Dynamics of Classical and Quantum Fields: An Introduction

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Elasticity
Lecture - 15
The Stress Function Method

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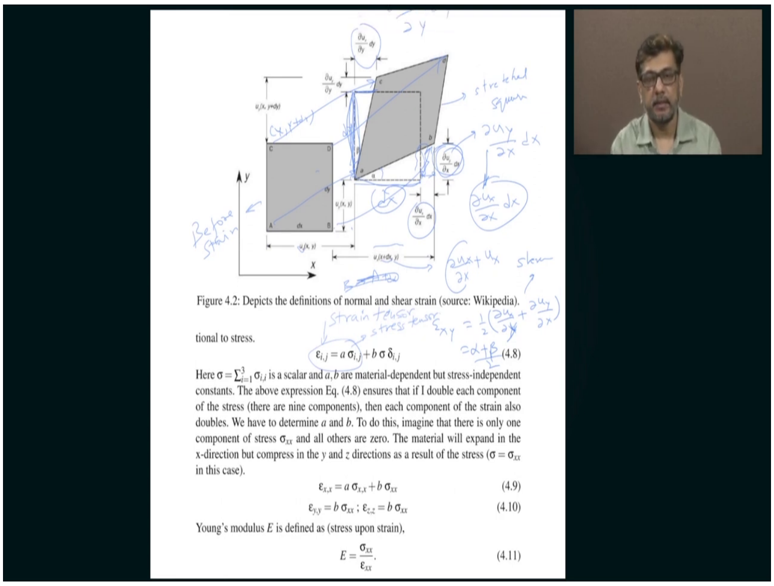

Ok. So in the last lecture I had stopped here where I described this diagram which shows how a piece of an elastic material in two dimension would deform under the application of stress. So, that means, that it is going to exhibit what is called strain and strain is described as the displacement of the various points in the elastic material in a very non-linear way ok.

So, I have explained to you what is the meaning of the various terms in this strain tensor. So, this epsilon i j was my strain tensor. So, I explained to you that the diagonal components tell you how the size of the of a piece of materials of an elastic material changes under the application of stress.

So, the diagonal elements like epsilon xx epsilon yy epsilon zz would correspond to the how much the size changes. Whereas, epsilon xy epsilon yz and so on they the off diagonal ones tell you and that is called the shear strain and that tells you how the shape of the of that piece of material changes.

So, under the application of stress the strain that appears in the elastic body also means has two aspects to it. One aspect is that it changes the size of the of that piece of the elastic body and it also changes the shape. So, and these two are captured by this strain tensor ok. So, now, I told you also that we will be focusing restricting our attention to what are called linear elastic materials and linear elastic materials are those where the strain tensor is proportional to the stress tensor.

So, now we are going to implement that in a somewhat less obvious way and for reasons that will become clear soon because you see you will actually need two coefficients. So, I am going to describe to you what they are. So, I am going to say that my strain tensor. So, for a linear elastic material my strain tensor so, this is my strain tensor for a linear elastic material my strain tensor is proportional to the stress tensor. So, you see notice that you might be thinking that why did I not stop here.

So, I should have simply done this. Well, the reason will become clear soon because there are two things that happen and one is that when you apply stress an elastic object will suppose you stretch a rubber band, the rubber band will increase it is length in the direction in which you are stretching it, but it will also decrease it is length in the perpendicular direction.

So, you actually need two numbers to describe what is happening. So, these are two independent things that the amount by which you stretch and the amount by which the material compresses in the perpendicular direction can be different. And they can be completely independent. So, that is the reason why you need two different coefficients to describe it.

So, bottom line is that you can ensure linearity see I told you linear elastic materials are those whose strain is proportional to stress. So, instead of doing it the obvious way, which is epsilon i j equals a times epsilon i j. So, this would unnecessarily constrain ourselves to material very peculiar type of materials.

So, if you want to be more general you add a term which is also proportional to stress and this is basically the trace of this matrix and then you put a chronicle delta here. So, the reason why this is done is because. Firstly, epsilon i j will remain symmetric and. Secondly; it is linear in the sense that if you double the stress the strain also doubles. So, it is linear. So, now, I am going to tell you why we needed this both these a and b why we could not do with just a which would have been the obvious choice.

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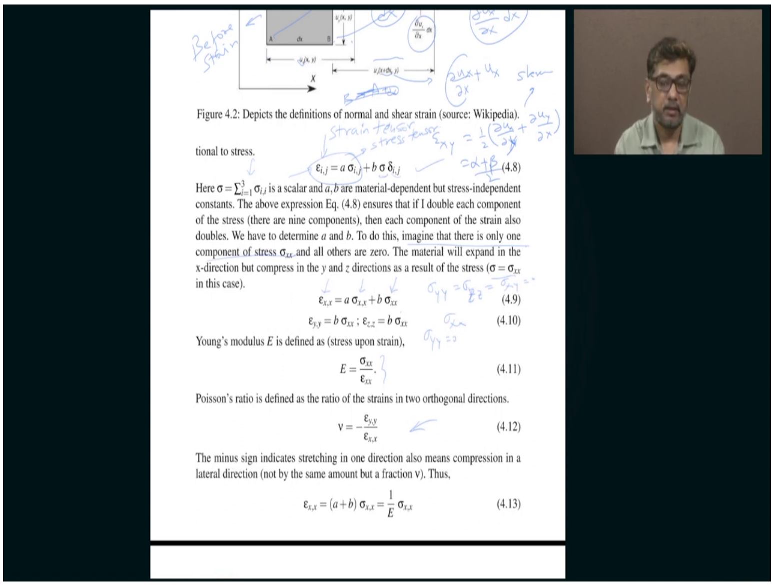

See the reason is that. So, imagine that there is only one component of stress. So, namely epsilon xx so; that means, you apply stress in the x direction and that is in other words you apply a force that perpendicular to the y z surface ok. And that is it and everything else is 0. So, in that case the material will expand suppose you stretch it in the x direction. It will expand in the x direction, but it will compress in the y and z directions because that is typically what we our intuition tells us and what our experience also tells us.

So, in that case so, let us try and see what; that means, to means how does that connect to this assumption or answers we made here so. So, now, if you set i and j to both to be equal to x then you see you get this equation, which basically tells you that. So, I told you that there is only sigma yy equals sigma zz equals everything else equals zero. So, everything is 0 except sigma xx.

So, in which case sigma trace of sigma is also sigma xx; so, which is sigma without any subscript is basically the trace of sigma matrix, but since the only element that is sigma xx that traces sigma xx itself. So, you see you have this sigma xx and then you have this sigma xx because of i equals j. And then you will you have this relation. That is epsilon xx is a plus b times sigma xx. But now if you look at sigma yy you see that epsilon yy is basically because sigma yy is now 0. So, now, it will just be b times sigma xx.

So, now you see sigma yy and sigma zz are basically b times the applied stress. See whereas, sigma xx is a plus b times the applied stress ok and all others are 0 because the shears. So, there is no chance of any shear strain because there is no shear stress in the material. So, this is all this is the whole story. So, now, the question is we can now make some statements about what the physical meaning of these coefficients a and b are the small letter a and small letter b.

So, the answer is the following that you define something called Young's modulus and Young's modulus is defined as the stress sigma xx, which you have applied divided by the strain ok. So, that is called Young's modulus. So, in other words the stress that you applied in the strain in the same direction in which you have applied. So, it is sigma xx divided by epsilon xx. So, that is called Young's modulus whereas, there is another coefficient which tells you.

So, you see this Young's modulus basically tells you how much the elastic object stretches or compresses in the same direction in which you applied the stress. But whereas, the other one this equation 4.12; it tells you the amount by which the elastic material stretches or compresses in the direction perpendicular to the applied stress. So, you see I told you if you take a rubber band.

So, imagine a reasonably thick rubber band and you stretch it in the usual way with your fingers then it is going to increase it is length in the same direction in which you are stretching it.

Whereas if you look closely you will see that the thickness of the rubber band has actually reduced because you have stretched it. So, that is what the second relation tells you is basically what is called Poisson ratio. So, Poisson's ratio is tells you the ratio of the amount by which the elastic body has compressed in the perpendicular direction and the amount by which it has stretched in the original direction.

So, that ratio is basically called Poisson ratio and that is unrelated to Young's modulus. So, that is the reason why you needed these two different parameters called small letter a and small letter b. Because now we are in a position to relate these two seemingly ad hoc parameters coefficients, which we introduce namely small letter a and small letter b to more physical quantities such as, Young's modulus and Poisson ratio ok.

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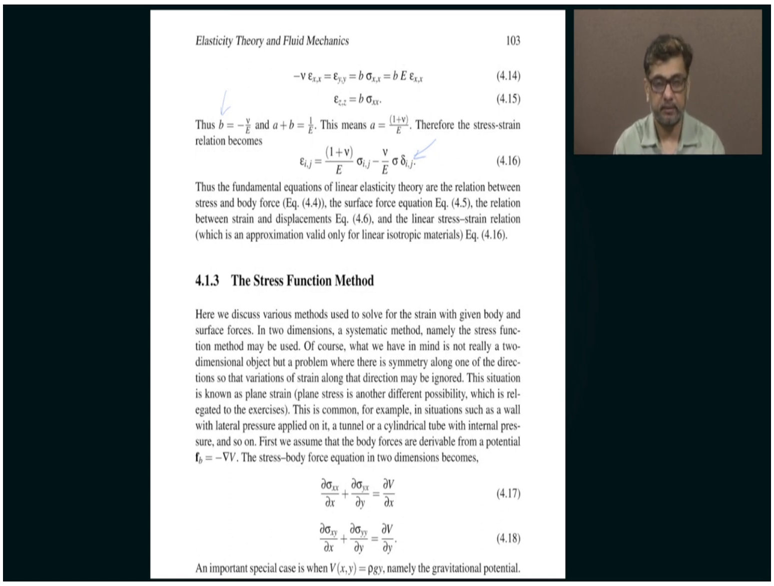
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$$-\nu\, \varepsilon_{x,x} = \varepsilon_{y,y} = b\, \sigma_{x,x} = b\, E\, \varepsilon_{x,x} \quad (4.14)$$

$$\varepsilon_{z,z} = b\, \sigma_{xx}. \quad (4.15)$$

Thus $b = -\frac{\nu}{E}$ and $a + b = \frac{1}{E}$. This means $a = \frac{(1+\nu)}{E}$. Therefore the stress-strain relation becomes

$$\varepsilon_{i,j} = \frac{(1+\nu)}{E}\sigma_{i,j} - \frac{\nu}{E}\sigma\, \delta_{i,j}. \quad (4.16)$$

Thus the fundamental equations of linear elasticity theory are the relation between stress and body force (Eq. (4.4)), the surface force equation Eq. (4.5), the relation between strain and displacements Eq. (4.6), and the linear stress–strain relation (which is an approximation valid only for linear isotropic materials) Eq. (4.16).

4.1.3 The Stress Function Method

Here we discuss various methods used to solve for the strain with given body and surface forces. In two dimensions, a systematic method, namely the stress function method may be used. Of course, what we have in mind is not really a two-dimensional object but a problem where there is symmetry along one of the directions so that variations of strain along that direction may be ignored. This situation is known as plane strain (plane stress is another different possibility, which is relegated to the exercises). This is common, for example, in situations such as a wall with lateral pressure applied on it, a tunnel or a cylindrical tube with internal pressure, and so on. First we assume that the body forces are derivable from a potential $\mathbf{f}_b = -\nabla V$. The stress–body force equation in two dimensions becomes,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = \frac{\partial V}{\partial x} \quad (4.17)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \frac{\partial V}{\partial y}. \quad (4.18)$$

An important special case is when $V(x,y) = \rho g y$, namely the gravitational potential.

So, it is easy to do that because we already have these relations and when you do that you get this relation that is b comes out as the ratio of the Poisson ratio and the Young's modulus. Poisson ratio divided by Young's modulus with a minus sign is your b whereas, 1 plus Poisson ratio divided by Young's modulus is a.

So, you put that in and you get this very important relation which tells you the relation between stress in a material and the strain that you apply. So, if you have a linear elastic material you apply stress called sigma ij the material is going to exhibit strain, which is described by a tensor epsilon i j and the parameters that you have to specify to describe this is there are two parameters one you have to specify the Young's modulus.

That tells you the amount by which the size of the material changes and indirectly the Poisson ratio tells you the amount by which the shape of the material changes ok. So, now, we are going to see how to apply this stress strain relation to find the you know how a body deforms. So, there are very many interesting things you can do with this relation. For example, you take an elastic take up for example, a ball made of solid rubber.

And you just place and imagine that is a relatively heavy ball and you just simply place it on a table its. So, if you just hold it in your hand it is like a perfect sphere, but if you place it on a table it is going to press on to the table because of it is weight and it is not going to look like a sphere anymore it will look like more or less a sphere, but not exactly a sphere. So, it is going to deform under it is own weight.

So, we want to know. So, this subject teaches you how to find the shape of the rubber ball the solid rubber ball when it is placed on the surface of a table for example. So, these are the kinds of interesting things you can do with this these types of relations like 4.16.

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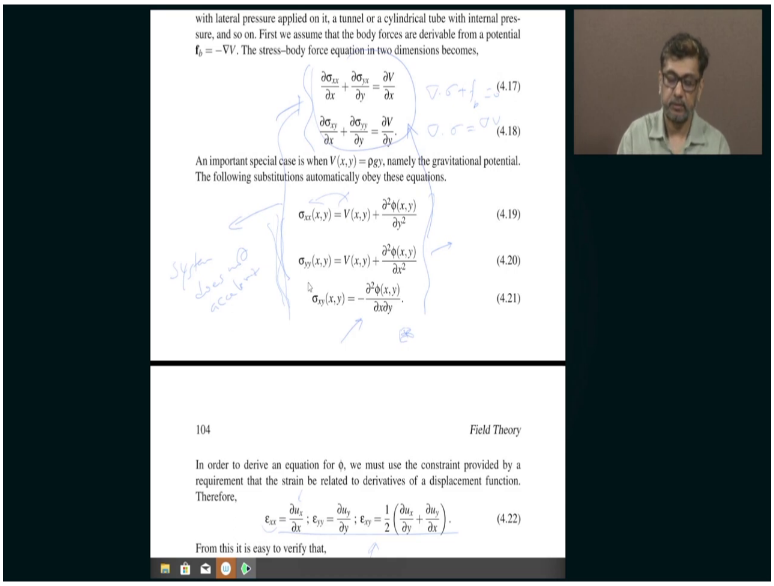

So, I am going to teach you certain very standard methods to solve for these shapes under the applications of known stresses. So, first let us assume that the body forces. So, remember I told you that just like you know if you have a rubber ball, which is like a sphere the body forces typically refer to the weight of the rubber ball so, the weight per unit volume as it were.

So, you assume that it is derivable from a potential. So, it is a conservative force of weight. So, there is a potential energy V and negative gradient of that is the force. So, if that is the case then clearly you can see that the remember I told you divergence of sigma plus f b is 0. So, that is basically tells you that this is so, this relation just tells you that divergence of sigma is grad V ok.

So, that is what I have done here. So, divergence of sigma equals grad V. So, I have just assumed that you have a two dimensional material with two dimensional body forces and you have applied a stress. But then you have applied of stress in their body forces and they kind of balance off. So, that overall the material does not accelerate it just deforms. So, now I am going to consider a special case where the material you know the gravity acts in the y direction ok.

So, as a result you have negative y direction. So, as a result your potential energy density is rho g y ok. So, now, if that is the case then you can clearly see that you can integrate these relations. So, these relations can be integrated in this way. So, the reason why you can do this is that you can introduce a function called phi x y and you can see that these substitution if you substitute this into this equation it is going to be an identity ok. So, you can verify that yourself.

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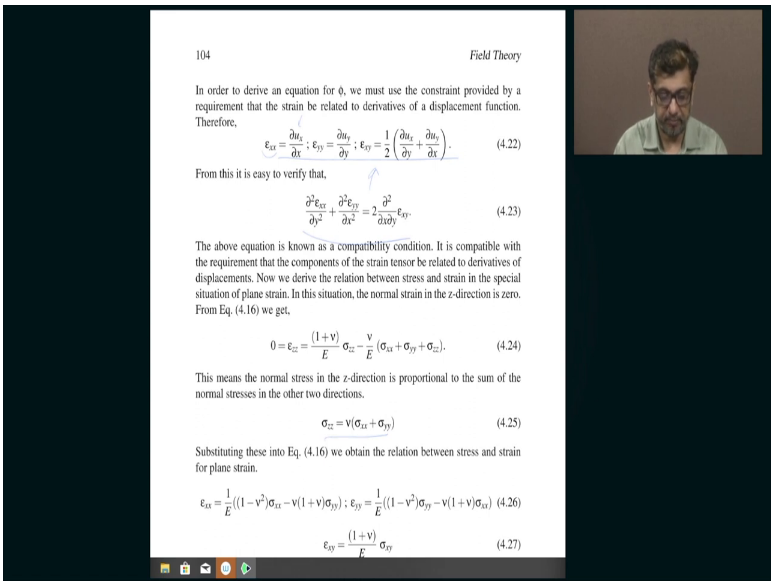
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In order to derive an equation for ϕ, we must use the constraint provided by a requirement that the strain be related to derivatives of a displacement function. Therefore,

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}\ ;\ \epsilon_{yy} = \frac{\partial u_y}{\partial y}\ ;\ \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right). \quad (4.22)$$

From this it is easy to verify that,

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2}{\partial x \partial y}\epsilon_{xy}. \quad (4.23)$$

The above equation is known as a compatibility condition. It is compatible with the requirement that the components of the strain tensor be related to derivatives of displacements. Now we derive the relation between stress and strain in the special situation of plane strain. In this situation, the normal strain in the z-direction is zero. From Eq. (4.16) we get,

$$0 = \epsilon_{zz} = \frac{(1+\nu)}{E}\sigma_{zz} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}). \quad (4.24)$$

This means the normal stress in the z-direction is proportional to the sum of the normal stresses in the other two directions.

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \quad (4.25)$$

Substituting these into Eq. (4.16) we obtain the relation between stress and strain for plane strain.

$$\epsilon_{xx} = \frac{1}{E}((1-\nu^2)\sigma_{xx} - \nu(1+\nu)\sigma_{yy})\ ;\ \epsilon_{yy} = \frac{1}{E}((1-\nu^2)\sigma_{yy} - \nu(1+\nu)\sigma_{xx}) \quad (4.26)$$

$$\epsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad (4.27)$$

So, now the question is that we have to solve for phi? So, this is like kinematics you know what in classical dynamics we would have called this kinematics. It is just a it is just a substitution, which solves for certain equations. But then you have to invoke the dynamics, which is the stress strain relation in order to actually find that phi and therefore, get explicitly the strain ok. So, this is the applied stresses right. So, the applied stresses.

So, these two ok what these two together ensure is that the system as in system does not accelerate. So, that means, it simply deforms. So, these two ensure that this because the body forces are compensated by the stresses. So, they do not so, the material no part of the material accelerates they just deform and remain in equilibrium. So, now, the question is how does now the important question is the now that we have figured out that there is a well defined way in which the strain or rather the stress is related to the body forces given by this.

So, the question is what does this mean when it comes to describing the strain in the material ok. First we use the definition of the strain tensor in terms of the displacements. So, remember that epsilon xx is defined as the rate of change of the displacement in the x direction with respect to x. Similarly epsilon yy is rate of change of displacement in the y direction with respect to y and x y is the you know the democratic mean of this derivative with respect to x and y and y with y and x you know.

Because remember that it was basically alpha plus beta see the reason why it is alpha plus beta you might be wondering why did I select alpha plus. So, remember what is alpha and beta these two is the sum of these two angles divided by 2. So, that was epsilon xy was alpha plus beta by 2. So, you might be wondering that why did I not only select this one. Firstly, it is not symmetric if I do not do that, but then you might think that why is it important for it to be symmetric. See the reason is because look alpha is defined like this beta is defined like this.

Now, if beta is equal to minus alpha what this means is this square actually does not deform, it simply rotates. See, beta is minus alpha what does that mean; that means, this square has actually not changed the shape at all it has simply rotated in anti clockwise is not it. So, then we do not want to consider that like a we do not want to think of that as a strain. We only want to think of it as a strain only when that shape actually changes. If it simply rotates we do not want to consider that a strain.

So, that is the reason why we do alpha plus beta by 2 because if alpha equals minus beta it is a simple rotation without changing shape. So, it is not as therefore, a strain there is just a rotation. So, now, that is out of the way let us proceed and you can see that because of this definition this identity is always valid. So, this is called the compatibility condition ok. So, this is just comes from the definition of epsilons in terms of the displacements ok.

So, now keep in mind that we have z that the normal strain. So, remember that the normal strain in the z direction is 0 because we are thinking of plane strain ok. So, there is no strain in the z direction. So, therefore, the stresses in the z direction if they exist should be given by this relation ok. So, the we assume that the strains take place in the plane the x y plane only.

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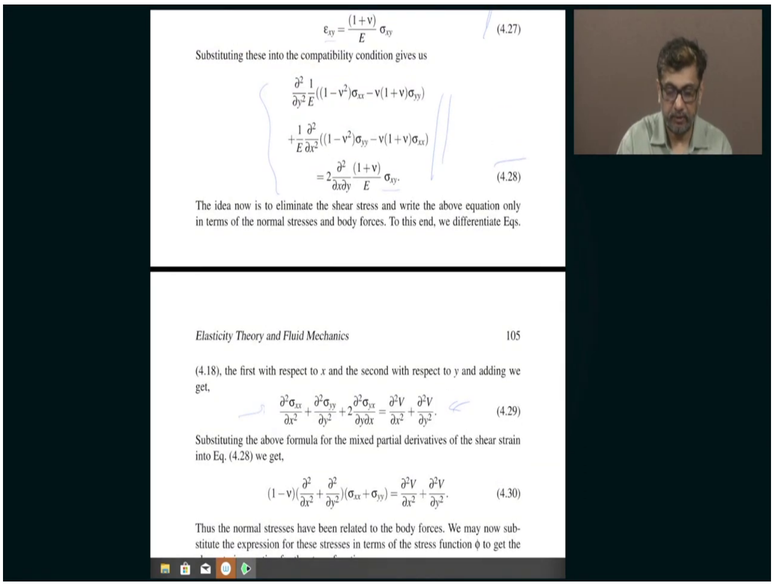

So, in that case you can now go ahead and find out the components of the strain tensor and you will see. So, from the stress strain relation you can deduce these results ok. So, the epsilon xx will be proportional to a combination of x sigma xx and sigma yy and epsilon xy will be directly proportional to sigma xy.

So, in other words the shear strain is proportional to the shear stress whereas, the normal strain is proportional to the normal stress. So, the normal stress in the normal strain in the x direction is not only proportional to the normal strain in the x direction, but it is there is also a part of it which is proportional to the normal strain in the y direction. I told you why that is well, if you try to stretch something in one direction it compresses in the perpendicular direction. So, both get mixed up ok.

So, now, we go ahead and substitute these two relations or these three relations into this compatibility identity. So, this compatibility identity forces this to be valid ok. So, now, yeah it is a long story. So, what we have to do is that we have to eliminate the shear stress from this and see if we can only write the normal stress in the body forces. Because now so, this is the compatibility condition gives you one constraint on the various components of the stresses that you have applied. But then remember that there are also conditions, which tells you how the stresses are related to the body forces ok.

So, now if you go ahead and combine these two, you will see that so, if you take 4.18 ok. That is what this was. So, if you just take these relations ok and then combine it with the 4.28. So, what you are going to get is basically this relation ok. And if you go ahead and combine this mixed partial derivative of the so, you substitute this formula of the shear strain into 4.28 ok. So, you just substitute that here and you will see that.

So, it is a lot of algebra ok. So, I am not going to describe to you all the steps because then it is to be quite distracting and boring so, but you just have to work it out on your own and because all the steps are explicitly given. So, you just have to follow the logic.

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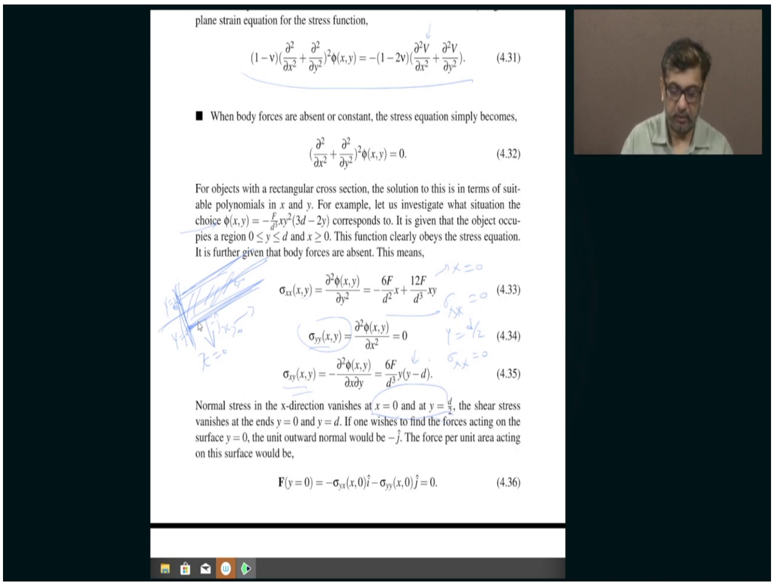

So, now once you do that then you will see that this is the equation that you finally, get and then if you go ahead and eliminate the sigma's the components of sigma and express it in terms of this phi. So, this is reminiscent of this you know writing the electric field in terms of scalar and vector potentials.

So, that is similar to what we do in electromagnetic theory we write. So, this is something like a potential function for the stress. So, we have introduced a kind of potential function. So in fact, that is called the stress function that is why it is called the stress function method. So, that is the. So, it is the analog of scalar and vector potentials.

So, that stress function now obeys a certain equation namely this. So, this is the body force potential energy and this is your stress function. So, if you know the stress function you also can find the stress just by this relation. So, if you find phi then you can just substitute here you know all the stresses. If you know all the stresses you also know all the strains because have the stress strain relation, which tells you all the strains ok. So, now, you have so, this is a very general result ok 3 4.31 is very general.

So, now the question is when body forces are absent or if they are constants then you can clearly see that the stress function of base this you know double Laplace equation ok. So, rather than solving this we can just postulate that we can try out certain forms of phi x y. So, let us try out arbitrarily if phi x y where this what does it correspond to.

So, just imagine that there is an object that occupies a region of you know it occupies this much region from y equal to d and y equal to. So, it occupies this region. So, it occupies x greater than 0, but y is between 0 and d. So, there is this elastic material here. So, and then imagine that there is a stress function given by this relation. So, now, the question is if the body forces are absent then the stress components are now given by these relations ok.

So, we also have to take into account the fact that at x equal to 0 the normal stresses vanish ok. So, we assume that for example, that there is no stress. So, the normal stresses vanish at x equal to 0 and at y equal to d by 2. Because if you set x equal to 0 and you get sigma xx equals 0, but then if you set y equals d by 2 then also sigma xx is 0.

So, basically it is saying that the normal stresses that is sigma yy is anyway 0 that that part of the normal stress is 0. So, there is only one other normal stress which is sigma xx and that is 0 whenever x is 0 so; that means, here there is no normal stress, but then there is also no normal stress here along this entire line. So, there is no normal stress on this line there is no normal stress on that line ok.

So, yeah that is just an observation an interesting observation. The other thing is that regarding shear stresses there are no shear stresses at the ends; that means, that there is no shear stress here there is no shear stress anywhere here or anywhere here ok, because that comes from here. So, this is not particularly illuminating because it is just an observation from the chosen randomly chosen phi we have just randomly chosen some phi and we have just said that this is what the stresses induced in the material are going to be.

But then it is more illuminating to find say for example, the forces acting on say some particular surface like y equal to 0. So, if I want to act know the forces active on y equal to 0 then what I do is I do sigma dot the normal to that surface ok. So, the normal to that surface would correspond to minus j hat because that would be the inward no I mean basically the outward normal to y equal to 0 would be here like this is the outward normal which is minus j hat.

Because this is my surface this is my x z plane. So, the force acting per unit area on that surface is sigma dot minus j hat. So, if you work that out you will see that it is 0 at y equal to 0 ok so, that there is no forces acting on that surface.

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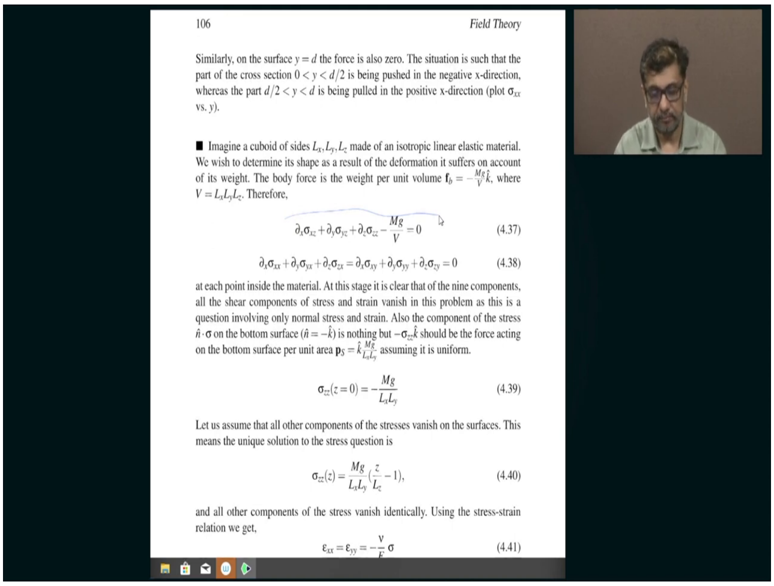
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Similarly, on the surface $y = d$ the force is also zero. The situation is such that the part of the cross section $0 < y < d/2$ is being pushed in the negative x-direction, whereas the part $d/2 < y < d$ is being pulled in the positive x-direction (plot σ_{xx} vs. y).

■ Imagine a cuboid of sides L_x, L_y, L_z made of an isotropic linear elastic material. We wish to determine its shape as a result of the deformation it suffers on account of its weight. The body force is the weight per unit volume $\mathbf{f}_b = -\frac{Mg}{V}\hat{k}$, where $V = L_x L_y L_z$. Therefore,

$$\partial_x \sigma_{xz} + \partial_y \sigma_{yz} + \partial_z \sigma_{zz} - \frac{Mg}{V} = 0 \tag{4.37}$$

$$\partial_x \sigma_{xx} + \partial_y \sigma_{yx} + \partial_z \sigma_{zx} = \partial_x \sigma_{xy} + \partial_y \sigma_{yy} + \partial_z \sigma_{zy} = 0 \tag{4.38}$$

at each point inside the material. At this stage it is clear that of the nine components, all the shear components of stress and strain vanish in this problem as this is a question involving only normal stress and strain. Also the component of the stress $\hat{n}\cdot\sigma$ on the bottom surface ($\hat{n} = -\hat{k}$) is nothing but $-\sigma_{zz}\hat{k}$ should be the force acting on the bottom surface per unit area $\mathbf{p}_S = \hat{k}\frac{Mg}{L_x L_y}$ assuming it is uniform.

$$\sigma_{zz}(z=0) = -\frac{Mg}{L_x L_y} \tag{4.39}$$

Let us assume that all other components of the stresses vanish on the surfaces. This means the unique solution to the stress question is

$$\sigma_{zz}(z) = \frac{Mg}{L_x L_y}\left(\frac{z}{L_z} - 1\right), \tag{4.40}$$

and all other components of the stress vanish identically. Using the stress-strain relation we get,

$$\varepsilon_{xx} = \varepsilon_{yy} = -\frac{\nu}{E}\sigma \tag{4.41}$$

So, similarly at y equal to d also the forces are 0 ok. So, what this means is that the situation is such that the part of the cross section between 0 and d by 2 is being pushed in the negative x direction right.

So, in other words this portion of the material is being pushed in the negative x direction ok. See whereas, this portion of the material is being pushed in the positive x direction right. So, that is what this physically this peculiar choice of phi corresponds to this situation. So, you have to imagine that this is like a 3D thing I mean it is the physics happens in 2D, but the material is still 3D.

So, you are stretching you are stretching the material in the plus side direction when the points in the material are between y equal to 0 and y equal to d by 2 and you are stretching in the negative x direction for the bottom half of the y values and the top half of the y values you are stretching in the positive x direction.

So, it is kind of you are trying to tear that you know that elastic material apart in the x direction as it were. So, so that is what this situation corresponds to; that means, somebody is trying to tear this apart by pulling in opposite directions. So, that is what this phi corresponds to yeah. So, typically that is not how the problem description is posed in real life.

Somebody is going to tell you what force is what I am trying to do I am going to be told that look I am trying to tear this apart by pulling this in the x direction this way in negative x direction that way and now tell me what stresses are in the material. So, that is of course, the correct way of posing this question, but that is a harder question to answer because it involves solving differential equations.

So, here I have in this example I have done the reverse. I have started with the solution of the differential equation, which completely opaque and very hard to know what it is all about what it corresponds to so, but then I am going to substitute this absolutely seemingly random choice for that potential stress function.

And then I substitute in my stress strain all these formulas that I have derived and then I figure out what the stresses induced are and then I infer what that really corresponds to by looking at the forces acting on particular surfaces. So, this is just pedagogy there is just to tell you what is the physics behind these relations. So, this is just you know interesting examples that get you familiar with the use of stress strain relations and these consistency conditions and so on.

So, there let us go to the another example. So, imagine there is a cuboid; that means, a kind of a rectangular shape 3D material with sides L x and L y L z made of linear material like this. So, this is the standard this is the more interesting question rather than just you know randomly postulate f i and say that what does this correspond to rather than that we ask ourselves. So, imagine that there is an elastic material of a certain shape.

And then you are holding it in your hand and it does not do anything. But now you just place it on a table and because of it is weight it is going to deform in a certain way and that is precisely what we want to find out how does it deform. So, this is a more interesting question to ask and answer. So, the body force per unit volume is therefore, given by this it is constant it is minus Mg times k hat. So, well assume that k hat is the vertical direction and minus k hat is the direction of acceleration due to gravity.

So, now you have your relations which the body force you know the analog of continuity equation in electrodynamics or fluid mechanics which will come to later, but bottom line is this is basically the equilibrium condition. It tells you the forces due to stress how to balance out the body forces else the material will accelerate ok. So, that is clear what that is.

So, now, the thing that is clear is that the because you know there are no shear stresses nobody is trying to change the. So, the so, there is no you know forces acting on the x direction and you know sigma x y type of thing is not going to be there because there is no there is no kind of rotating kind of stress that is because sigma x y is some kind of a torsion type of stress somebody is going to actually twisting. So, is basically that is the better word. Somebody is trying to twist that material.

So, see if sigma xy is nonzero; that means, somebody is trying to twist that material. So, nobody’s trying to twist that material; so, there is a there is a elastic body which is you it has some weight and you just place it on a table and it is changing it is shape. So, nobody is twisting anything. So, all the sigma xy and xz and all those things are 0. So, because of that we can state that basically you have this relation that.

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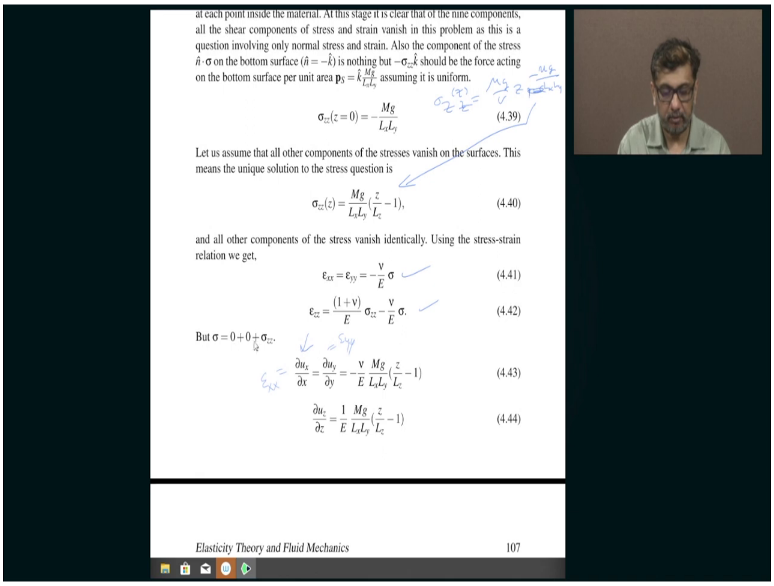

So, therefore, all these other ones are 0. So, only this is there. So, because of that you can immediately find out that sigma zz is basically Mg by V into z plus constant. But then we are going to assume that at z equal to 0. So, we have to assume that at z equal to 0. So, that z equal to 0 is the force acting on the bottom surface per unit area. So, we have to assume that there is a force acting on the bottom surface per unit area, which is basically holding up that material right.

Because you are placing it on the table; so, z equal to 0 is the bottom surface of that material and it is being held up by some forces and that force per unit area is Mg divided by L x into L y, which is the cross section of the portion that is sitting on the table. So, that is the force. So, there is therefore, there is a stress which is sigma zz that is acting on the bottom and that is clearly nothing but minus sigma zz k hat. So, that is the force acting.

So, this should therefore, be equal to. So, at z equal to 0 it should be this. So, this is going to be M g by L x L y ok. So, then so, if you combine these two you will get this relation that sigma zz is basically Mg divided by L x L y into z by L z minus 1 and all other components of stress vanish identically. Because you know you have a L x L y L z type of rectangularish material which is just sitting on the table and it will simply compress in the z direction.

So, but then you see even though the strain all the other components vanish I am sorry the stress all the other components vanish except sigma zz the strain you know you will have components in y direction also. Because I told you that if material compresses in the z direction it will expand in the x and y directions ok.

So, here the material is trying to compress in the z direction. So, that is what we are going to try and find out. So, you see from these relations it is clear that ok. So, this is your sigma xx sorry epsilon xx ok. And this is also equal to epsilon y y is going to be this because we know what sigma is trace, but then all the other sigma's are 0. So, you get this.

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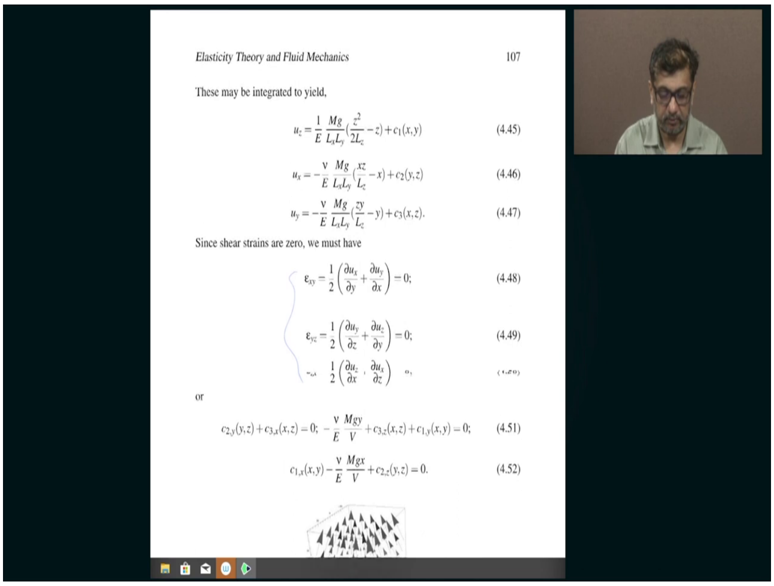

So, now if you integrate you will be able to see that these integrations will give you all these integration type constants. And then you have to also keep in mind that nobody is trying to twist anything. So, all the shear components are 0 and that will basically tell you all these unknown constants. So, the integration constants when you are partially differentiating with respect to with respect to z the integration constant could depend on x and y.

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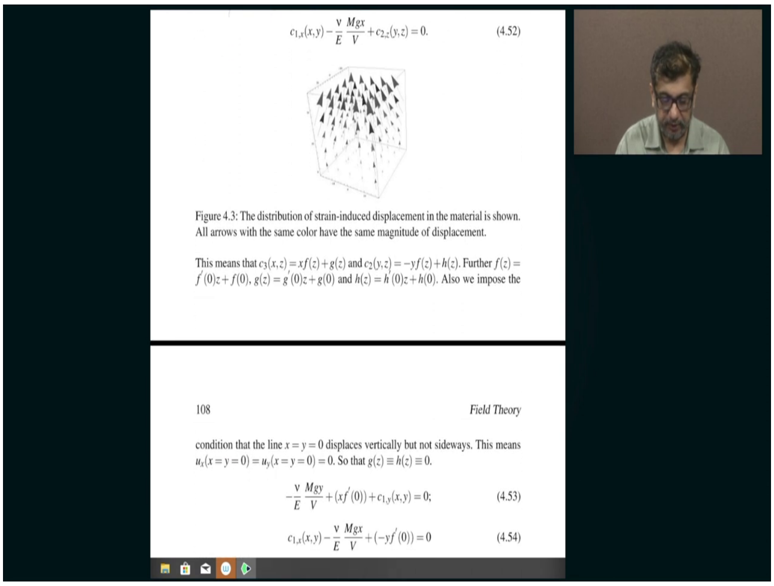

So, you will have to follow a rather lengthy procedure to find out the various integration constants ok.

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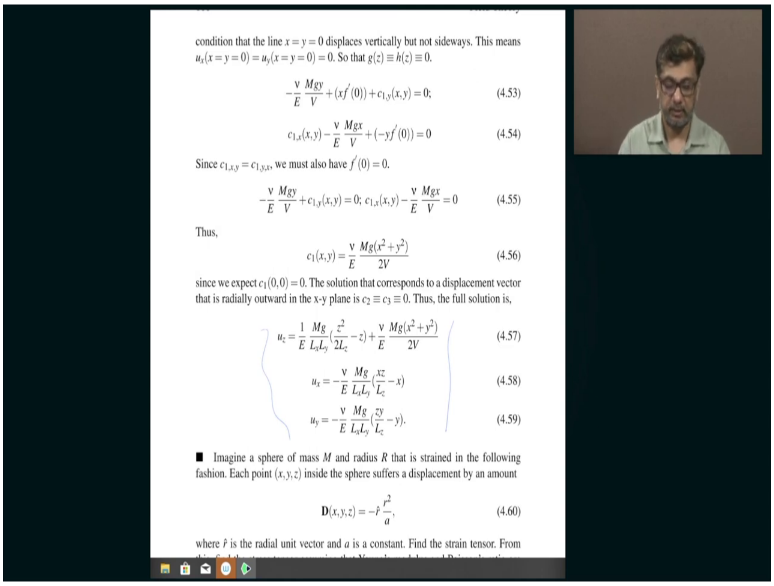

So, I am not going to so, it is just a lot of tedious algebra. So, I am not going to spend too much time on that. Bottom line is that after doing all that this is the full solution. So, this is the amount of displacement. So, remember what u of x is the amount by which a material at point x y z has displaced in the x direction so, that elastic material. So, this will basically tell you what is the final shape of that cubicle elastic material of length L x L y L z when you are just placing it on a table and allowing it to deform under it is own weight.

So, this is very interesting because it tells you how the material has deformed. So, this is a very precise mathematical vein which is it has deformed. So, it is nice to know that it is possible to do this ok. So, here for example, in figure 4.3 I have explained how the displacement vectors look like ok. So, unfortunately this is in black and white, but in color you will see that you have different you know if you have arrows of the same color they; that means, they have the same magnitude of the displacement.

So, bottom line is that this is what it looks like. So, this is an induced displacement. So, you can just you know just try and see if you can visualize this in some way you know use some software like Mathematica or MATLAB to plot try to visualize the strain try to plot it in 3D and see how that that object will look like physically when you. So, basically this tells you like the exact mathematical way in which an elastic object of length L x L y L z has deformed because you have placed it on a table.

So, it is nice to know I mean tries to create a kind of image of that in 3D using this formula. So, I think I would encourage my listeners to try and do that using some software like Mathematica or MATLAB. So, maybe we can discuss that in one of our tutorials later on ok.

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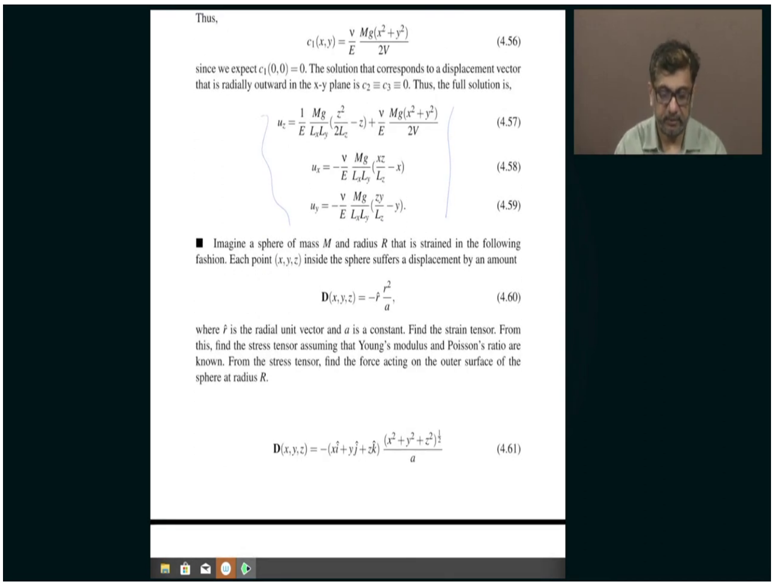

So, the next example which I am going to quickly discuss is a sphere of mass M and radius R that is strained in a certain way. So, imagine that the displacement is directly proportional to the square of the radius. So, the further away it is from the center the more it strains and the strain is in the. So, in other words somebody is trying to compress that sphere and the compression is larger when you are far further away from the center ok.

So, there is more displacement in the sense that fear kind of displaces more further away than. So, it is like you know take a solid rubber ball and try to squeeze it in and so, that is what this means. So, that is what 4.6 means. Somebody is trying to squeeze a rubber ball from all directions. So, now the question is what is the strain? So, here also we do not expect some shear things nobody is trying to twist anything.

So, we expect that to be the case. But then remember the geometry the sphere. So, , even though nobody is trying to twist anything, but because it is a it is a sphere there are going to be. So, we will have to let the equations play out and we have to decide you know the equations tell us what is going to happen. So, this is your displacement and if this is your displacement you can figure out all the strain tensors ok and then from there you can get your stress tensors ok.

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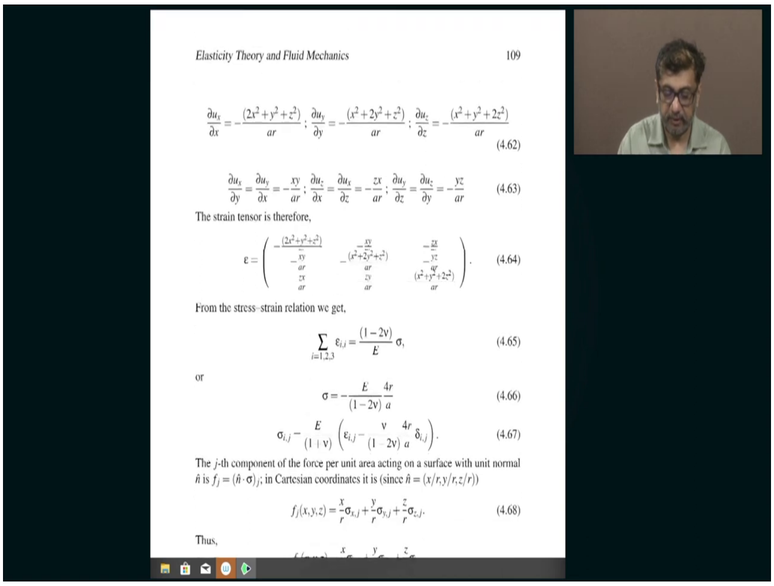

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$$\frac{\partial u_x}{\partial x} = -\frac{(2x^2+y^2+z^2)}{ar};\ \frac{\partial u_y}{\partial y} = -\frac{(x^2+2y^2+z^2)}{ar};\ \frac{\partial u_z}{\partial z} = -\frac{(x^2+y^2+2z^2)}{ar} \quad (4.62)$$

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x} = -\frac{xy}{ar};\ \frac{\partial u_z}{\partial x} = \frac{\partial u_x}{\partial z} = -\frac{zx}{ar};\ \frac{\partial u_y}{\partial z} = \frac{\partial u_z}{\partial y} = -\frac{yz}{ar} \quad (4.63)$$

The strain tensor is therefore,

$$\varepsilon = \begin{pmatrix} -\frac{(2x^2+y^2+z^2)}{ar} & -\frac{xy}{ar} & -\frac{zx}{ar} \\ -\frac{xy}{ar} & -\frac{(x^2+2y^2+z^2)}{ar} & -\frac{yz}{ar} \\ \frac{zx}{ar} & \frac{zy}{ar} & \frac{(x^2+y^2+2z^2)}{ar} \end{pmatrix}. \quad (4.64)$$

From the stress–strain relation we get,

$$\sum_{i=1,2,3} \varepsilon_{i,i} = \frac{(1-2\nu)}{E}\sigma, \quad (4.65)$$

or

$$\sigma = -\frac{E}{(1-2\nu)}\frac{4r}{a} \quad (4.66)$$

$$\sigma_{i,j} = \frac{E}{(1+\nu)}\left(\varepsilon_{i,j} - \frac{\nu}{(1-2\nu)}\frac{4r}{a}\delta_{i,j}\right). \quad (4.67)$$

The j-th component of the force per unit area acting on a surface with unit normal $\hat{n}$ is $f_j = (\hat{n}\cdot\sigma)_j$; in Cartesian coordinates it is (since $\hat{n} = (x/r, y/r, z/r)$)

$$f_j(x,y,z) = \frac{x}{r}\sigma_{x,j} + \frac{y}{r}\sigma_{y,j} + \frac{z}{r}\sigma_{z,j}. \quad (4.68)$$

Thus,

So, so, this is in this a different kind of question in the sense that somebody has told you what the display usually it is the other way around somebody tells you what body force is acting, what stresses are acting and you are supposed to figure out the displacement that is what we did here in the earlier question. We just found out how the material deforms under it is own weight because somebody told you there is a body for the stress.

So, whatever it is that if this is the problem description that somebody has told you what the displacement is then we are supposed to find out what are the stresses that are involved which leads to this displacement ok. So, now, you see you can figure out the strain tensor, which is going to have all kinds of components ok. So, now, from the stress strain relation you can figure out the suppose you want to figure out the force acting suppose you want to find out the force acting per unit area in the radial direction ok.

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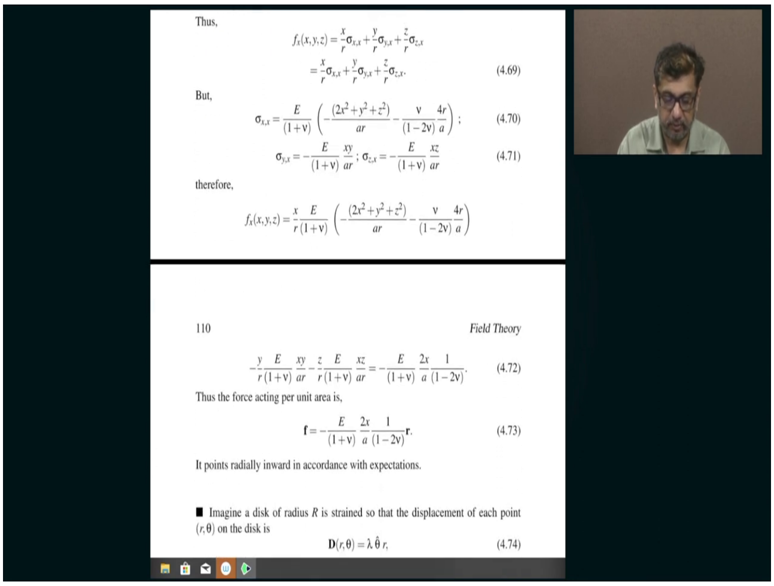

So, what you have to do is first figure out this stress tensor and then because now you can figure out the stress tensor just by simply stress strain relation because you already know the strain tensor. So, from stress strain relation you immediately get this stress tensor. So, from the stress tensor you can figure out the force acting on any surface.

Specifically if you choose your unit normal to the surface to be xyz basically some point xyz then you can figure that out. And so, you will basically be able to show that the force acting per unit area at any point R is basically proportional to R and, but in the opposite direction to R. So, in other words that is what I told you somebody is trying to compress. So, so that is what this means.

So, that there is a force acting per unit area, which is proportional to R; that means, there is more force acting as you go further away from the center and it is kind of symmetrical in the radial it is like being compressed from all directions ok.

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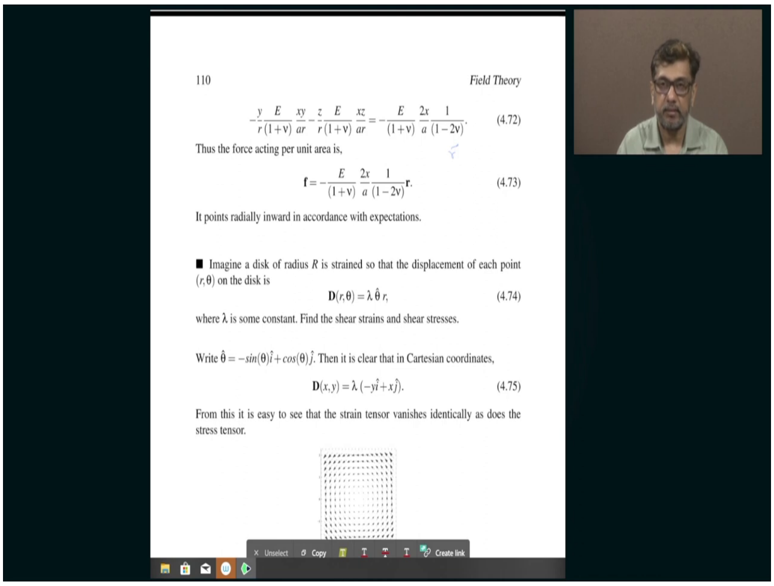

So, I think well we could keep giving such examples. So, there is another example involving a disk where ok. So, this is a different example where the displacement is in the angular direction, but the magnitude is proportional to the radial distance. So that means, that magnitude wise it is more and more the displacement is more and more as you go further away from a certain center, but the displacement is not in the direction of in the radial direction it is exactly perpendicular to that. So, that is basically somebody's trying to twist that material.

And they are trying to twist it in such a way that the twist is more when they are further away from the center. So, now, here also you can figure out various things, but then you see. So, what is going to happen here for this particular example that you will see that the strain tensor vanishes identically as well as the stress tensor. So, what is going to happen for this particular example? So, if yeah this is the reason why I have given this example.

So, if you choose the displacement to be in the angular direction, but then the magnitude is exactly proportional to the distance then you will see that the strain tensor and stress tensor strength and surface vanishes identically and therefore, the stress also vanishes identically.

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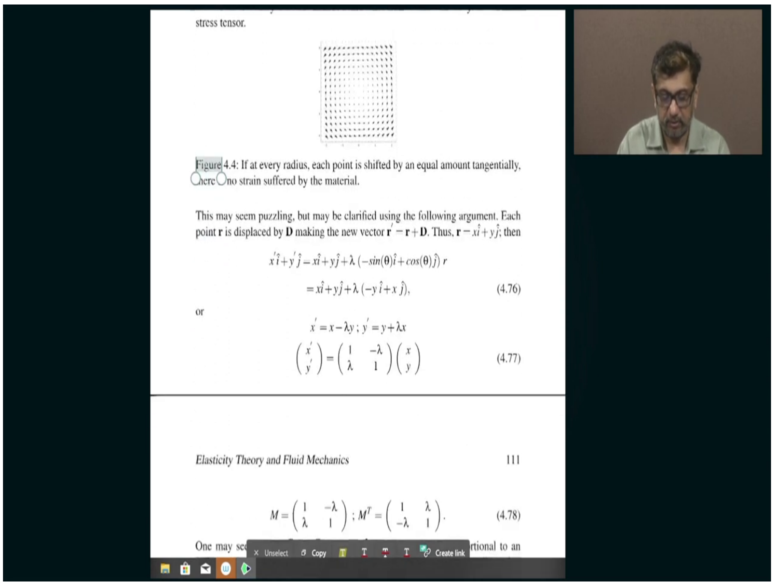

So; that means, basically what is happening is that the whole material is getting rotated by a certain amount. So, so this displacement just corresponds to a simple rotation of the whole material. So, as a result no strain is accumulating in the material and therefore, there are no stresses. So, this is just to illustrate that just because some displacement happens does not mean there is a strain in the material.

So, the displacement can be an overall displacement of the entire material. So, this is an example where you have a 2D material that is not twisting in the it is like the different portion of the material are not relatively twisting. The whole thing is twisting together. So, that is just a simple rotation. So, when that happens there is no strain in the material. So, if there are no strain there are no stresses ok.

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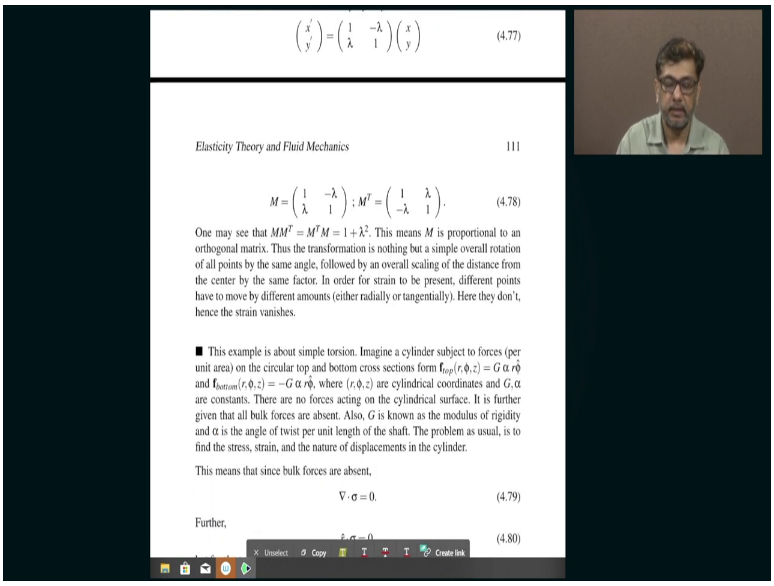

So, now I am going to stop here and in the next class I will try and explain to you certain other approaches certain other ways of solving for stresses and strains in elastic material. So, this is the earlier method was called the stress function method and there are other methods also depending upon the problem description.

So, once we are done with elasticity theory which is what we are discussing now, we will move on to fluid dynamics of fluid mechanics basically that corresponds to their description of elastic materials that do not support shear stresses. So that means, that the moment you put a shear stress instead of something compensating and the material coming to an equilibrium it will actually accelerate right.

So, that is what a fluid is an elastic medium tries basically it will not accelerate rather it will deform, but a fluid will accelerate if you apply shear stresses. So, that is the big difference. So, I am going to stop here and in the next class I will be discussing some more techniques for understanding elasticity theory followed by fluid mechanics ok.

Thank you.