Dynamics of Classical and Quantum Fields: An Introduction

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Fluids
Lecture - 18
Bernoulli's Principle

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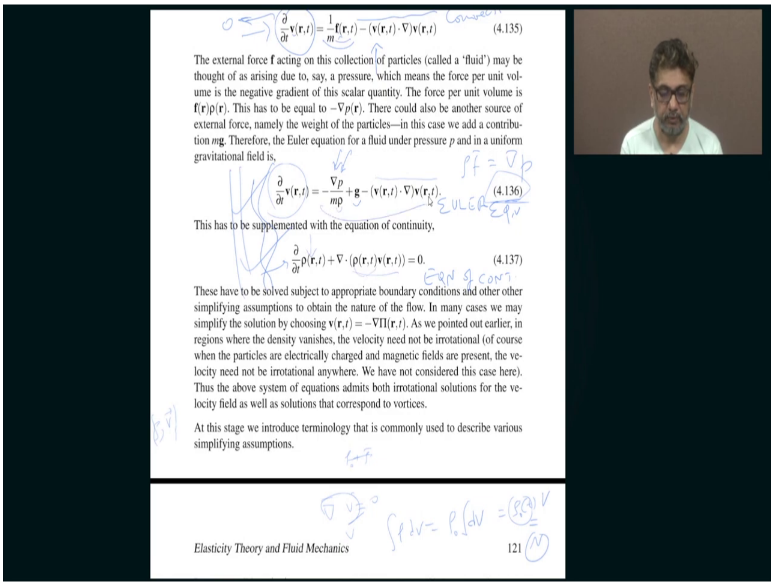

So, today, let us continue our discussion of the fluid Dynamics Equations. So, we have painstakingly derived two equations; one is called the Euler equation which tells us how the velocity of the fluid changes with time, the other is the more familiar continuity equation which tells you that in any region, the total mass of the fluid is either constant or if it changes, its because of fluid coming from outside the of the of that region.

So, basically these are the two equations. So, this is the Euler equation and this is the Euler equation, this is the continuity equation.

So,

the question is we of course, in general you can see first of all that these equations are non-linear. They are non-linear because the unknowns are basically rho and v and the equation for the unknowns appear you know it is not linear. In other words, first of all there is even if you assume this rho in the denominator is approximately constant, there is still this term the convective derivative which is basically non-linear and that is one of the major source of non-linearity.

The other source of non-linearity could be this; but then typically even you can assume that in many examples we assume rho is you know like some constant plus some fluctuations. So, even then if you choose to ignore that, then of course, you can linearize these equation in very many different ways; but you have to do it in a way that is mathematically acceptable that is not always easy.

But bottom line is that if you decide to just look at the formalism, these are the equations that you have to deal with and there have been no approximations made except the only physical assumption that we have made is that there is no internal friction; that means, there is no viscosity. So, these are equations of fluid dynamics that describe fluids with no viscosity; that means, ideal fluids ok.

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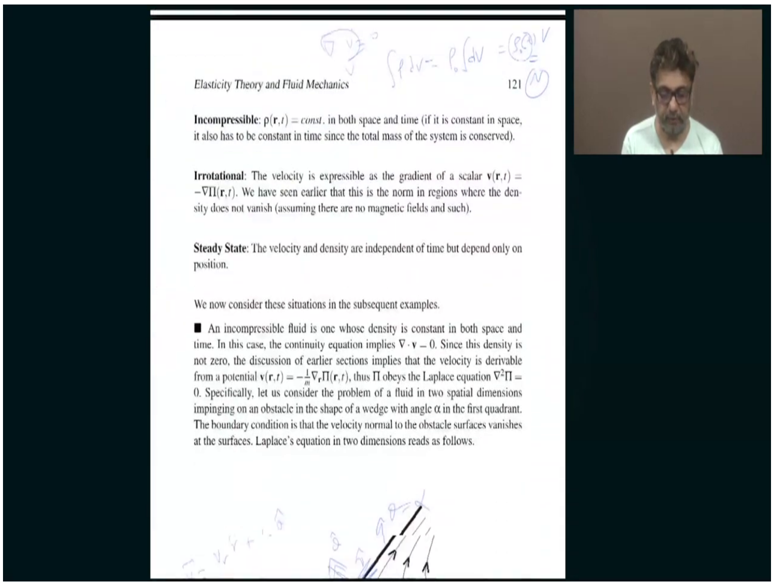

So, even within these ideal fluids, you can have situations that make our life simpler and those are listed here. So, the situations that are likely to simplify these equations even further are the following. So, there is this concept called the incompressible fluid. So, the incompressible fluid means that the density of the fluid is constant. So, in fact, you can easily see that it is sufficient for us to demand that the density of the fluid does not change from point to point that immediately guarantees that it cannot change with time also.

Because if you look at the total number of particles is the integral of the density with respect to volume, see if the density is constant, it goes outside the integral and you get total volume times that. So, now the total volume is anyway fixed, it is given. The density if it is dependent on time, then it will mean that basically the. So, if density depends on time, so this is basically the total number of particles which cannot change with time. So, density also cannot change with time; is not it?

So, so that means, that its sufficient for us to demand that the density of the particles should be independent of position that automatically guarantees that it should also be independent of time because total number of particles are anyway independent of time ok. So, the other thing is irrotational. So, irrotational like I told you that velocity can always be written as the gradient of a scalar at all points, where the density of the fluid does not become 0 ok. And that is another simplification that we can readily exploit. The other thing the third example is called steady state.

So, steady state is an assumption that your velocities and densities are independent of time. So, that means, if whatever disturbances are there have kind of died down and so, the velocity and density have reached some steady state. So, in other words, it changes from point to point in space, but it does not change with time ok.

So, these are the three different types of simplifying assumptions that we can exploit and some of them are not assumptions like irrotational is just an observation that you can always write velocity easily. See when you can always write velocity as the gradient of a scalar whenever the density is not 0 at that point.

So, let us try to simplify these two equations that is Euler equation and equation of continuity using these further simplifying assumptions. Namely, let us start with for example incompressible. So, if you start with incompressible; that means, it is a fluid whose density is constant both in space and time, in which case you see the equation of continuity will immediately tell you that means, the divergence of velocity should be 0 because rho is constant.

But then, let us also keep in mind that rho is constant and therefore, not 0. Because if it is 0 and constant, then you do not have any particles left. So, it has to be necessarily non-zero. So, because it is non-zero velocity can always be written as the gradient of a scalar and further velocity is also divergence free; that means, its divergence of velocity is 0. So, that means, that del squared of that scalar potential of the velocity; that means, velocity is a gradient of scalar. So, that scalar quantity has this property that del squared of that quantity is 0.

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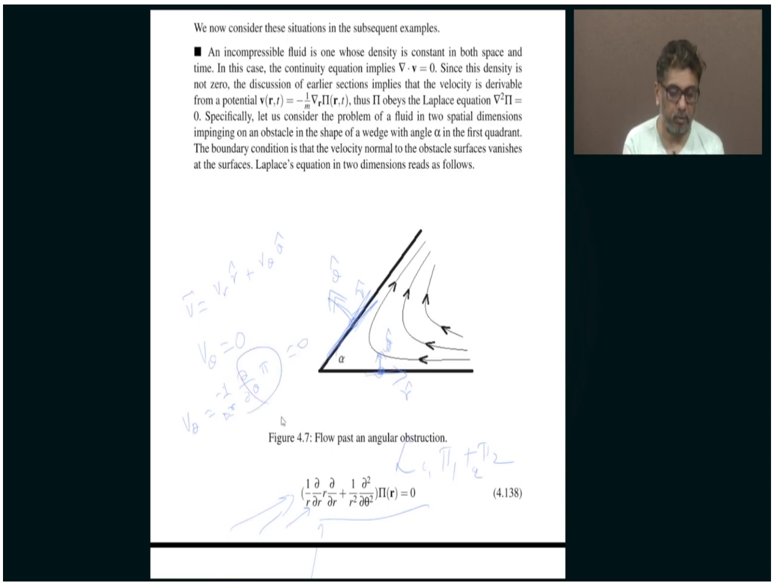

So, now, so in other words, solving for the properties of a fluid whose density is totally uniform just involves solving a Laplace equation. So, now, let us consider a specific geometry because you have to specify of course, Laplace equation is second order partial differential equation.

So, that means, you have to specify various boundary conditions before you solve it. And so, boundary condition mean you should explain what the geometry you are looking at. So, specifically let us focus on two spatial dimensions. This is del squared pi could be its always valid in all dimensions.

So, now, I am going to focus specialized to a case, where the problem is in two dimensions. So, in two spatial dimensions, there is a an obstacle in the shape of a wedge which has an angle alpha ok. So, now, the boundary condition is that the velocity normal to the obstacle vanishes at the surfaces ok. So, the idea is that you see the fluid flows along the surface; when it near the surface, it flows along its kind of. So, in other words the net velocity of the fluid perpendicular to the surface is 0 ok. So, that is the boundary condition that we are looking at.

So, clearly in this case, the polar coordinates is natural because the geometry is that of you know some kind of an angle. So, that means, some there is some angular region in two dimensions. So, plane polar coordinates are natural in this case. So, the Laplacian and plane polar coordinates is given by this.

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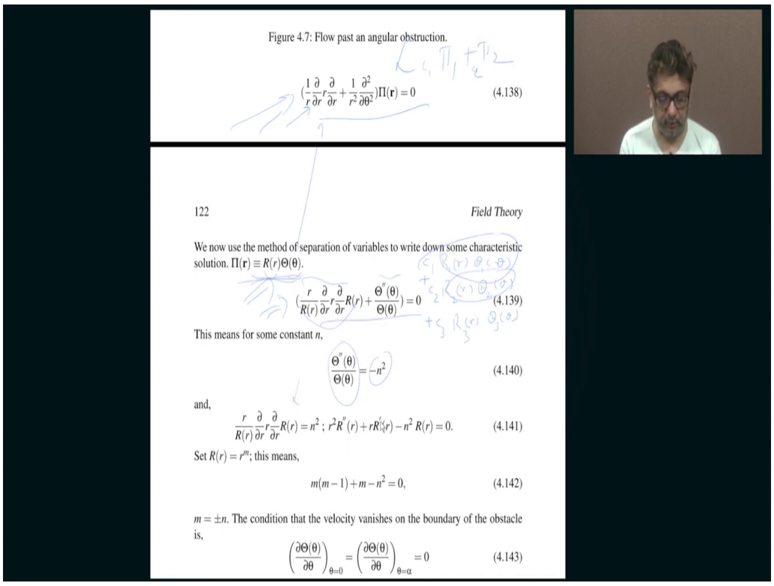

So, now as usual you might have encountered this method many many times in your electromagnetic theory and PDE courses. So, how do you solve a PDE? Typically, you solve by separation of variables. So, if you have two independent variables r and theta, you assume that your dependent variable namely pi can be written as the product of you know I mean some function of r separately, from function of theta separately. Of course, this is not a I mean this is you might think that this is unacceptably simple assumption.

So, you might think that it will miss out a whole bunch of other solutions which may not be expressible in this form and of course, that is true. But the implication is that we do not stop here. In other words, suppose you get a whole bunch of you will not get one solution like this. You will get a whole bunch of them. So, you will get this, you will get this, you will get a whole lot of these solutions. So, the claim is that the most general solution can be written as the linear combination of all these because after all this particular equation is linear in pi.

So, that means, pi is a solution pi 1 is a solution, pi 2 is a solution, c 1 pi c 2 pi 2 is also. So, in other words, this is a linear equation; the Laplace equation is linear. So, the implication is that even though it seems that writing it in this separated form, so that means, we have some separation of variables; we have separated them into r and theta seems like unreasonably excessively simplistic.

But keep in mind that is just an intermediate step in the final calculation; namely that we provisionally assume that this is the case and then, we generate a whole bunch of these types of solution and then, there is a mathematics theorem which will guarantee that the most general solution to the Laplace equation is in fact given by the linear combination of all these different solutions that you generate by assuming or imposing separability ok. So keeping once you have that at the back of your mind, so you will probably feel a little more assured in going ahead.

So, let us go ahead and substitute the unsorts namely this separability unsorts and so, when you separate them, you get this relation and clearly, what this means is that you know this is only a function of r, this is only a function of theta. So, both had better be constants and those constants had better add up to 0.

So, I am going to call this has to be a constant, I am going to call that constant and n minus n squared minus n squared because I want the solutions to be trigonometric because I want you know you know why it has to be trigonometric because the theta that you are talking about this the lowercase theta is the angle. So, there has to be periodicity. So, it had better be trigonometric. And this has to therefore, necessarily be plus n squared, the other one because they have to add up to 0 ok.

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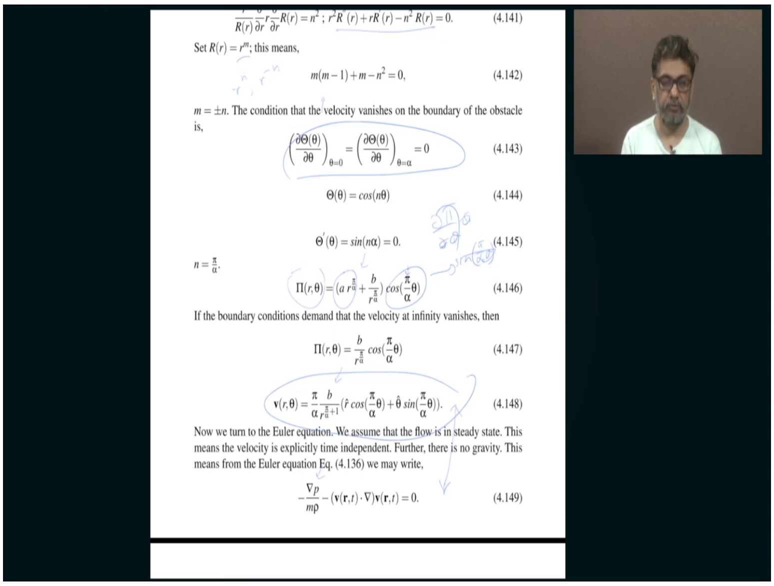

So, then we go ahead and so this is a basically a homogeneous equation, whose solution is given by some homogeneous unsorts and then, so you have two linearly independent solution r, r raise to n and r raise to minus n. So, the general solution is linear combination of these two.

Now, the keep in mind that I have said that the velocity vanishes on the boundary of the obstacle. So, let us see bottom line is that you expect the not the velocity, it is the normal component of the velocity. So, you see the normal component of the velocity is basically the its it is. So, if you write it in terms of r and theta. So, it is the velocity is nothing but you have v r r cap plus v theta theta cap.

So, so, this is your theta. So, you see at any point on the boundary, the normal component is basically in the angular direction ok. So, here for example, here you are talking about the normal component is in the angular direction ok. So, this is radial, this is angular.

So, I mean here the radial is this, this is the angular. So, here the radial is this, this is the angular. So, the claim is that the angular component of the velocity should be 0 because that is normal to the surface. So, it is the normal angular component is normal to this surface at this point of the boundary and this point of the boundary also this angular component is normal to this.

In other words, inverse v theta has to be 0 and what is v theta? v theta is basically $\frac{1}{r}\frac{d\Pi}{d\theta}$ because v_θ is the angular part of grad the angular part of grad is $\frac{1}{r}\frac{d}{d\theta}$. So, with a minus m like that ok. So, this has to be 0 on the boundary. So, that is why I have said that basically that this has to be 0 on the boundary. So, that two boundaries; one is at theta equal to 0 that is the this one.

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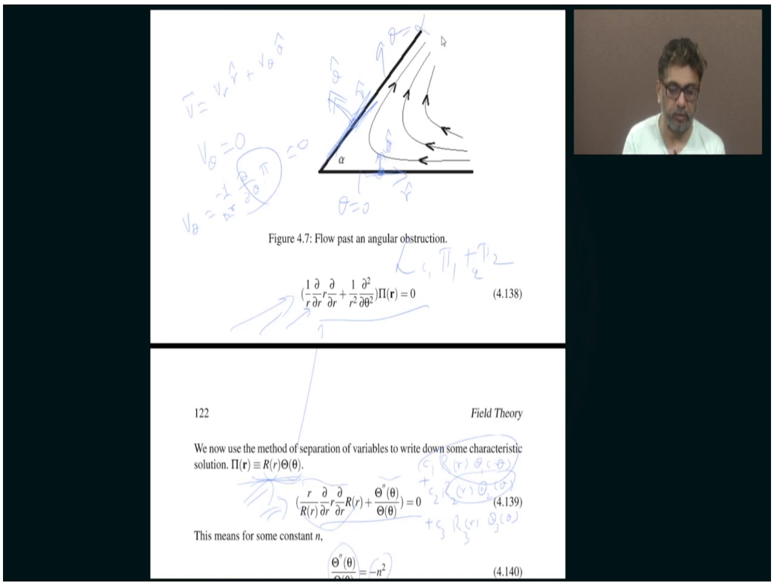

So, this is theta equal to 0 and this is theta equal to alpha. So, there are two boundaries. So, that means, d capital theta as a function of lower case theta should be 0 at lower case theta equals 0. And similarly, the derivative with respect to lower case theta should be 0, also when the lower case is equal to the second boundary which is at alpha. So, these two imply necessarily that the angular dependence has to be like that because this will ensure that this is you see if you take the derivative d pi by d theta, it becomes sin. So, this becomes sin of pi alpha into theta.

So, which is 0 when this theta is 0 and it is also 0 and theta is alpha and the rest has to be basically equal to n n minus n and that n is necessarily this. This is what n is. So, that is pi by alpha. So, this is your answer ok and then, we also demand that the velocity should vanish at infinity because we do not expect the velocity to keep growing as you go further and further away from the vertex.

So, therefore, we expect this to become 0 because alpha is positive and we expect that to be 0. So, that is the answer ok. So, and then well that is the answer for this pi which is potential of the velocity potential. So, you take the gradient, you get the velocity and so, this is your final answer.

So, let us see what all equations we have exploited to we have spent a lot of time, it looks like we have fully solved the problem, but actually no. The reason why it is no is because we still do not know what this b is and also, we have only exploited one of those equations.

So, remember there were two equations; one is the Euler equation the other is continuity equation. So, all we have done is exploit this equation divergence v equals 0 and the fact that v is the rotational that is all we have exploited. So, we have completely ignored the important equation which tells you how velocity is supposed to change with time which of course, it does not change with time because you assumed steady state ok.

Well, let us see; yeah. So, we have assumed steady state. So, if steady state, so this is 0 ok. So, now, let us go ahead and exploit Euler equation, see where it takes us. So, velocity is independent of time. So, that is the left hand side is 0; left hand side of 4.136. So, therefore, this is saying that basically. So, we will assume that there are no body forces like we will ignore the weight of the fluid and all that. That does not make sense in this 2D geometry.

But however, there is pressure. There has to be because you see these two are not if you if you randomly choose everything 0, then these two will become self-contradictory. So, we have to assume that there is some kind of a pressure gradient and that should be something that your theory predicts. So, in other word, tells you what. So, therefore, there ought to be a pressure gradient that is set up in the fluid in order for the fluid to behave in this particular way.

So, that is the in the correct interpretation of this problem. So, the correct interpretation is that imagine there is a wedge with angle alpha and there is a fluid flowing you know into and out of this wedge in such a way that the velocity of the fluid is 0 perpendicular to the surfaces at those surfaces and 0 at infinity and it is it has uniform density and its steady state.

So, the so, given all these assumptions the question is what sort of pressure gradients have to be set up in the fluid in order for the fluid to be to behave in this particular manner.

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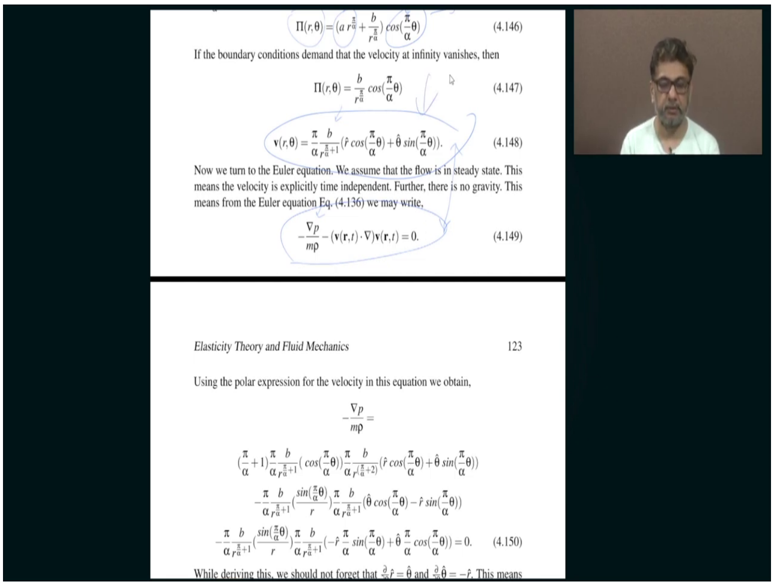

So, the answer is exactly obtained by solving this equation, given the fact that we have already reached this far.

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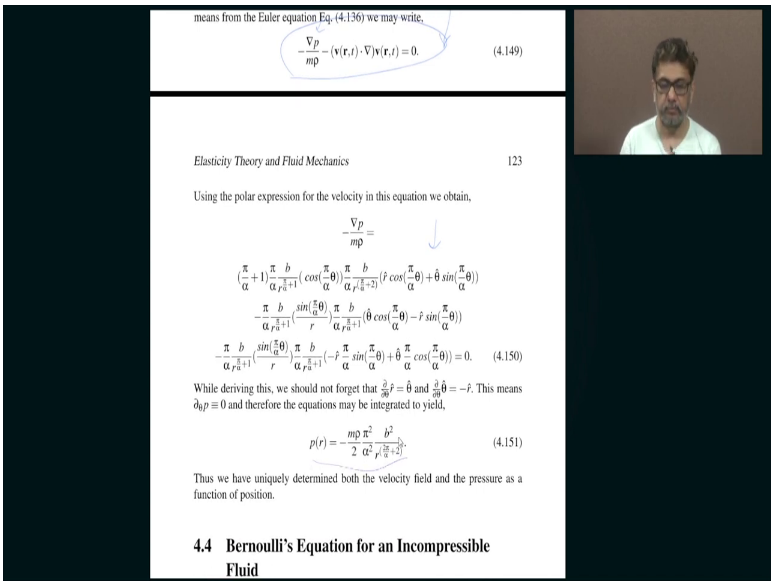
means from the Euler equation Eq. (4.136) we may write,

$$-\frac{\nabla p}{m\rho} - (\mathbf{v}(\mathbf{r},t)\cdot\nabla)\mathbf{v}(\mathbf{r},t) = 0. \qquad (4.149)$$

Elasticity Theory and Fluid Mechanics 123

Using the polar expression for the velocity in this equation we obtain,

$$-\frac{\nabla p}{m\rho} =$$

$$(\frac{\pi}{\alpha}+1)\frac{\pi}{\alpha}\frac{b}{r^{\frac{\pi}{\alpha}+1}}(cos(\frac{\pi}{\alpha}\theta))\frac{\pi}{\alpha}\frac{b}{r^{(\frac{\pi}{\alpha}+2)}}(\hat{r}\,cos(\frac{\pi}{\alpha}\theta)+\hat{\theta}\,sin(\frac{\pi}{\alpha}\theta))$$

$$-\frac{\pi}{\alpha}\frac{b}{r^{\frac{\pi}{\alpha}+1}}(\frac{sin(\frac{\pi}{\alpha}\theta)}{r})\frac{\pi}{\alpha}\frac{b}{r^{\frac{\pi}{\alpha}+1}}(\hat{\theta}\,cos(\frac{\pi}{\alpha}\theta)-\hat{r}\,sin(\frac{\pi}{\alpha}\theta))$$

$$-\frac{\pi}{\alpha}\frac{b}{r^{\frac{\pi}{\alpha}+1}}(\frac{sin(\frac{\pi}{\alpha}\theta)}{r})\frac{\pi}{\alpha}\frac{b}{r^{\frac{\pi}{\alpha}+1}}(-\hat{r}\,\frac{\pi}{\alpha}\,sin(\frac{\pi}{\alpha}\theta)+\hat{\theta}\,\frac{\pi}{\alpha}\,cos(\frac{\pi}{\alpha}\theta)) = 0. \qquad (4.150)$$

While deriving this, we should not forget that $\frac{\partial}{\partial\theta}\hat{r} = \hat{\theta}$ and $\frac{\partial}{\partial\theta}\hat{\theta} = -\hat{r}$. This means $\partial_\theta p \equiv 0$ and therefore the equations may be integrated to yield,

$$p(r) = -\frac{m\rho}{2}\frac{\pi^2}{\alpha^2}\frac{b^2}{r^{(\frac{2\pi}{\alpha}+2)}}. \qquad (4.151)$$

Thus we have uniquely determined both the velocity field and the pressure as a function of position.

4.4 Bernoulli's Equation for an Incompressible Fluid

So, we have found velocity through continuity equation and then you simply substitute that here and you get this result for your pressure ok. So, this basically tells you what the pressure is. So, it of course, tells you in terms of some constant b which is still undetermined and that has to be further supplied by somebody. So, that cannot be determined. Somebody has to say this is the velocity at this point and in terms of that you can express b in terms of that.

So, bottom line is that the pressure has to have a position dependence and it has to be in this way ok. So, that is the that is the story of this fluid flowing through a wedge ok. So, that was just an example which illustrates how you can determine the specific manner image of fluid flows you know given the Euler equation and the continuity equation.

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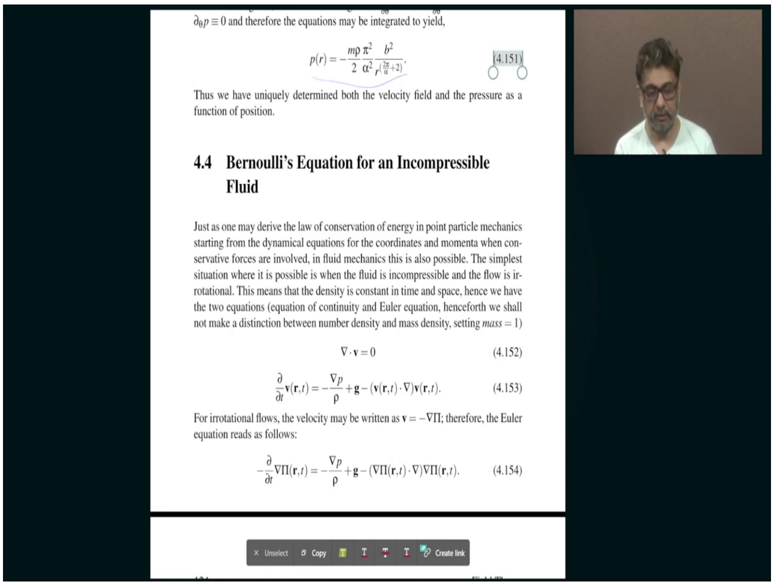

So, I am using further simplifying assumptions. So, now, let us come back to some generalities. So, I am going to discuss a very important result which is familiar to school students. Of course at the school level, you kind of simply told that is what it is and you are forced to memorize that formula and pretty much every formula that you encounter at the school level, you are simply told that is how it is and you are supposed to memorize it. But this course is one where you pull back the screen and you see the bizarre behind.

In other words, I am going to tell you how those formulas came about that you are. So, familiar with that you have memorized from your school days and a I am going to be able to tell you how to derive them now. So, one such equation that you would have memorized long ago would be Bernoulli's equation for an incompressible fluid. So, the Bernoulli principle as it is sometimes called. So, just like you know in classical mechanics, you see if you have Newton's laws of motion and you have a forces which are derivable from a potential, the there is a quantity namely the total energy of the system is conserved. So, the kinetic plus potential.

So, the kinetic energy basically energy is a scalar quantity; whereas, the velocity and position are vector quantities. So, similarly, even here in fluids you see our velocity or the vector quantity and you had this rate of change of velocity equals something which is the Euler equation.

Now, I want to derive a scalar quantity corresponding to these Euler equation continuity equations that is finally, conserved. So, that means, I want a scalar conserved quantity analogous to what I would do with energy in case of point particle classical mechanics. So, I am going to focus or restrict my attention to incompressible fluids in which case divergence of v is 0.

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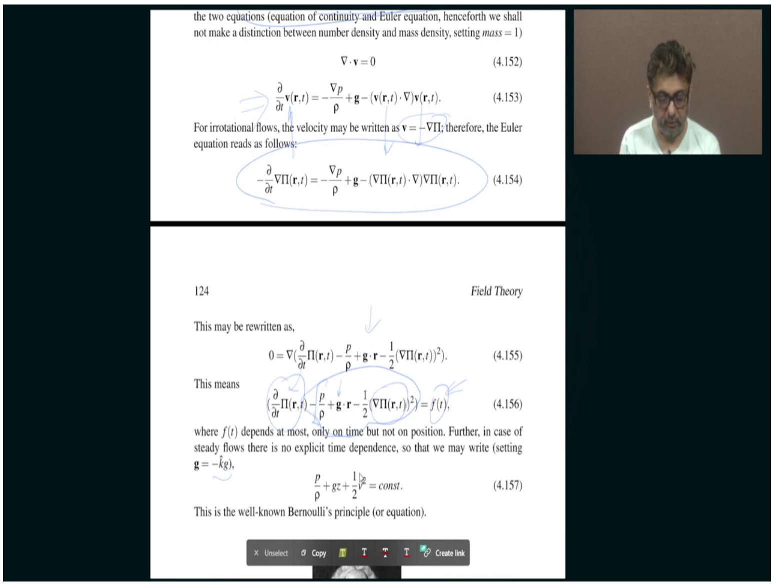

So, you see in that case my continuity equations can be. So, sorry my Euler equation was this now. I know for a fact that in regions where the density is non-zero and in fact, that is pretty much everywhere because density is a constant. So, I am I am going to assume that density is constant ok.

So, I am going I assumed that density is constant in time and space. So, in that case, you have density is non-zero. So, velocity is expressible or derivable from the gradient of a scalar. So, you just go ahead and substitute that here and all of a sudden, you will be able to rewrite this equation. So, this is basically the Euler equation rewritten in terms of this scalar potential whose gradient is the velocity ok.

So, this can be rewritten in this way and so, it is basically tells you the gradient of something involving the scalar potential is 0. So, therefore, that something has to be independent of position ok. So, now, we will further assume that we are talking about steady state.

So, in the case of steady state, there are no explicit time dependences. So, if that is the case, then you can immediately convince yourself that. So, if g is you know g is the acceleration due to gravity which I have included just for good measure. So, that is in the negative z direction suppose in which case you can clearly see that this equation and this is nothing but velocity.

So, this basically becomes so when you have explicit time independence. So, then, this is derivative is 0 and this is independent of time is an absolute constant because anyway it was independent of position, it worst case it dependent on time; but even that is not there because its steady state. So, now, so what this means therefore, is that this equation tells you that this thing put together is constant.

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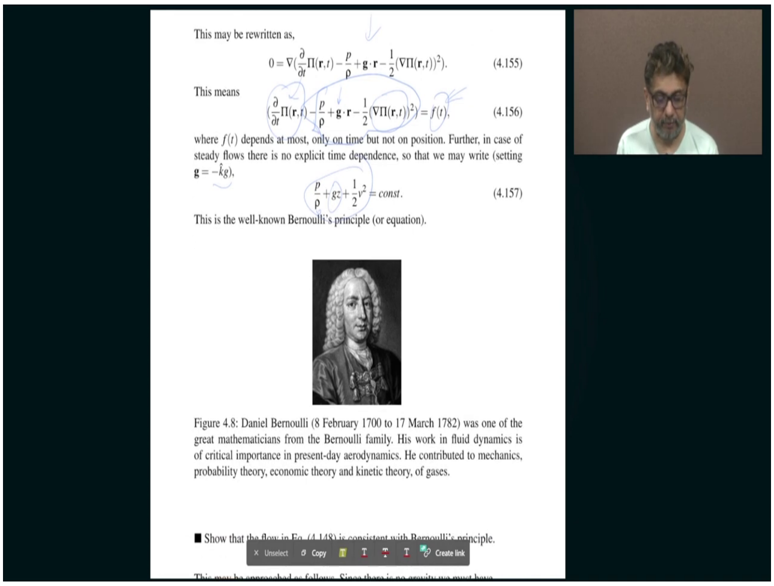

And what this thing is basically energy. Well, energy divided by density. So, energy per unit volume divided by density alright. So, this is basically tells you that whatever that is constant and that is called Bernoulli's principle. So, what it effectively tells you that if you ignore this term, if you ignore the acceleration due to gravity what this says is basically and rho is constant. If rho is constant what it basically says that in regions, where the fluid flow is fast if the velocity of the fluid is speed of the fluid is very high the pressure is low and vice versa.

In fact, I am reminded of this principle every day in my room, where I have my cupboard, where I have my clothes there and it is a wooden cupboard and I do not lock it many times and whenever I switch on the fan, the cupboard door swings open. So, the reason is not because of some mysterious ghost; it is because of Bernoulli principle. So, the fan outside creates airflow with a high velocity and therefore, the pressure outside is less than what is inside. So, inside the cupboard, there is no airflow is not is airflow is static. So, the speed is 0.

So, the pressure inside is high inside the cupboard; but outside, the airflow has a high velocity. So, the pressure is low. So, the in pressure inside the cupboard is more than the pressure outside and if I do not lock the cupboard, the door will swing open. So, that is what happens every time and that is the reason why you have to lock your cupboard because you know otherwise you can lose your belongings or some insects can get in and so on.

Anyway whatever it is. So, that is the; so, you can see the Bernoulli's principle in action in your daily life also ok. So, now, I am going to so, this is another interesting. So, now, that I have told you what Bernoulli's principle is. I am going to ask you to go back to the earlier example involving this wedge and I want you to convince yourself that the wedge equation which basically tells you what the pressure is and what the velocity is.

So, you see we have derived pressure and velocity both by solving Euler equation and continuity equation and so on so forth. So, we know both from this earlier wedge example; the pressure and the velocity. So, if you know both. So, the question is that Bernoulli principle also involves both. So, namely this and this.

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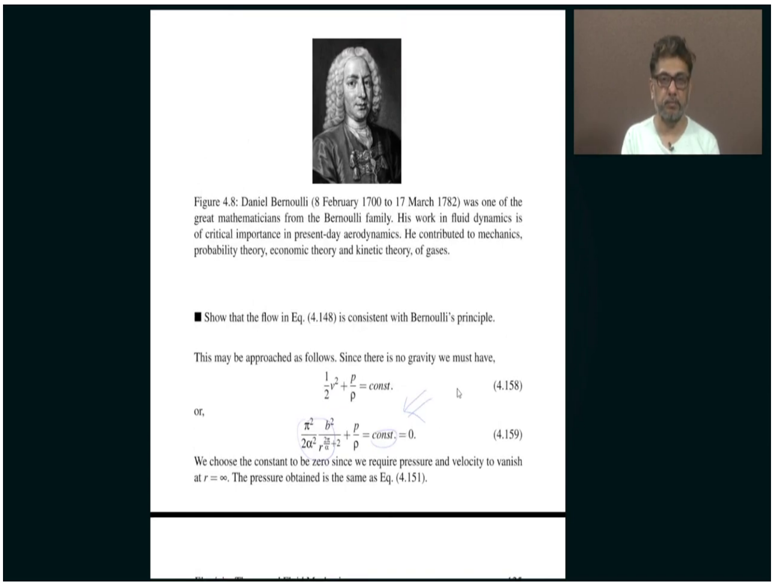

So, the question is that is the earlier derivation of this wedge problem, where we derive the formula for pressure in terms of r and theta and velocity in terms of r and theta; is that consistent with Bernoulli principle. So, let us see if it is or it is not. So, you just substitute v squared, see what is v squared because this is v and this is a unit vector. So, that is the square of that is 1.

So, therefore, v squared is simply equal to so, this ok. So, now, you can see that this is; so, we want the constant. So, this so, what Bernoulli's principle, it says is that half v squared plus P by rho is constant. But then that constant has to be 0 because at r equal to infinity, we expect both the pressure and velocity to be 0. So, if that is the case this is 0. So, therefore, pressure is minus rho times this. So, that is exactly what we got here.

So, in our speed in, we could have derived this 4.151 from Bernoulli's principle; but rather we did not. We actually derived it by substituting the solution of this Laplace equation which you obtain from continuity equation into Euler's equation. So, by substituting Euler's equation, you got the pressure. You could have done it more easily by simply invoking the Bernoulli principle. But well that we had not derived that yet so. So, now, that we have derived Bernoulli's principle your ae free to do either.

So, its typically easier to invoke the Bernoulli principle. Just like it is easier to invoke energy conservation, when you want to solve for the motion of particles because that is already the first integral of the motion; whereas, you see Newton's second law involves two derivatives with respect to time; energy conservation is half mv squared plus potential energy equals constant.

But then, half m v squared v is first derivative in position. So, you already integrated once with respect to time. So, basically it is already a first integral. So, it is always more convenient to start with energy conservation. So, here too, it is always easier to start with Bernoulli principle because that is precisely the analog of energy conservation in fluids ok. I am now I am going to stop this is a good place to stop because the next topic is going to involve viscosity.

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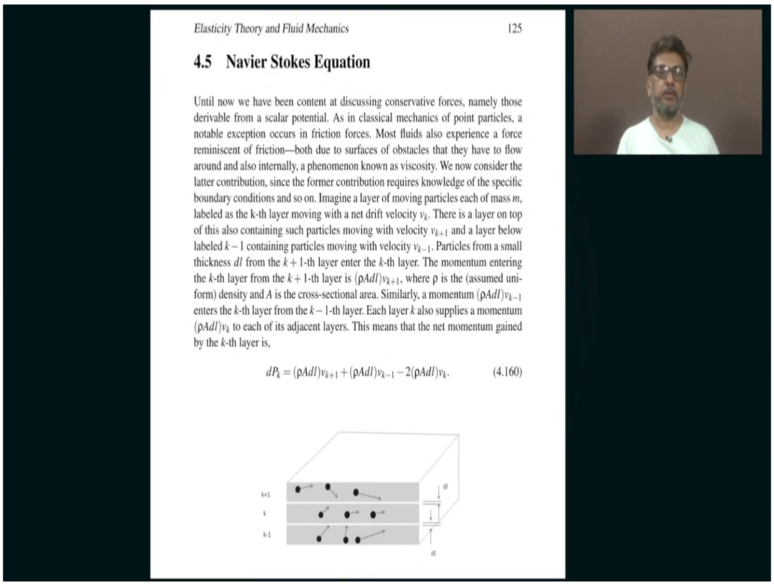

So, the Navier Stokes equation is the next topic, where I am going to describe what I have ignored till now; namely, the fact that fluids do not have to necessarily be ideal. So, that means, the different layers of a fluid can exert friction on each other and that leads to some further modifications of the equations that we have been writing down. So, I am going to stop now. I hope you will join me for the next class, which is all about this very famous Navier Stokes equation ok.

Thank you.