Dynamics of Classical and Quantum Fields: An Introduction
Prof. Girish S. Setlur
Department of Physics
Indian Institute of Technology, Guwahati

Elasticity
Lecture - 16
Strain Energy

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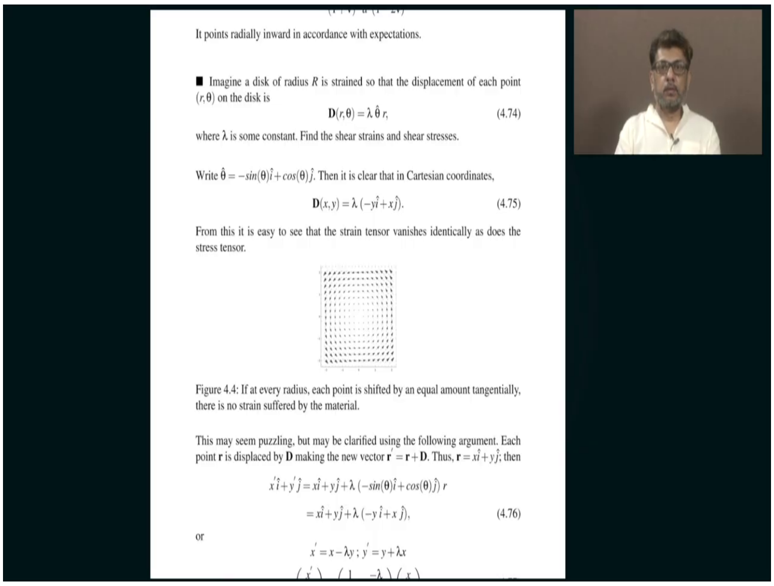

Ok. So, in the last class I gave you this peculiar example of strain suffered by a disc of a radius r. So, the idea was that. So, this example even though that there is a displacement that is given namely this. So, each point at r comma theta. So, there is a disc of radius r. So, it is a two dimensional problem. Small r is the distance from the centre and theta is the angle from x axis. So, that specifies some point uniquely. Now, that point under some strain it will get displaced. So, that displacement of that point is suppose it is given by this formula.

That means the displacement happens tangentially in the direction of theta and it is proportional to the distance from the centre. So, now, you will see that even though there is a displacement it does not necessarily mean there is a strain in the material. So, this particular example is meant to show that displacements can actually lead to zero strain, which implies therefore, that because there is a displacement and because there is no strain it means that the whole material has actually rotated by a fixed amount.

So, to see that you can rewrite this in cartesian coordinates. So, you write your theta hat as minus sine theta i and cos theta j.

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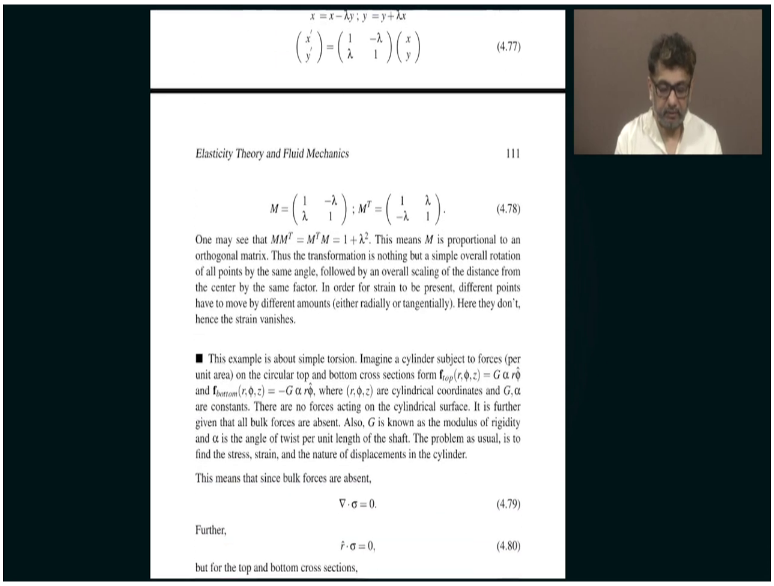

And you will see that this is write writable in a cartesian coordinate like this. So, then you find out what is the new position of that point. So, the earlier position was x y and the this was the displacement. So, and that displacement can be written in cartesian form like this. So, you see the new displacement will become like this $x' = x - \lambda y$ and $y' = y + \lambda x$. So, this can be written basically as a matrix like this right. And this is essentially it is a proportional to an orthogonal matrix.

So, what this means is that there is a rotation ok. So, there is an overall rotation by some angle and there is a shift I mean the distance from the centre is kind of uniformly shifted in all directions. So, there is a scaling. So, the distance shifts in and there is a uniform rotation. So, as a result there is no strain in the material. So, strain happens when you know different parts of the material shift by different amounts, but here the whole material is shifting the same way in all locations. So, therefore, there is no strain suffered by the material ok.

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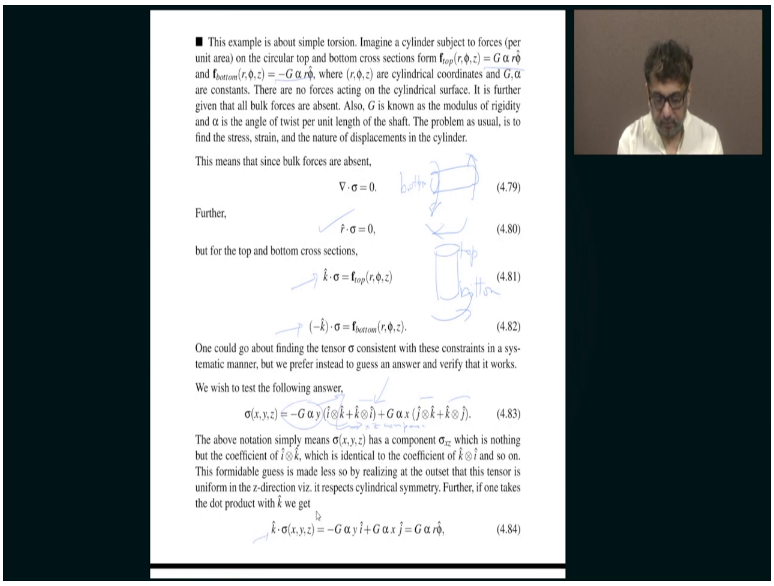

So, now let us discuss another example which also involves something like twisting. So, the earlier example seemed like something is getting twisted, but then that twisting does not result in strain. It just results in overall rotation, but the next example is you have a twisting which actually results in a in strain and stress. So, this example is this imagine a cylinder which is subjected to two forces. So, you have a cylinder like this basically what you do is you twist this cylinder here in this direction and in this direction you twist it in. So, basically you twist it in opposite direction holding the two ends.

So; obviously, you are causing a strain and that is called torsion. So, that is called simpler torsion means twisting. So, basically the top part of the cylinder. So, you can think of this as the bottom I mean I should have written it like this maybe then it would make sense. So, this is top this is bottom. So, the idea is that you are twisting this. So, you are applying force per unit area which is in the angular direction so; that means, in the tangential direction phi hat and it increases as you go away from the centre

So, the cylindrical coordinates are of course, natural here. So, and their coordinates are basically r theta r phi and z, r corresponds to a distance from the axis the central axis to the point you are interested in the shortest distance from the central axis to the point you are interested in and phi is the angle made with some arbitrarily chosen x axis you know in a plane that is parallel to the top and bottom faces. And the z of course, is along the central axis. So, now, given those coordinates then we say that imagine that the force that you apply per unit area on the top face of the cylinder is proportional to r, but in the twisting direction phi and the bottom you twist it the opposite way.

So, it is minus. So, it is minus G alpha r phi for the bottom plus G alpha r phi for the top. So, now, obviously, we expect strain and stress here, because you are actually twisting something I mean it is not reasonable to suspect that this whole material all parts of the material twisting the same way. In fact, they are not. We expect the centre portion to remain untwisted, because the top is getting twisted in one direction it is the other, the bottom is getting twisted in a different direction. Anyway bottom line is that. So, you got to solve for the stress tensor.

So, the stress tensor obeys a whole bunch of equations that I have pointed out one is the equilibrium condition. That is that because the cylinder does not move under the application of this external twisting forces rather it deforms see unlike a rigid body a rigid body when you apply a force or a torque or whatever it will start to angularly accelerate, but here the response is different and namely it deforms. So, this is not a rigid body it is a it is a deformable elastic body. So, it rather than angularly accelerate it deforms.

So, clearly you need a balancing force. So, in other words you have to make sure that if there are bulk forces they have to be such that the divergence of the stress plus the bulk force is 0. We just showed some time back that is the condition that is a necessary condition for this elastic body to be in equilibrium, means not accelerate. So, clearly now we are assuming that the weight of this cylinder can be ignored maybe you are doing this in outer space or in a you know in the international space station you are doing this experiment.

So, everything is freely falling. So, everything is weight less. So, in that case there is no there are no body forces. So, then divergence of sigma is 0, rather than divergence of sigma plus fb being 0 is divergence of sigma is 0, because fb is the force per unit volume the body force, which is absent. And secondly, this is some assumption that is consistent with what we have kind of seen earlier that the stresses basically are you know the forces that you apply are in the angular directions. So, clearly the radial forces are absent, because you know nothing is being pulled in the radial direction.

So, it is you are pulling it angularly ok. So, there will be components in the z direction, because you see you are pulling angularly in one way when z is plus l by 2 and you are pulling angularly in a different opposite way when z is minus l by 2. So, you expect components along z also, but you do not expect components along r, because you are pulling the same way for a fixed r it is the same.

So, that is the reason why we assume this I mean if it is not immediately obvious to you. Why this is a reasonable statement. You just you know you just go along with what I have said and then you work backwards and convince yourself that this is in fact, correct ok.

So, now clearly by definition the, because stress is given I mean we have assumed that sigma is a stress. So, if stress is it is not really given, but we have assumed we have a symbol for it namely sigma. So, if stress is denoted by sigma which is a symbol then the k component of the stress on top of the cylinder or to be equal to the force per unit area that you are applying on top of the cylinder. So, that is the this relation. So, similarly on the bottom it has to be this. So, put them all together and then you can convince yourself that this makes sense ok.

So, this is again I have rewritten in terms of the cartesian just like I did earlier that phi hat can be rewritten in the cartesian i and j. So, when you do that you will be able to rewrite it like this. So, this circle with a cross inside it is called a tensor product you do not have to be scared by this. This all this means is that this refers to the x z component. So, this is basically I mean this is the x z component. So, this one. So, this is the x z component of sigma. So, this is the z x component of sigma and this is the j z component this is the z j component ok.

So, there are all these components. So, this has been you know like this is consistent with these conditions ok.

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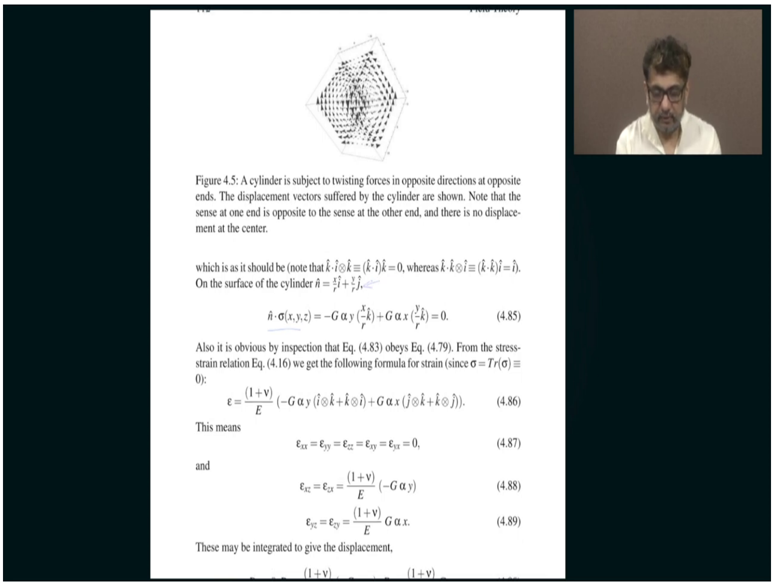

So, now you can go ahead and ask yourself what is the stress on the surface of the cylinder ok? So, the reason why we need this on the surface of the cylinder clearly you do not expect stresses. So, the stresses propagate in the z direction. So, there is a twisting in the phi direction and they propagate in the z direction. So, on the surface it is got to be 0 and we impose this condition that on the surface of the cylinder there are no stresses ok.

So, on the surface of the cylinder this is what that is. So, if that is the case then you can rewrite now remember what we wrote about this, this is the strain tensor. So, the strain tensor was basically related to the stress tensor in what way. So, remember what we wrote about the strain tensor. So, the strain tensor had a well defined relation in terms of this stress tensor ok. See namely this ok. let us look at for the most general case.

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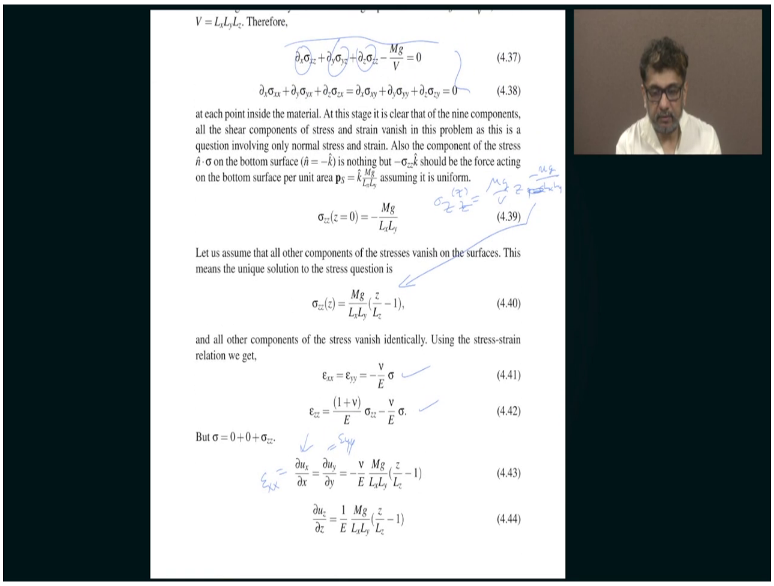

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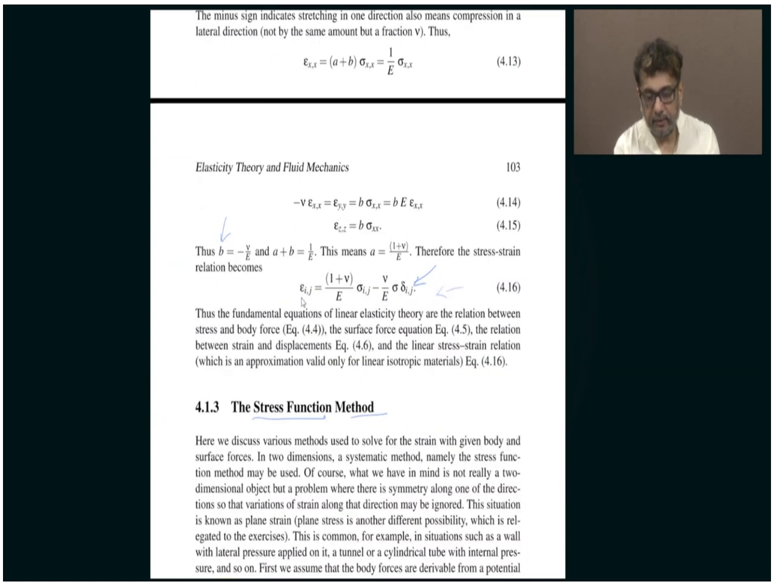

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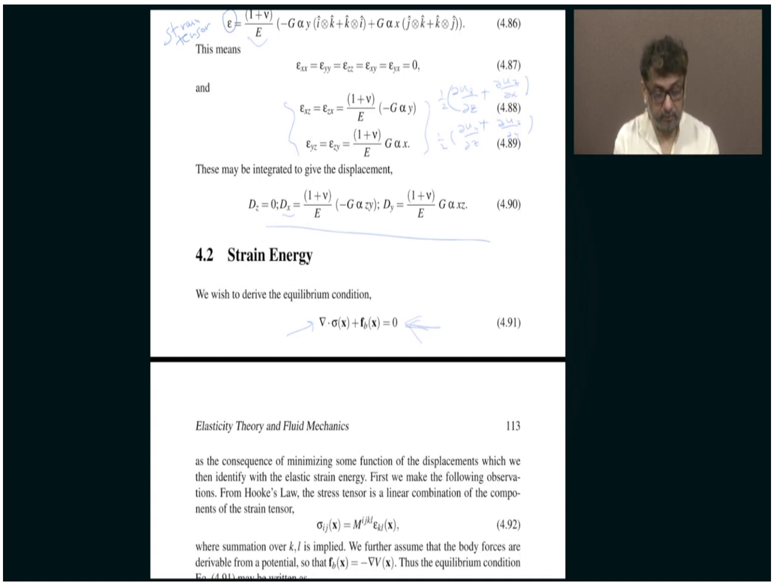

Yeah this one. So, the strain is related to stress in this way this is the most general one. So, the stress strain relation. So, if you use that relation. So, you will basically since the diagonal components are 0 right, because there is no diagonal components of the stress. So, therefore, the diagonal components of the strain are also 0. So, and the off diagonal components have this coefficient here. So, which involves the Poisson ratio and Young's modulus. So, this clearly shows that all the diagonal components of strain are 0 and specifically along the x y directions also the strain are 0.

Only the x z and z x components and y z and z y components are nonzero ok. So, now, you can. So, now, this you know how this is related to. So, this is basically ux by z plus d u z by x.

So, similarly here this is $\frac{1}{2}\frac{du_y}{dz}+\frac{du_z}{dy}$ is it not. So, you simply integrate these two relations and you will get this displacement. And this is interesting, because what this says is basically that at any given point xyz the point I mean no point displaces vertically. So, that is to be expected, because you are taking a cylinder you are twisting the bottom portion in a anticlockwise way and the top portion in a clockwise way.

So, there is no reason why anything should shift vertically. So, they will shift tangentially. So, that is what is happening here. So, the amount by which it shifts in the x direction is proportional to z and y and it is in one sense. So, it basically it tells you exactly how that cylinder is getting twisted; means how all the points in the cylinders are getting displaced. So, this is clearly an example where there is a strain and stress. So, the strain that appears in the material and the stress is similar. So, because of this twisting you the material undergoes a strain and the strain undergoes this displacement.

So, it is interesting to know that you can explicitly calculate. So, in other words once you figure this out. You can actually you know explain what the shape of that cylinder will look like. You can probably even plot the shape of the cylinder. So, just imagine. So, I would encourage you to use some software like MATLAB, Mathematica and you know plot some before and after pictures. So, before you apply any of these twisting forces what does the cylinder look like and what does it look like after you apply the twisting forces ok.

So, that is an activity worth doing I encourage you to do that and I think I will leave that to some of the exercises. So, now, let us revert to some more general statements regarding the formalism. So, I told you that there are some general constraints obeyed by the stress and strain tensors.

And one of them is this equilibrium condition, which ensures that the elastic material does not accelerate upon application of stress rather it deforms. So, the condition that ensures that is to is the condition which says that the divergence of the stress tensor should be compensated by the body forces if it does not get compensated in this way.

The material will accelerate in addition to being deformed. So, we will of course, not consider that possibility. We will consider pure deformation.

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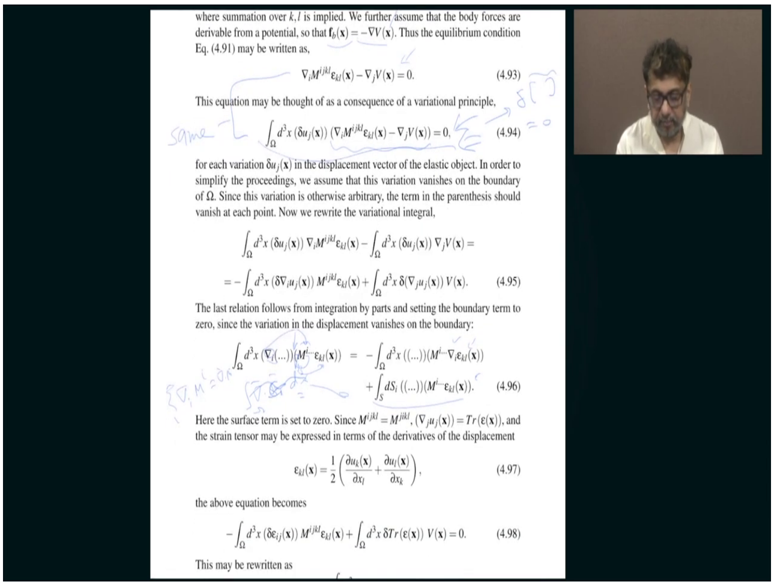

So, now, point is that I have been repeatedly telling you that any such you know differential equation can be thought of as a consequence of an extremum problem; that means, you take some functional and you minimize with respect to some some other parameter or some other function and then you ask what is the condition under which that function becomes or functional becomes minimum and. So, we showed that Lagrange I mean basically the for example, Maxwells equations can be thought of as the Euler Lagrange equation of some suitable Lagrangian.

And of course, the Euler Lagrange equations themselves are a consequence of the extremum principle that is minimizing the action. So, similarly here also this equilibrium condition can be thought of as a consequence of minimizing some internal stress induced energy of the system ok.

Elastic strain it is called it is called strain energy ok specifically, because of course, stress induces strain and strain you know the strain in the material is some kind of a potential energy. It has a very anthropic relation also; that means, we all know that when a strain builds up you know when we are stressed we are also strained like psychologically and the strain that builds up is some kind of a potential energy and then to release strain we shout at people and that sort of thing.

So, that is the psychological aspect of stress and strain, but here it is a very physical manifestation, but the parallels are quite striking. So, you have stresses that are applied on the material, which induce strain and strain builds up and manifests itself as a form of potential energy. So, now, the question is what we want to do is? We want to find the potential energy that is you know pent up in the material, as a result of this strain.

So, we make this very general statement that you know any 2 by 2 matrix its components can if it is linearly related sigma and epsilon are 2 by 2 matrices that are linearly related and clearly I should be able to write them with some proportionality coefficients, which this capital M is what that is and then further I assume that the body forces are derived from a scalar potential that I am means I am assuming conservative body forces typically the weight of the material itself or if it is a fluid or something it would be some pressure even if it is a solid it could be some external pressure.

But typically it is the weight of the material. So, I go ahead and substitute these two this is just the stress strain relation, which just says that stress is proportional to strain and then I substitute that here and then I substitute the body force there and I get this equals 0. So, now, you see this relation may be thought of as the consequence of some variational principle. So, what you do is that you multiply this by some delta u which can be arbitrary you integrate over some region and then you say that is equal to 0.

And then you say that this is this relation is valid for every delta u. So, if this is valid for every delta u; that means, that this itself has to be 0. So, these two statements are the same ok. So, now, the reason why I wrote it like this is, because I can go ahead and rewrite this 4 point 9 4, as a delta of something. So, I want to write this as delta of something equals 0. So, that will means basically I am minimizing this something. So, I want to write this as delta of something equals 0.

So, then I do all my integration by parts for this. So, you see I do my integration by parts and your gradient will come on this and so on and so forth. And yeah and then I have this gradient, which I split it up into you know this is like a you know the Gauss theorem. So, this is like the divergence of something. You know this is the divergence of some something else. So, this is d 3 x, but then this will be the surface integral of that. So, basically that is what I have done here ok well.

Firstly, right now it does not look like divergence. So, that is why I have to integrate by parts I have to first differentiate with respect to this. So, if I integrate by parts this derivative goes and sits here in the middle here, but then this derivative sitting in the middle means that it will first give me divergence it will also give me some term which is not a divergence which is this one. So, I have to make sure I subtract that out. So, when I do that, I get these two terms. So, this is a consequence of applying Gauss's theorem. So, this was a divergence of M i right. So, which is basically divergence M.

So, I think you should think deeply about this, because some of these steps can be a little bit tricky. So, I will allow you to think deeply about this, but this is straightforward calculus. So, there is no physics here it is just calculus

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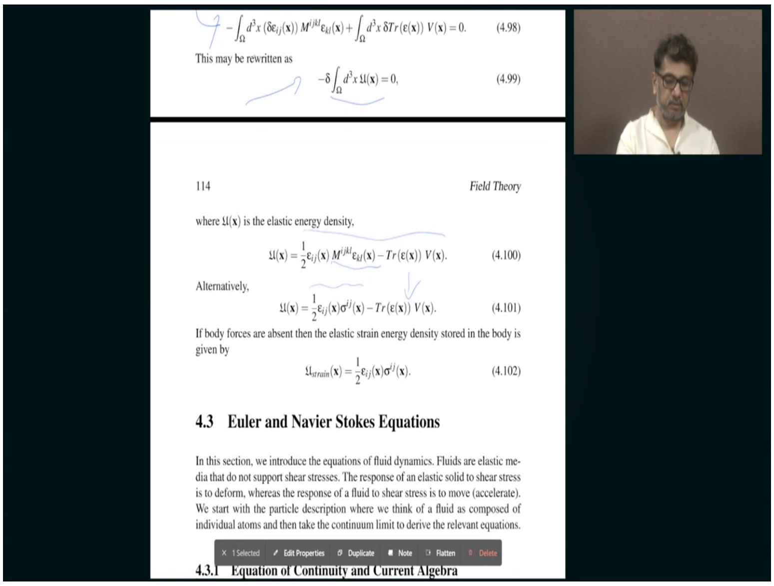

So, now, I go ahead and use my symmetry properties that, because these matrices epsilon and sigma are symmetric. And furthermore this is this is basically the trace of epsilon is just divergence of or basically yeah. So, the divergence of u which is the displacement. And then I also use this general statement about epsilon.

So, when I do all that I substitute all that here and then I will be able to show that this equation. So, in other words this equation, which becomes this. So, this equation. So, this equation. So, I substitute instead of this I substitute this and then finally, all this becomes this and this can be written finally, like this ok.

So, it is. So, that is what I promised earlier that I am going to rewrite this equation, which see this was my condition the equilibrium condition I multiplied this whole thing by delta of u and integrated and said. So, these two are the same provided this is valid for any delta of u.

So, since that is valid for any of delta of u, I can simply rewrite this as the variation of some something equals 0. So, which is what I was driving at and that something is basically the elastic energy density which is given by this ok. And since this is equal to epsilon. So, I can rewrite this as the elastic energy density.

So, it has two parts one is basically the trace of sigma times epsilon times sigma is one of them and the other is due to the body forces if body forces are absent the strain energy is simply given by the trace of 1 half of the trace of the strain times the stress.

So, and that is your strain energy density or the elastic energy density. So, that is the energy density, that is kind of pent up in your elastic material, because it is holding on to strain ok. So, minimizing this will amount to enforcing the equilibrium condition ok alright.

So, I think this completes the discussion of elasticity theory. So, I think the bottom line is that there are many steps in especially in the derivations, that are quite technical and they are going to be necessarily hard to follow unless you actually sit down and work out all the steps which is the reason why I have brought out this book.

So, it is imperative that you go through the book and make sure that you work out all steps and if you have doubts you send me a message asking me how I got from here to there. So, it is only when you work out all the steps on your own, that you can actually understand. So, it is impossible for you to simply understand just by staring at some equation for a few minutes.

So, you have to work it out ok. And that applies to everyone including people who have worked in this field for a long time ok. So, they also get stuck, because human mind is evolutionarily not equipped to handle equations you know we were hunters gatherers in the distant past and our bodies and minds are evolutionarily adapted to do very mundane things.

So, it is with great difficulty we are able to reach up to this level. So, effort is necessary. So, you have to put in effort in order to overcome your evolutionary adapted brains ok. So, I am going to stop here

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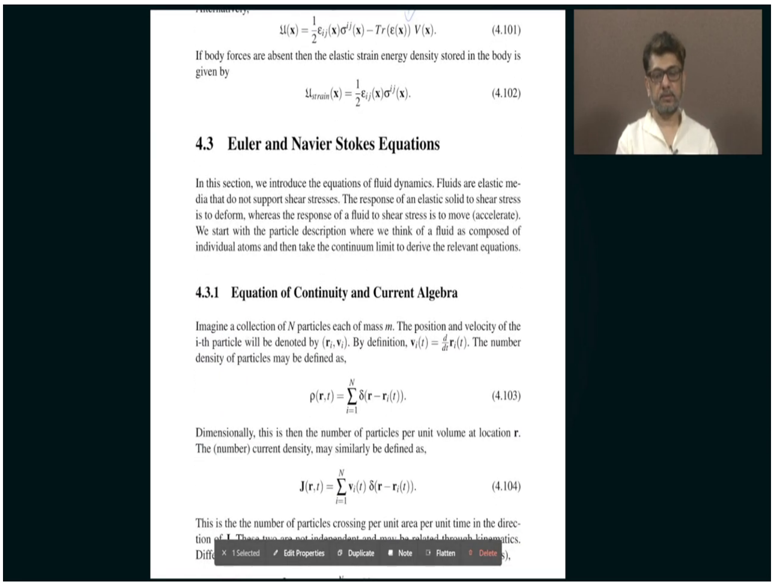

And in the next class I am going to discuss a new topic, which is Euler basically the fluid mechanics right. So, I am going to discuss the theory of fluids, which is you know elastic materials which are also modelled like elastic materials, but then elastic materials that do not deform under shear stress, but rather accelerate under shear stress. So, in the earlier case I assume that no type of stress will induce an acceleration in the material. So, that would correspond to in a perfectly elastic medium.

But fluids are elastic only in some limited sense. If you apply shear stress they will accelerate ok. So, that is how you define a fluid in elastic medium that suffers acceleration instead of deformation upon application of shear stress ok. So, I am going to stop here and in the next class let us discuss fluid mechanics.

Thank you.