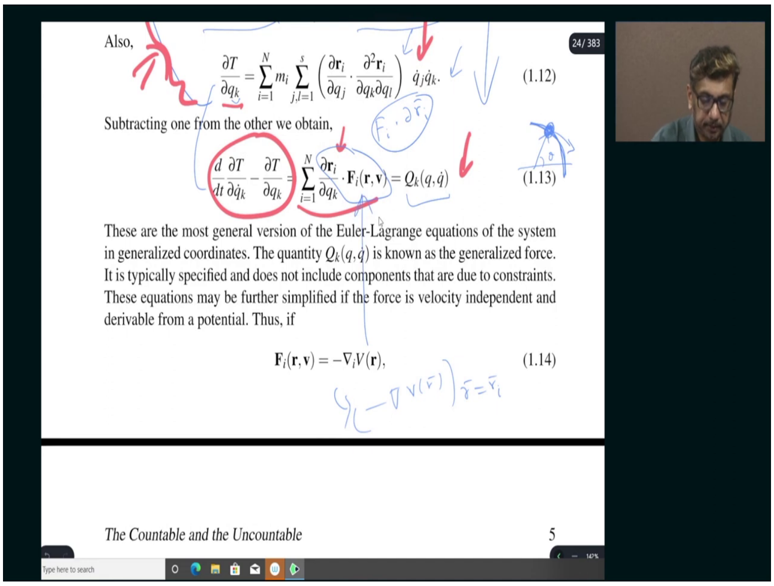

And then by this I mean you take ∇ and then ok and then you make r go to ri, that is what I mean by this, ok. So, having done that, you can see that this if I substitute this here ok, so what I get is this relation.

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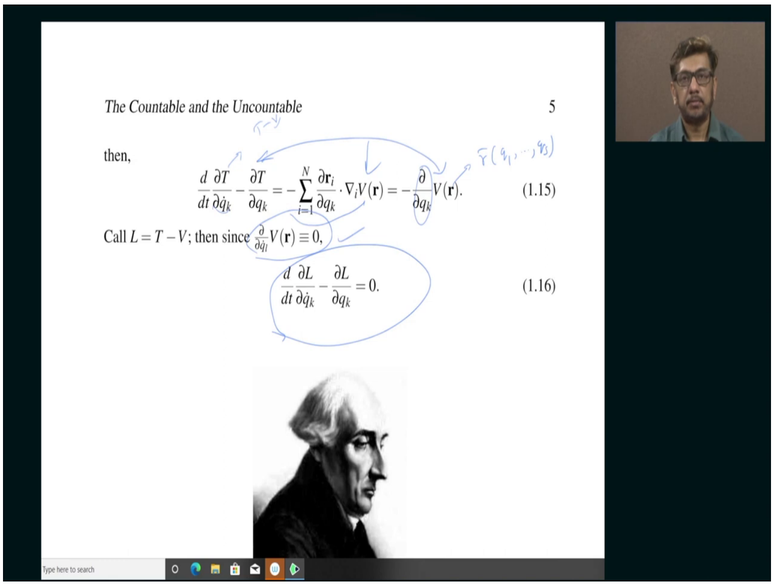

So, now I can go ahead and, ok. So, it is going to be that and now if you stare at this what is this; again by chain rule this is nothing but the partial derivative of the potential energy with respect to qr, qk, but remember that this r depends upon q1 up to qs, ok. So, now one more step and we are there. So, the last step is to realize that the potential energy function by definition does not depend on the velocities of the particles.

So, therefore, it does not depend on the rate of change of the generalized coordinate with time. So, we are obviously forced to make the statement and if we make the statement; then it is clear that I can simply rewrite this as T-V if I want, because $\frac{\partial}{\partial \dot{q}_i}V(\text{r}) \equiv 0$.

So, then I bring this to this side this becomes T - V as well, because these two if I bring the right hand side of 1.15 to the left hand side it becomes T - V; but this was T to begin with, but I can simply replace T by T - V, because differentiating with respect to $\dot{q}$ will anyway destroy this V there, because V does not depend on $\dot{q}$.

So, having done all that, we are now finally here and what we have written down involves a quantity called L, which is the difference between the kinetic energy of the system of particles and the potential energy acting on that system and this is a peculiar quantity; it is peculiar, because we are usually familiar with T + V that is the total energy of the system.

So, T - V is called the Lagrangian and the equation that replaces Newton's second law for systems where the forces are derivable through a scalar potential is now called the Lagrange equations of motion and they involve only scalar quantities; remember that L is scalar, because the difference between the kinetic energy of the system and the potential energy acting on the system. So, that is the beauty of the Lagrangian approach to classical mechanics what it does is that, it trades one sort of ignorance with certain assumptions.

So, in other words the ignorance is the ignorance of the forces of constraints. So, most of the meaningful problem descriptions do not specify forces of constraint; because people who are observing systems of particles do not have any means to measure what forces of constraints compel those particles to move in a peculiar or particular way, all they can do is observe that those particles are in fact moving in that constrained manner.

So, what Lagrangian mechanics does is that, it utilizes that knowledge that the particles are thus constrained and then somehow bypasses the need to know all the forces that are acting on the system and is able to determine the trajectories of each of those particles even though all the forces acting on the particles are not specified, ok. So, I hope you have sort of understood what Lagrangian mechanics is; in any case this was supposed to be a prerequisite, but I hope I have succeeded in refreshing your memory.

And I am going to stop here and in the next class, I am going to continue the description of point particle classical mechanics through a description of Hamiltonian mechanics; because we are going to be using both the Hamiltonian mechanics as well as Lagrangian mechanics later on for many applications involving the continuum counterparts of the systems that we have just studied, ok. I am going to stop here, hope to see you for the next class.

Thank you.