Dynamics of Classical and Quantum Fields: An Introduction

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Fluids
Lecture - 17
The Euler Equation

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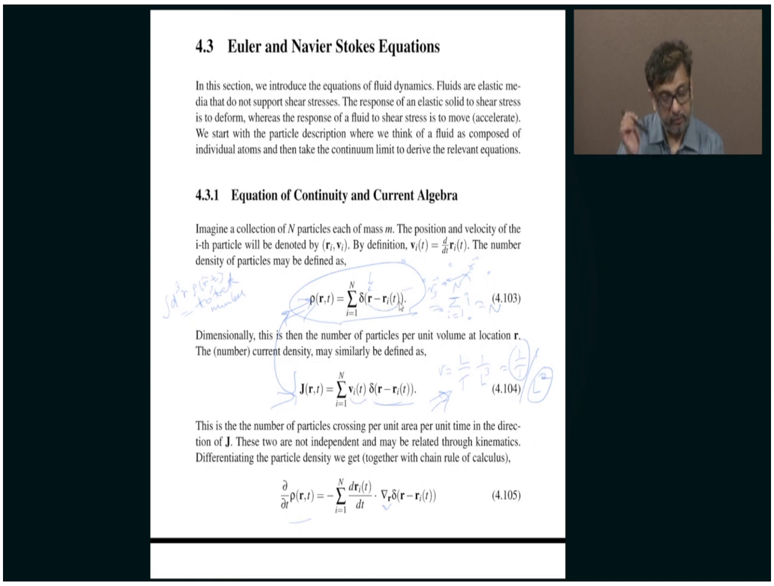

Ok, so today let us start a new topic which is understanding how fluids behave that is fluid dynamics. So, the subject of fluid dynamics of course is a very old one and it is associated with the names of Euler and many others including Navier and Stokes and so on, ok. So, what I will do is that, I will be discussing two aspects to the fluids one is a fluid which does not exhibit viscosity.

So, viscosity is a phenomenon where the layers of the fluid you know exert a friction force on adjoining layers. So, there is some kind of an internal friction. So, that is an important topic, but it is a difficult topic. So, I will relegate that to the end of this discussion. So, the initial topics will be basically the ideal fluid where there is no viscosity. So, for an ideal fluid you encounter what are called Euler equations which are basically a combination of kinematics and dynamics.

So, the kinematics will tell you how the conservation laws appear I mean just like in the case of electromagnetic theory you had conservation laws, where charge was conserved here also you know the net mass of a fluid within a volume; if it is not conserved it is because fluid is either flowing in or out of the system. So, that is a kind of equation of continuity. So, that is going to be our focus.

And then, we will also understand what are the role of forces and how fluids respond to forces. So, the basic difference between an elastic body and a fluid is that an elastic body deforms under the application of shear stress, whereas the fluid flows under the application of shear stress; that means it accelerates. So now, we will have to understand fluids. Because you see the formalism of elasticity theory that we just completed discussing is not particularly suitable to discuss fluids.

Because, if fluids do not deform under the application of shear stress. So, they accelerate or flow and there was no room for such a description in the earlier topic namely the study of deformable elastic bodies it was assumed that the elastic body simply deforms it does not accelerate. So, a rigid body by contrast always accelerates it does not deform. So, fluid is something in between, ok.

So, that is the reason why we need a somewhat new approach. So, what we will do is that, we will take this approach that the fluid is described initially it is more convenient intuitively easy to understand when you think of fluid as being composed of point particles. But, in fact they are that is the correct description anyway, but then you see in the end we really do not want to dwell on that aspect.

That we do not want to in some sense in the end we want to hide that that quality of the fluid namely that it is actually composed of elementary entities, discrete objects. So, rather we want to think of fluids as something continuous. So, but then that is more easily achieved by first thinking of fluid in terms of discrete entities that have their own dynamics and then, finding a way to hide that information and only display the information that corresponds to the continuum description, ok.

So, how do you do that? So to do that, we start off with this definition of the density of a fluid at some point r at a given time t. So now, like I told you just now I am going to think of this fluid as consisting of large number of discrete entities. So, discrete particles for example, capital N of them and let us assume that each of them have mass m. And then, clearly from our classical mechanics background we can reel off a bunch of facts about these collection.

This collection of particles namely that if r i is the position of the ith particle at time t the velocity of the ith particle is going to be the rate of change of r i with t. So, and that is what we call as v i. So now, given the fact that we have this r i and v i in other words something like phase space. So, we can construct certain objects which are going to be of immense value later on when we try to describe the fluid.

The first quantity of interest is what is known as the density of particle at a given position r at time t. So, clearly you see if you think of fluid as being composed of discrete particles; you know it you might think that is quite an absurd concept to introduce, because after all, what is density is just the in this case is the number density the number of particles per unit volume.

But then, I am not talking about a finite volume, I am talking about at some point. So that means, that you choose a point r and you look at a really really tiny volume delta V and you count how many particles there are which is delta N and your delta N divided by delta V is your density there. But, if your model your particles are discrete that limiting process is meaningless.

Because, the moment you go if your volume is smaller than the inter particle separation cubed for example, then you would necessarily get 0 as your answer most of the time. Because, most of the points do not I mean the most whatever point r you select in space will typically be unoccupied by a particle. So, if the particles are point objects. So, they occupy only some point.

So, even though that seems absurd, but it is also true that the moment that r becomes equal to the location of one of the particles the density is in fact infinite. So, in other words it has this property that the density is infinite when r is exactly one of the locations of the particle and it is 0 when r is not one of the locations of the particle. So, that should immediately make you recall an important concept that you ought to know by now and that is the concept of the Dirac delta function.

So, the Dirac delta function is precisely that concept which has this property. Namely it is 0, if you know the argument is not 0 and it is infinite when the argument is 0. So, I am going to go ahead and define the density of particles at position r at time t to be the sum of all the Dirac delta functions for the particles at various locations. So, this ensures that if I am looking at position r the density is going to be 0 if r is not equal to any of those positions where the particles are located.

So, clearly you know we expect that to be the case. So, if this is where all your particles are there, then if your position r is this, then clearly density is 0. But it is so if r hits one of them say r 3 so, if this r is exactly r 3 then clearly the density is infinite. So, that is what this definition ensures. So then, now you can see that if you integrate over all r. So, if I do this, right.

So, if I find out the total number of particles. So, this is the total number of particles because, recall that rho r comma t is the number densities the number of particles per unit volume at position r at time t. So, this is the total number. So, now if I integrate this out what do I get? I basically get so integral of the Dirac delta is 1 and this is just one being summed over N times.

So, that is basically N and that is exactly what we expect because, we expect the total number of particles to be capital N. So, this is consistent with what we expect. So, we will go along with this. So, in addition to the number density of particles we also have to introduce what is called the current density. So, that is the number of particles flowing per unit area per unit time you know in some particular direction, right.

So, that is what current density is. So, it is some kind of a flux, the flux of particles tells you how fast the particles are moving across a given cross sectional area. So, to define that we clearly so, you must know that is you know it is just this is what it is going to be because, what this is basically this will tell you it is. So, V has dimensions of length by time and this has dimensions of density.

So, that is 1 by length cubed, right. So, this is basically 1 by time by length square. So, in other words dimensionally what this is saying is the number of particles flowing per unit time per cross sectional area L square at least dimensionally, but physically also that is the meaning it has. In fact, to convince yourself that this has that physical meaning you what you have to do is you have to relate these two namely you have to relate the current density to the number density.

And that is when you get to know the real physical mean, but dimensionally it is clear that, that is likely to be the physical meaning of that. So, let us go ahead and relate rho and J. So, how do you do that, you see look rho depends on time because the position of the particle depends on time.

So now, I am going to go ahead and differentiate with respect to time and then you will see that by chain rule I have to first differentiate the argument which is same as differentiating r because, it appears as a difference and that is going to be the gradient; times the derivative of the argument of the delta function with respect to time and that is $-\frac{dr_i(t)}{dt}$.

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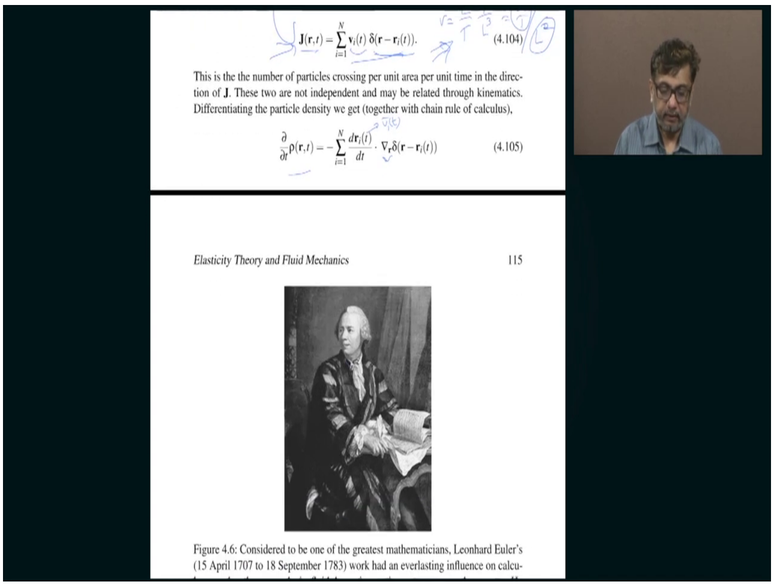

And what is our $\frac{dr_i(t)}{dt}$ is precisely the velocity of the ith particle. So now, that is going to be what it is for $\frac{d\rho}{dt}$, ok. So, this is I forgot a step. So, this is basically v_i. So now, however if I take the divergence of J, I get the same thing without the minus sign. Because, if you take the divergence of J you will get v i dot grad acting on the delta function.

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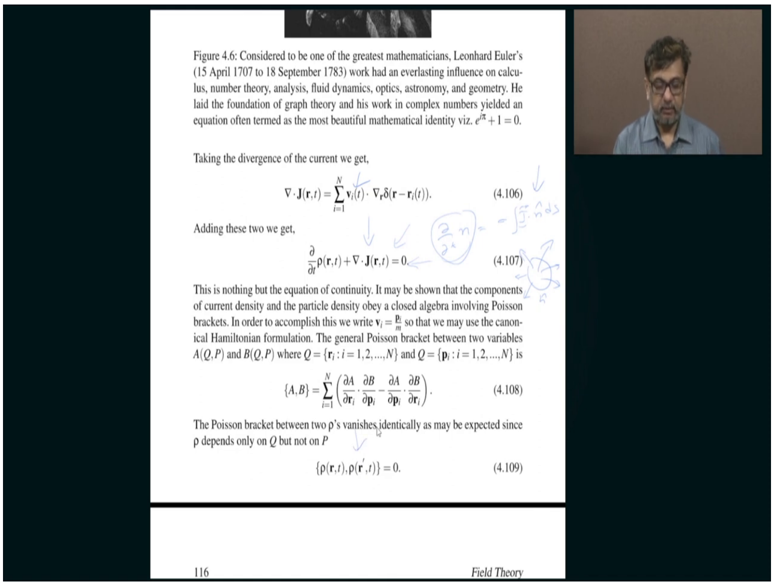
Figure 4.6: Considered to be one of the greatest mathematicians, Leonhard Euler's (15 April 1707 to 18 September 1783) work had an everlasting influence on calculus, number theory, analysis, fluid dynamics, optics, astronomy, and geometry. He laid the foundation of graph theory and his work in complex numbers yielded an equation often termed as the most beautiful mathematical identity viz. $e^{i\pi}+1=0$.

Taking the divergence of the current we get,

$$\nabla\cdot\mathbf{J}(\mathbf{r},t)=\sum_{i=1}^{N}\mathbf{v}_i(t)\cdot\nabla_{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}_i(t)). \quad (4.106)$$

Adding these two we get,

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t)+\nabla\cdot\mathbf{J}(\mathbf{r},t)=0. \quad (4.107)$$

This is nothing but the equation of continuity. It may be shown that the components of current density and the particle density obey a closed algebra involving Poisson brackets. In order to accomplish this we write $\mathbf{v}_i=\frac{\mathbf{p}_i}{m}$ so that we may use the canonical Hamiltonian formulation. The general Poisson bracket between two variables $A(Q,P)$ and $B(Q,P)$ where $Q=\{\mathbf{r}_i:i=1,2,...,N\}$ and $Q=\{\mathbf{p}_i:i=1,2,...,N\}$ is

$$\{A,B\}=\sum_{i=1}^{N}\left(\frac{\partial A}{\partial\mathbf{r}_i}\cdot\frac{\partial B}{\partial\mathbf{p}_i}-\frac{\partial A}{\partial\mathbf{p}_i}\cdot\frac{\partial B}{\partial\mathbf{r}_i}\right). \quad (4.108)$$

The Poisson bracket between two ρ's vanishes identically as may be expected since ρ depends only on Q but not on P

$$\{\rho(\mathbf{r},t),\rho(\mathbf{r}',t)\}=0. \quad (4.109)$$

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So, now if you add these two you will clearly get 0 because, they are the same apart from a sign. So now, this is the equation of continuity and this will tell you the precise meaning of that. So, if you know this we have encountered this again and again in many contexts for example, in electromagnetic theory and so on.

So, you integrate over r this is just going to be the number of particles so, if you integrate over some volume. So, this is the number of particles and this is nothing but the flux J dot n because you can use Gauss's theorem and so what this says is the so, if suppose this is J and this is suppose J is parallel to N ok, what this is saying is that. So, J is the number of particles flowing out in this case.

So, flowing in the direction of J which is N so, N is out. So, so it is the number of particles flowing in the direction of J per unit area per unit time and if I multiply by d s.

So, what I am doing is basically I am counting how many particles are flowing out per unit time ok and so, clearly that with a minus sign is exactly the rate of change of the number of particles with time, because if particles are flowing out N decreases with time. So, dn by dt is going to be negative and magnitude wise it is going to be exactly that it is the number of particles flowing out per unit time.

So, that is the reason why J has that interpretation. So, I dimensionally gave you the interpretation of why J has the interpretation of number of particles flowing in the direction of J per unit area per unit time and the correct technical reason is because of this. So, you derive the continuity equation and you integrate over a volume and this rigorously has that interpretation not just dimensionally, but physically also. So, that is as far as the kinematics is concerned.

So, you seen we have not had any occasion to use the information of the forces that may or may not be acting on the particles. So, I have not even told you if forces are acting they could be acting, they could not be acting. So, far I have not used that information at all so, but I am going to use that later on. But, even before you use that information there is one important set of properties that you can derive about this rho and J which involves again only kinematics and that is what is known as current algebra.

So, basically current algebra is in this context a relation which tells you that the Poisson bracket of the density and the various components of the current they are all related linearly to each other, ok. So, that is called a Lie algebra. So, you see it is an algebra in the sense that the Poisson bracket operation is a linear one. So, the Poisson bracket of A plus B with c is Poisson bracket of A with C and B with C.

So, that is the reason why it is called a Lie algebra, but you do not have to get stuck on the terminology. Bottom line is that what you do is you evaluate the Poisson bracket of because after all rho and J involve the positions of the particles and we know from classical mechanics, how to calculate Poisson bracket, if you know the velocities and positions of the particles rho and J depend on the velocities and positions of the individual particles.

Then you can easily convince yourself the Poisson bracket of rho and rho even when r and r dash are different, but so, long as times are the same they are going to be 0.

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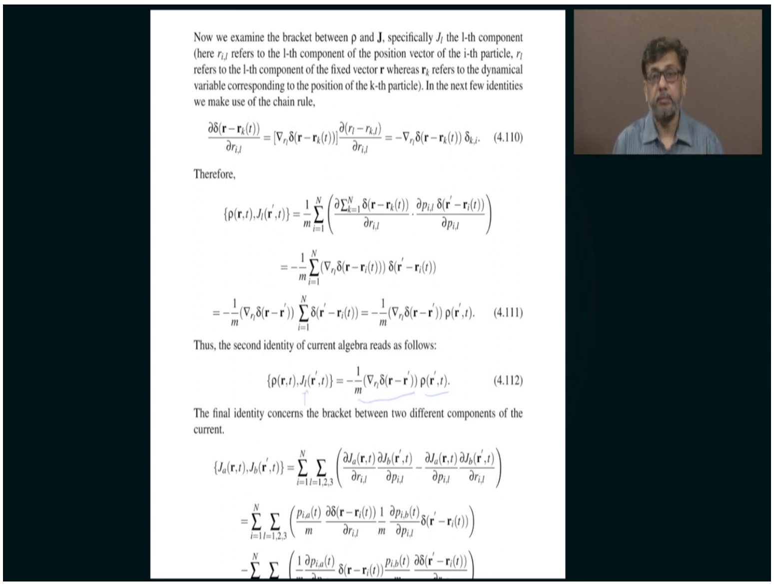

And similarly so, I am not going to dwell on the you know the derivation because it is just a tedious algebraic manipulation which you have to simply go through patiently. So, bottom line is that so, it has this relations so the Poisson bracket of rho and rho with two different r and r dash are 0.

Whereas, the Poisson bracket of rho and an appropriate component of the current is again proportional to rho and the proportionality will depend on r and r dash, ok. And of course, the component that you are talking about in this case l and one should not forget the mass each particle has some mass. So, remember the four things in the four quantities that we are talking about one is density, the other is say the x component of current, third is y component of current, fourth is z component of current.

So, there four objects. So, we have to find out the Poisson bracket between any two of them. So, rho rho is 0 Possion bracket rho rho is 0 rho with any other component of l is determined by this formula and lastly J with other components of J.

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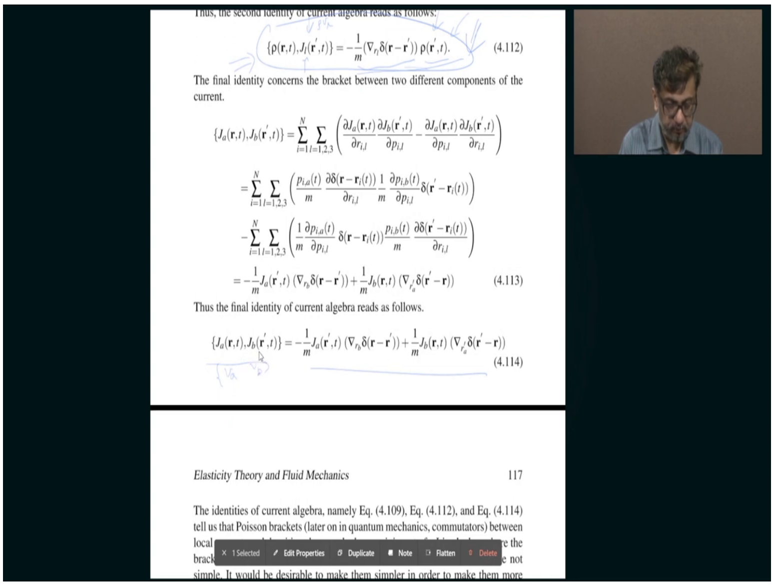

So, that also can be evaluated and when you evaluate the answer comes out to be this. So, it is basically a linear combination of these two J's themselves, but with appropriate r and r dash and so on, ok.

So, you might be wondering why I mentioned this. I mentioned this firstly, because it is interesting in it is own right. In the sense that, the Poisson bracket of rho and various components of J they form a closed algebra means, the Poisson bracket does do not involve anything other than rho and the various components of J; you do not have to invoke new notions. Because that is going to be very useful because, remember I told you that in the end I want to hide the fact that a fluid is contained or made of discrete number of particles.

But then, till now I have not had you know a proper way of doing that and the first inkling that it might be possible to do that comes actually from the equation of continuity it is self. You see even though rho and J themselves depend on the discrete particles the definitions depend on the discrete particles involved, but the equation of continuity has no mention of that. There is no mention of r i or v i in equation 4.107.

So, in other words the information that the fluid actually consists of discrete particles is hidden in the background. So, eventually that we are actually going to forget about it altogether, ok. So, this is one equation that allows us to later on forget about the fact that the fluid is made of discrete particles. So, similarly with Poisson bracket as well you can see that these Poisson brackets also do not have any information that the rhos and Js are actually in the end made of the discrete particles that we were initially talking about.

So, even though notice that in order to derive these Poisson brackets we had to explicitly make use of the fact that the rhos and J's are described in terms of discrete particles. So, we had to make use of it, but having made use of it the end result is that the Poisson brackets do not retain any information about the discrete of the underlying system. So, that is very interesting.

So, these two aspects namely the equation of continuity and the Poisson bracket between rho and various components of J will allow us to eventually be faithful to our original intention namely that we want to describe fluid as something continuous that flows, ok rather than some grainy discrete collection of particles, ok.

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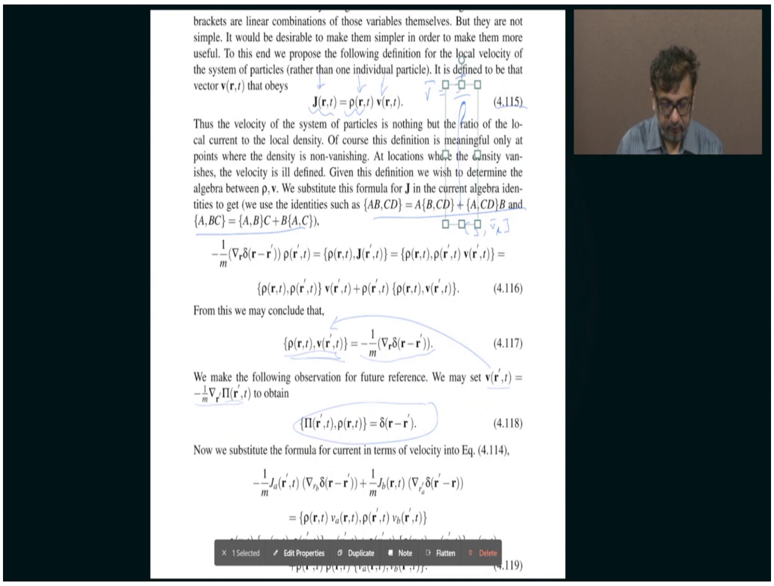

So, having done all this let us go ahead and see, but ok so, in order to motivate the next discussion I have to point out one thing.

And that is that the Poisson bracket that we derived even though they look very nice they are still not simple enough, because you see the Poisson bracket of rho and rho is 0 that is nice that is simple it cannot be any simpler than that, but then rho and some component of J is proportional to rho yes this is also simple and I doubt you can make it any simpler than this, but what is not simple is this J with J.

So, I would much rather so even this if I could I would much rather make it simpler. So, in other words what I ideally want is I want to redefine rho and J or any one of them in terms of other quantities such that the Poisson bracket of those will involve some constants on the right side. So, I do not want it to again involve rho and J or something similar. So, I do not want dynamical variables on the right hand side.

So, first identity does that already is identically 0, second one does not do that. So, I want to avoid that. So, in other words I want to redefine J and rho in such a way that the right hand side of the Poisson bracket is actually something unrelated to the dynamical variable. So, in other words it involves only r and r prime which we regard as something fixed somebody has told us what they are.

So, how do we do that? So, the way we do that so, the reason why we want that is because it makes you know calculations successively simpler. So, the way to do that is to define what is called the velocity of the fluid rather than the velocity of the individual particles which we have identified as v i; we now define velocity of the fluid and that is defined as basically through this relation.

So, the current density is postulated to be equal to the particle density times the velocity of the fluid. So, in other words V is by definition J by rho of course, this makes sense only for point's r where rho is not 0. So, now the question is why is this simpler to do. So, what I am going to do is rather than find the Poisson bracket of the various components of J and rho I am going to try and find the Poisson bracket of rho and the various components of the velocity of the fluid.

So, that would make more sense. So, I am going to use identity such as this and then, go ahead and substitute this definition here, ok. So, if you substitute that here you will see you will get exactly what I was talking about. So, you just substitute this relation 4.115 you substitute it in this relation. So, you just write J l as rho times v l and then you work out therefore, what should the Poisson bracket of rho and v l be.

So, you will find in fact it is this. Namely the Poisson bracket of rho and the velocity of the fluid is something unrelated to any of the dynamical variables it depends on r and r prime. So, they are just fixed positions in space, alright. So, this achieves what I had set out to do, but I am a little greedy I want it to be even simpler and so, instead of having a gradient of a function I want a delta function on the right side. So therefore, I am going to define a new quantity called pi.

I am going to say surmise that the velocity of the fluids therefore should be proportional to the gradient of that scalar, ok. So, this is called irrotational. So, this would correspond to an irrotational fluid, ok. So, this is the velocity is derived from a potential because the curl of V is 0 so, it is irrotational. So now, we go ahead and substitute this here and you can convince yourself the pi and rho have this property the Poisson bracket of pi and rho is basically the Dirac delta function, ok.

So, the question last question is that how old this so, instead of J and J a and J b we are now compelled to look at this V a and V b. So, is that going to be something simple.

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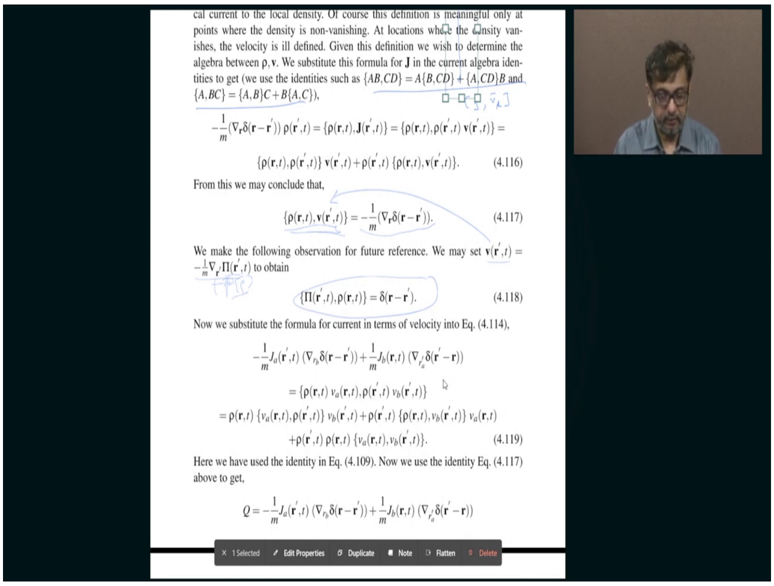

cal current to the local density. Of course this definition is meaningful only at points where the density is non-vanishing. At locations where the density vanishes, the velocity is ill defined. Given this definition we wish to determine the algebra between $\rho, \mathbf{v}$. We substitute this formula for $\mathbf{J}$ in the current algebra identities to get (we use the identities such as $\{AB,CD\} = A\{B,CD\} + \{A,CD\}B$ and $\{A,BC\} = \{A,B\}C + B\{A,C\}$),

$$-\frac{1}{m}(\nabla_{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}'))\,\rho(\mathbf{r}',t) = \{\rho(\mathbf{r},t),\mathbf{J}(\mathbf{r}',t)\} = \{\rho(\mathbf{r},t),\rho(\mathbf{r}',t)\,\mathbf{v}(\mathbf{r}',t)\} =$$
$$\{\rho(\mathbf{r},t),\rho(\mathbf{r}',t)\}\,\mathbf{v}(\mathbf{r}',t) + \rho(\mathbf{r}',t)\,\{\rho(\mathbf{r},t),\mathbf{v}(\mathbf{r}',t)\}. \quad (4.116)$$

From this we may conclude that,

$$\{\rho(\mathbf{r},t),\mathbf{v}(\mathbf{r}',t)\} = -\frac{1}{m}(\nabla_{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}')). \quad (4.117)$$

We make the following observation for future reference. We may set $\mathbf{v}(\mathbf{r}',t) = -\frac{1}{m}\nabla_{\mathbf{r}'}\Pi(\mathbf{r}',t)$ to obtain

$$\{\Pi(\mathbf{r}',t),\rho(\mathbf{r},t)\} = \delta(\mathbf{r}-\mathbf{r}'). \quad (4.118)$$

Now we substitute the formula for current in terms of velocity into Eq. (4.114),

$$-\frac{1}{m}J_a(\mathbf{r}',t)\,(\nabla_{r_b}\delta(\mathbf{r}-\mathbf{r}')) + \frac{1}{m}J_b(\mathbf{r},t)\,(\nabla_{r_a'}\delta(\mathbf{r}'-\mathbf{r}))$$
$$= \{\rho(\mathbf{r},t)\,v_a(\mathbf{r},t),\rho(\mathbf{r}',t)\,v_b(\mathbf{r}',t)\}$$
$$= \rho(\mathbf{r},t)\,\{v_a(\mathbf{r},t),\rho(\mathbf{r}',t)\}\,v_b(\mathbf{r}',t) + \rho(\mathbf{r}',t)\,\{\rho(\mathbf{r},t),v_b(\mathbf{r}',t)\}\,v_a(\mathbf{r},t)$$
$$+\rho(\mathbf{r}',t)\,\rho(\mathbf{r},t)\,\{v_a(\mathbf{r},t),v_b(\mathbf{r}',t)\}. \quad (4.119)$$

Here we have used the identity in Eq. (4.109). Now we use the identity Eq. (4.117) above to get,

$$Q = -\frac{1}{m}J_a(\mathbf{r}',t)\,(\nabla_{r_b}\delta(\mathbf{r}-\mathbf{r}')) + \frac{1}{m}J_b(\mathbf{r},t)\,(\nabla_{r_a'}\delta(\mathbf{r}'-\mathbf{r}))$$

And you will see that. In fact, it is simple and in fact, you will be able to convince yourself that the Poisson bracket of pi and pi are 0. So, after a long calculation you will convince yourself that in fact the Poisson bracket of pi and pi are 0.

So, there is one settle point which I have been struggling with which I will of course, discuss later and that is this identification that the velocity is irrotational is not entirely obvious; it could be something I mean at this stage it is not obvious that the velocity is irrotational. You could always add something to it such that the Poisson bracket of rho after all Poisson bracket of rho with any other rho is 0.

So, even if you add some function of rho it is still valid. So, it is not clear at all that the velocity has to be necessarily irrotational, but later I will convince you that. In fact, it is the case that the velocity is irrotational in this example, ok. So, those technical details I will discuss later, ok.

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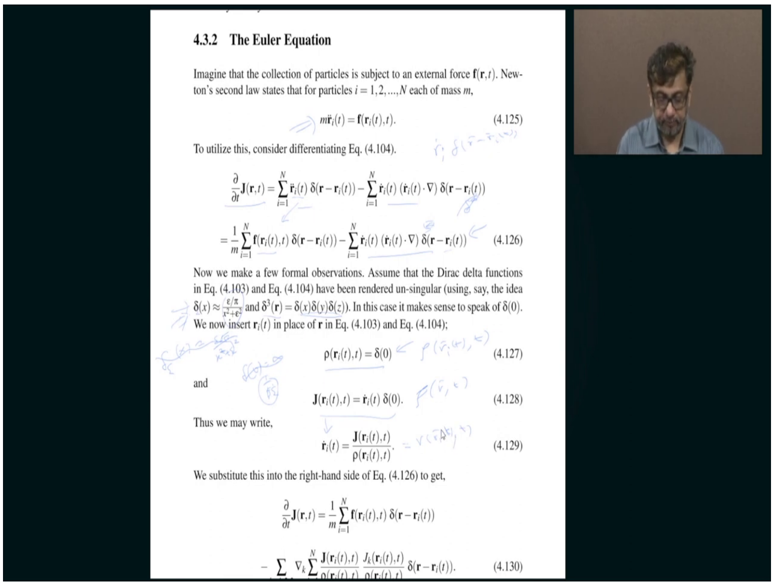

So, now let us move to something important and that is trying to understand how to incorporate forces that may be acting on the particles of the fluid. So, till now we had only discussed kinematics. So, in other words the continuity equation only it is basically a kinematic relation it is true regardless of whether forces are or are not acting on the particles.

Similarly the Poisson bracket identities are also kinematic. They are always true regardless of whether forces are there or not. However, the next topic is of course, I mean the fluid the behavior of the fluid is incomplete until you specify what forces are acting.

Because, merely knowing the continuity equation will not allow us to you know explain how the fluid flows because that is very incomplete information. So, we have to take into account the forces that are acting on the particles of the fluid. So of course, we are going to start with our familiar starting point which is Newton's second law we are going to write mass times acceleration.

So, mass time acceleration is force. So now, I am going to rather than so, in the case of continuity equation I inquired about the properties of the rate of change of the density of particles, the density of the fluid at position r at time t and how the density changes with time. So now, I am going to so that involved only kinematics. So, the answer to that question involved only kinematics.

Whereas the next obvious question is how does the current density change with time clearly involves more than kinematics it is going to also involve dynamics. Because, you know the current is basically related to the velocity of individual particles and rate of change of velocity is clearly acceleration and acceleration necessarily involves knowing what forces are acting, ok.

So, just let us blindly differentiate the current density with respect to time and what you will find is that remember that J was defined as r i dot which is velocity times the Dirac delta of r minus r i.

So, if I differentiate with time I get this this. So, I get these two relations, ok. So now, I am going to write acceleration as force divided by mass and so, now I am gradually allowing myself to simplify or explicitly write down what the rate of change of current with the time should be in terms of again variables that do not involve the explicit discrete positions of the particles or their velocities. So, I want to be able to answer this question. What is the rate of change of the current density of the fluid only in terms of properties of the fluid?

Namely the current density of the fluid and the particle density of the rho and J. I do not want to involve r i and v i. So, involving r i and v i even at the last step would mean that you are describing the fluid not in terms of continuous object that is a continuum rather than you are invoking it is discrete nature which is not something what we want to do. So, now, how do we proceed beyond this? So, we are going to assume that the Dirac delta function is so at this stage is very hard to proceed further until you make some concessions.

And the concession we are going to make is the following that, I am going to simplify this make by assuming this Dirac delta function is actually a limiting case of some well behaved object. So, we all know that in at least by physicists define Dirac delta function as the limit of a sequence of well behaved functions. So, just think of as a well behaved sequence of functions parameterized by epsilon where you define epsilon by pi x squared plus epsilon square.

And then, what physicists do is that we think of epsilon tends to 0 and then we say that look this is Dirac delta function the limit as epsilon tends to 0. Bottom line is that limit really does not exist as a bona fide function it goes outside the space of bona fide functions. So, they are called generalized functions. So, just like you know limit of a sequence of rational numbers need not be rational. So, it goes outside the space of rationals. So, similarly the limit of a sequence of bona fide functions like epsilon by pi divided by x square.

So, this is perfectly regular it is infinitely many times differentiable it is continuous whatever, but then it is limit as epsilon tends to 0 is anything but regular. So, just like if you can always construct a sequence of rational numbers that finally, converge to square root of 2. So, that goes outside this sequence of rational numbers. So, since we have made peace with that idea we should also be able to make peace with this idea that the limit of a sequence of perfectly well behaved functions may not be well behaved.

So, that is what we are going to do. So, then that case the 3 dimensional Dirac delta function. So, I have not put in that delta with a 3 reminder there, I have to put a 3 reminder around top. So, we will all assume that when the arguments of vectors it is really the three dimensional Dirac delta function. So, in which case this three dimensional delta function is just delta of x times delta of y times delta of z, ok.

So, the reason why I made this re interpretation is because, I want to be able to make sense out of delta of 0. See normally delta of 0 is actually infinite, but then before taking this epsilon tends to 0 delta of 0 is not infinite. In fact, delta of 0 is 1 by pi epsilon it is only when epsilon tends to 0 it is infinite. So, I am going to assume that provisionally epsilon is smaller, but not yet 0.

So, in which case it makes perfect sense to talk of delta of 0. So, in which case if it makes sense to talk of delta 0 then I can happily put rho of r i. So, I can ask what is this. So, I knew the answer to this which is the sum of Dirac deltas, but so that is clearly 0 unless r is equal to r i and when r equals r i is actually infinite, it is infinite because the answer is delta of 0 and delta of 0 is infinite.

But in my way of doing things now delta of 0 has been rendered finite because I have smeared it out I have used. That is why I have I have not yet taken the epsilon tends to 0 limit. So therefore, I am entitled to write this. So, it is a rho of r i equals delta of 0. Similarly, J of r i is r i dot times delta of 0. So, this allows me to write the velocity of the ith particle as the density of the fluid at r i divided by so, it is basically the velocity of the fluid at r i that is hardly surprising that is fairly obvious. So, it is the velocity of the fluid at position r i.

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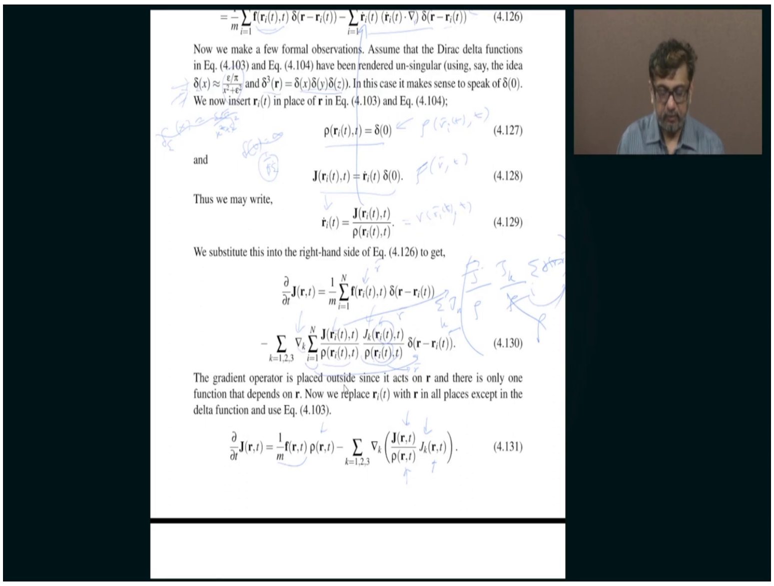

So, now the reason why I want to do this is, because I want to go ahead and insert this relation here, ok. So, I want to insert this relation here because, r i dot appears there and see I have managed to get rid of the r i with two dots which is acceleration I have written in terms of the force, but there is this annoying r i dot here and there I want to rewrite that in terms of the fluid.

So, remember what I am doing. I am consciously trying to get rid of the discrete particle description and trying to rewrite everything in terms of fluids. So, therefore, I have written r i dot which is the velocity of the ith particle now has the current density of the fluid at r i divided by the number density of the fluid at r i. So, when I do that I get this relation. So, it is the gradient with respect to r i. So, this is clearly in this acts only on r by definition, ok.

So, this can happily be outside. So, this is just the kth component and then this V k. So, it is basically this is V dot grad and this is V, alright. So now, because I have taken this gradient outside and this is inside. Now, I can go ahead and confidently replace the r i's by r. You see what I am doing. So, I purposely did this because, I want to systematically find a way of you know simply getting rid of the particle descriptions altogether. So now, because of this Dirac delta function r i is equal to r.

So, I can now eliminate that r i and then, I replace it by r, ok. So now, when I do that so, what I get here is basically. So, I get J by rho times J k by rho times sigma i delta of r minus r i, right. So, that is what I get. So, and then there is a sigma k and grad k outside, ok. So, that is what this is. Now, what is sigma i because now here all the r i's have become r. So, the sigma's can now be taken inside here and then sigma i of delta of r minus r is nothing but precisely the density again.

And these two densities cancel and you get only a J k there. So, otherwise it was J vector by rho times J k by rho times sigma i delta of r minus r i, but then sigma i delta of r minus r i is again rho. So, this rho cancels with this rho and you get this, ok. And similarly here this becomes f of r, r i becomes r and this becomes f of r t and this sigma of i becomes rho. So, you see it is quite amazing that with this simple device of you know re interpreting the delta function make rendering it un singular allows us to very cleanly derive.

What is eventually going to be called the Euler equation? Which is basically tells you how the current density in the fluid changes with time. So, you see that in 4.131 there is no reference to the underlying particle description of the fluid. So, it only involves the properties of the fluid and the forces that are acting on the fluid. So, it involves rho and it involves J rho is the number of particles of the basically it is the it is the number density of the fluid, ok.

So, and if you multiply by mass it becomes mass density the mass of the fluid per unit volume. So, that is even more compelling, because then mass of the fluid you can think of the fluid as a continuous entity and then you have you focus on any volume and you will find some mass there.

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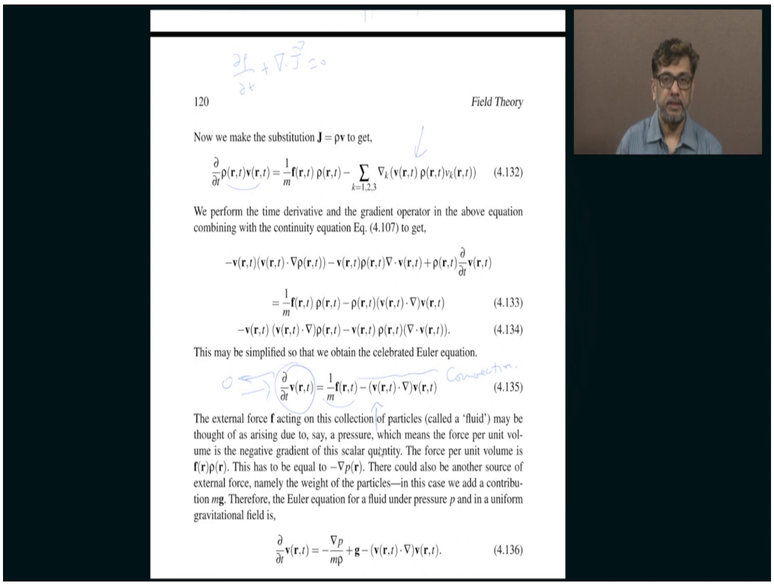

So, now we go ahead and so, we are almost there and this is this is all we need because, you see the equation of continuity was the other one which told me how the density of the fluid changes with time.

Now, this equation tells me how the current density changes with time. So, put together I have what I need, ok. So, now I am just going to the rest of the journey is kind of you know smooth sailing, because we already defined J as the particle density times the velocity of the fluid. So, in other words we know that we have introduced the concept called velocity of the fluid which allows us to write the current density of the fluid as rho times v.

So when you do that, you get this relation ok and then you expand it out and you cancel a whole bunch of terms and you get this very simple beautiful relation. So, what this tells you is that the rate of change of the velocity of the fluid is just the force of acting on the fluid divided by mass, but there is an extra term caused by the fact that the velocity of the fluid can change from point to point and this is what is called the convective derivative.

So, this is the convective derivative. So, what it says is that you see. So, you can have a situation where. So, how do you interpret this 4.135, what this is saying is that normally if you apply force there has to be an acceleration. So, if you take a bunch of particles you apply force they will accelerate, but a fluid can do something more interesting. So, you can have a situation where you apply force the velocity of the fluid remains independent of time.

So, you can have a situation where this is always 0 there is no explicit dependence on time. So, the velocity of the fluid remains independent of time, but the application of force will then mean that the velocity of the fluid changes from point to point. So, at different points the fluid has a different velocity. So, the so fluid can do something more interesting than what a point particle can do. A point particle will simply necessarily accelerate under application of forces.

But whereas, the fluid can do something more interesting, it can certainly accelerate, but it can also choose not to accelerate, but instead redistribute it is velocities in such a way that this equation is obeyed. So, in other words the application of force will cause the velocity of the fluid to become spatially dependent. So, in other words it will exhibit spatial in homogeneity, ok.

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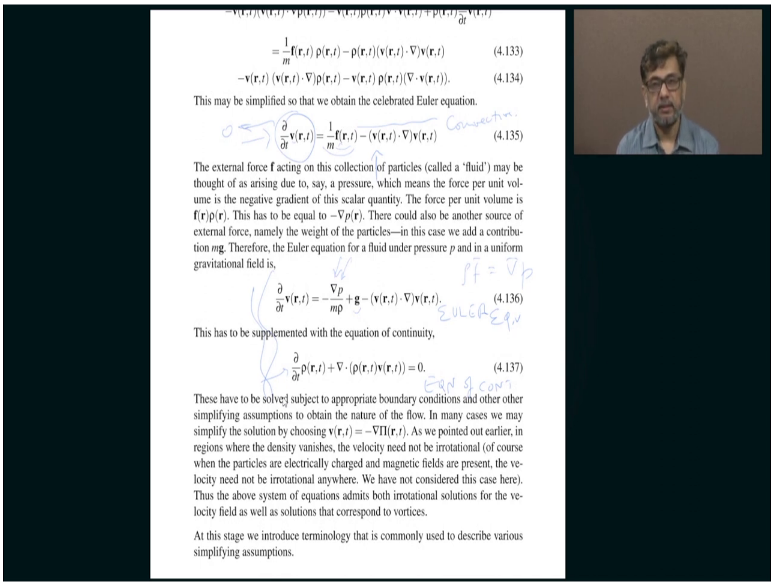

So, typically what happens is that the force that you are talking about is writable in this way. So, this is actually force per unit volume I should not sorry that is really the force yeah because, dimensionally this is velocity so this is force it is not force yeah.

So, it is actually the force and so this is this force can be written as so, just typically what happens is that you normally think of force as being derivable from a potential, but in this case we think of something else has derivable from a potential and that is rho times f is derivative from a potential and that is defined as grad P, ok.

Rho is the density of the fluid, ok. So, we think of that as the derivable from a potential and so that is called grad P, ok. So, and P is the pressure. So, that it has the interpretation of pressure. So, this is the pressure of the fluid and you can also have a force because of gravity. For example, you know if the fluid has mass and it is on earth it will exhibit acceleration due to gravity and then, you also have a relation which tells you how the density of the fluid changes with time and that will depend on the velocity of the fluid.

So, these two equations when solved with appropriate initial conditions will tell you how the fluid flows you know subsequent to the application of a force. So, you have some initial condition you apply some forces and then you want to know what the fluid does subsequently the solution of 4.136 which is what is called Euler's equation and 4.137 which is the equation of continuity.

So, this is the equation of continuity and this is the Euler equation. So, these two equations put together will tell you how the fluid flows subsequent to the application of some force with appropriate initial conditions, ok. So, I am going to stop here and in the next class I will give you some simple examples of fluids and we will try and work out the solutions to this in some simple contexts.

So, keep in mind that this description necessarily omits the important one important property of a realistic fluid namely that it has internal friction. So, I have only considered external forces external to the fluid to be acting on the fluid. So, I have not considered the possibility that you know different parts of the fluid can exert forces on other parts of the fluid. So that means, the adjacent layers of the fluid can rub against each other and cause an internal kind of friction it is completely omitted.

So, this f vector is external to the fluid. So, towards the end of this topic on fluids I am going to include this discussion of viscosity which involves taking into account the forces that are acting inside the fluid. So, different layers of the fluid exert forces on each other. So, ok I am going to stop now and in the next class let us continue our discussion of fluid dynamics.

Thank you.