Dynamics of Classical and Quantum Fields: An Introduction

Prof. Girish S. Setlur
Department of Physics
Indian Institute of Technology, Guwahati

Elasticity
Lecture - 13
Introduction to Elasticity Theory

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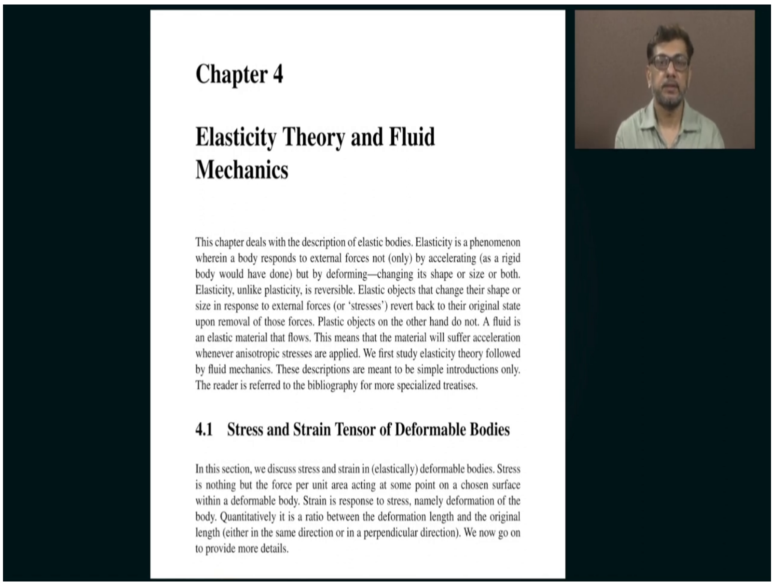

So, today we will discuss a new topic and that is the theory of elasticity and fluid mechanics. So, normally when you think of elasticity you think of a rubber band most of the time and when you think of fluid mechanics you think of you know flow of liquids in a pipe for example. So, you might be wondering why I have clubbed them together and discussing both in the same chapter.

So, the reason is there is a very good reason for that and that is that both have the same mathematical and conceptual underpinnings. So, they all come from the idea of stress and strain and how these bodies respond to the applied stresses. So, basically the subject matter of elastic bodies and elasticity and fluid mechanics both involve understanding how objects deform when stresses or forces are applied. So, I am going to read off this description in the introductory chapter here.

So, this chapter deals with the description of elastic bodies. So, elasticity is a phenomenon wherein a body responds to external forces not only by accelerating as a rigid body would have done, but by deforming. So, in other words it changes its shape or size or both. So, normally a rigid body with a rigid body by definition will not change its shape or size, but it will simply accelerate if you apply a force, but whereas, elastic bodies they also change their shape and size.

Elasticity and plasticity are similar, but different in the sense that plastic deformation is one which is not reversible. In other words if you a plastic object will remain deformed even after you remove the forces that you have applied which causes the deformation. So, we will of course, not be studying plasticity in this course. So, we are studying elasticity theory and fluid mechanics ok.

So, a fluid is an elastic material that flows. So, this means that the material will suffer acceleration when you apply shear stresses. So, these shear stresses are basically an isotropic stresses. So, I will have to precisely define what stress is it is a kind of force, so which I am going to do now. So, that is basically the content of this chapter. So, it is a description of elastic bodies and fluids and how they respond to various forces that are acting on them ok.

So, in order for us to proceed with the subject we have to define the core concepts of the subject and the core concepts of the subject are basically stress and strain ok. So, stress is nothing but the force per unit area acting at some point on a chosen surface. So, you just imagine a surface. So, you have to consider an imaginary surface ok, so it is just there in your mind.

So, just imagine that there is some surface in some deformable body then you choose a point on the surface and you ask yourself what are the forces acting on that point ok. So, there will be a force a vector and a vector has three components because we are working in three dimensions and those three components we conveniently choose to be perpendicular to that imagined surface that we have considered.

And the two other directions which are parallel to the surface; that means, that there is a tangent plane to the surface at that point you are interested in and that tangent plane will have two independent direction. Because it is a two dimensional tangent plane and the normal to the surface is one specific direction. So, a force acting at that point is basically has three components because we have again three dimensions.

And one components perpendicular to the surface and that is called normal stress and the forces that are parallel to the surface they are called shear stresses. So, that is basically the point. So, an elastic body will now decide to either deform or accelerate or both depending upon what the situation is ok.

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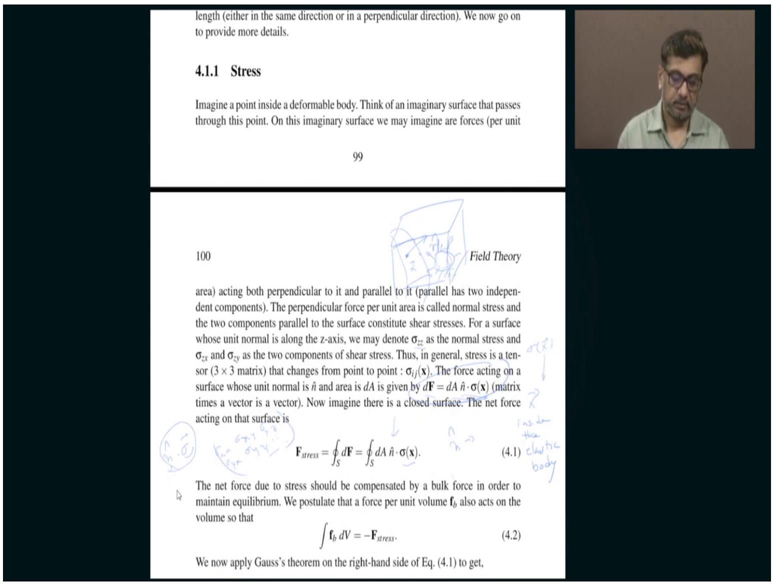

So, if it is a fluid it will typically accelerate it will always accelerate when shear stresses are applied, but for elastic solid it will deform ok. So, that is the basic idea of a stress, strain is basically defined as the amount of deformation the body undergoes relative to the displacement that you have considered so; that means, so I will precisely define what that is subsequently ok.

So, now, you know because you see the forces of stress clearly depend upon the imagined surface that you have considered. So, it is not like you know if you have a point particle there is just a force acting on that, but then here in within a solid with an elastic solid you can actually have different forces acting at the same point depending upon, which surface you have in mind that is passing through that point. So, you can have different surfaces passing through that point and these different surfaces have normals in different directions.

And you have these stresses which are parallel to the normal. So, in other words those are the normal stresses they can point in different directions depending on the surface you have in mind. So, let me explain to you this particular formula. So, basically it makes perfect sense to then talk of a matrix called sigma of x x. So, this, so where x is some point inside the elastic body. So, this n hat is normal to the imagined surface. So, if you have an elastic body here ok.

So, what you do is you imagine, so imagine a point called x inside it and you imagine some surface ok. So, let me consider this to be x. So, this is your imagined surface. So, this surface can be in any direction so; that means, this there is a normal to the surface called n hat. So, now, you can have a force acting on the surface which is basically. So, the force acting on the surface whose unit normal is basically this. So, this is the this is how you determine the force acting on the surface ok.

So, there is a unit normal and so you see you can have these are the types of forces. So, this is the normal stress that we are considering. So, the force acting normal to the surface would be given by this. So, if you imagine a closed surface inside the solid then you can have many of these normal forces acting and that will be the total normal force acting on the surface can be calculated that way.

So, if you calculate the total normal force acting on the surface it will be related to. So, this is called the stress tensor ok. So, stress tensor basically tells you what the. So, this is basically the force per unit area. So, this is a matrix the 2 by 2 matrix. So, it has components like sigma x x equal xy sigma xz sigma y y x y y y z like that. So, it has 9 components and you see this is a column matrix a unit vector. So, if you multiply these two you get another vector.

So, basically this is force per unit area acting parallel to n hat so; that means perpendicular to the surface. So, now, if you integrate over all the, so this force per unit area at some point you integrate over all the small small patches around those points say around the closed surface inside the material, then what you get is basically the net normal stress acting on that imagined surface ok within imagined closed surface within the deformable body.

So, now the point is that clearly if the deformable body just it does not accelerate. So, usual elastic solid will not accelerate when the net force is applied rather it will simply deform. So, basically therefore, there must be some kind of a body force within the solid that compensates for this applied stress otherwise it will accelerate. So, if you do not want it to accelerate there has to be some other force compensating for this stress ok.

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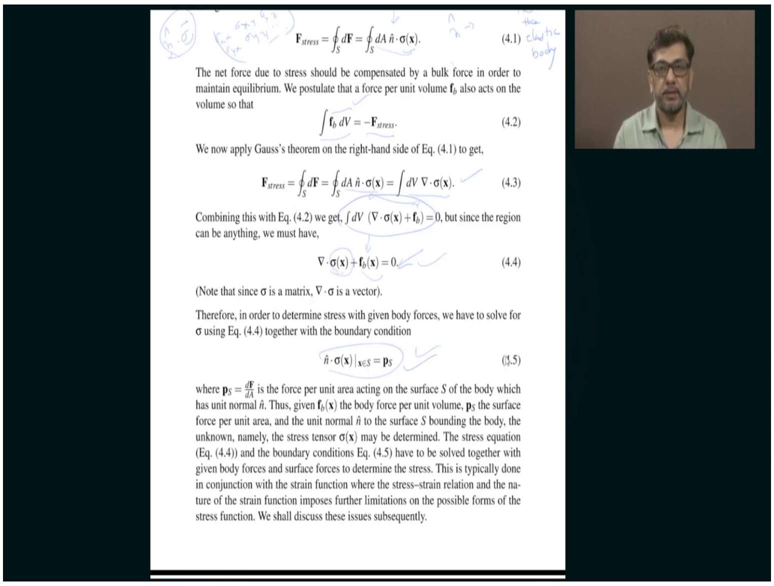

So, this normal stress around that is applied to this closed surface and that body force is basically force per unit volume which we think of as fb. So, now, you integrate this force per unit volume over the entire volume of that closed surface and then. So, that will be the net force acting due to the body forces. So, that could be say the weight of the. So, typically it is the weight of the stuff or the material inside that imagined closed surface. So, that would constitute a body force because that would amount to you know force per unit volume there.

So, if you integrate over the volume you get basically a net force and that force has to cancel out the force due to the stress. So, when these two cancel out then you can be sure that the imagined material. So, the imagined surface containing this elastic material will not accelerate rather it will deform.

So, it is important that these two forces perfectly balance out each other; so which is why we have to demand that the body force within the imagined surface should be minus F stress. So, F stress is the net stress acting on the surface of the normal stress acting on the surface of the and this imagined closed surface that we are thinking of ok.

So, now, you see you can use your well known Gauss's theorem which says that you can always write the surface integral of the normal component as the volume integral of the divergence which is very familiar; this is a very familiar theme in electromagnetic theory which I am sure many of you have encountered before. And so were going to use that, so it is of course, nothing to do with physics it is just a simple matter in calculus ok.

So, we simply use this Gauss's theorem which relates these two and then we combine this relation which is F stress and this relation. So, when you do that you get this result that is the volume integral of divergence of sigma which is still a vector because remember that sigma is a 3 by 3 matrix and you dot product it with a vector you will get another vector which is 3 component and fb is clearly a force per unit volume which is of course, still a vector.

So, now, these two put together has to be 0 because this volume can be anything. So, it is completely general. So, the integrand itself has to be 0 because it is not any specific volume it is any general volume. So, it is clear that this integrand itself has to be 0 ok. So, the bottom line is this that if somebody tells you what the body forces acting are that is typically the case when you know what the material is mass density.

And so then you are talking about the body force due to the weight of the material, then clearly you know what f b is or maybe somebody is applying some external pressure on the deformable body you are stretching a rubber band for example, if you are stretching a rubber band you know what forces you are applying. So, those are the body forces and there will be stresses because of the applied body force there. So, bottom line is that somebody tells you body forces you can solve for the stresses provided you also know the boundary conditions.

So, the so, somebody has to tell you the stresses acting on the boundary and the body force equation together with this known stresses acting on the boundary will uniquely determine the stress tensor of the body ok. So, that is the bottom line so, because this is the first order differential equation because if you want to think of sigma as your unknown this like divergence of sigma is something.

And so it is a first derivative divergence is a first order derivative and they will be in when you integrate you get integration constants and those constants are determined by 4.5 which is basically the boundary conditions ok. So, you need those things and then these two for 4.4 and 4.5 put together uniquely determine the stresses acting in the body ok.

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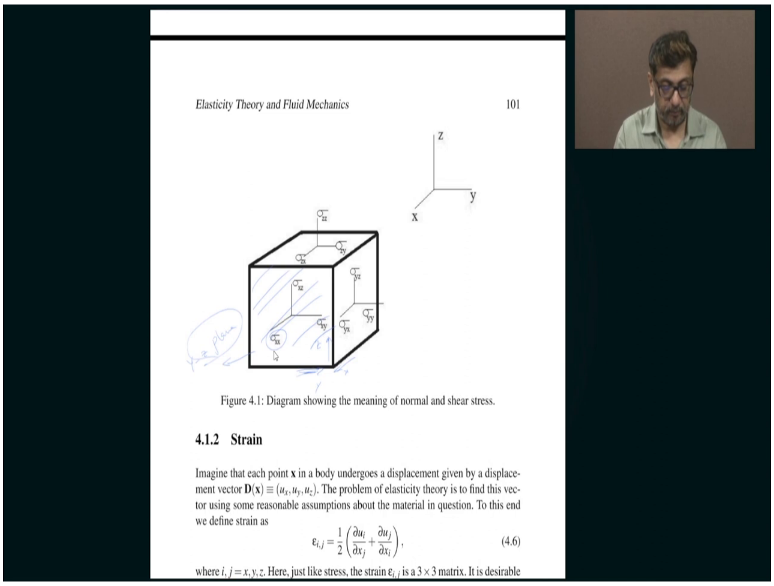

So, so much for the, so you can see this diagram clearly tells you what I am talking about for example, say if I am looking at this surface. So, you see, so this is my X direction right. So, this is my X direction and this is my Z direction and this is my this is my Y direction. So, you can see that sigma XX is perpendicular to the surface right. So, the surface is basically the YZ plane. So, this is the YZ plane.

So, sigma XX is perpendicular to the YZ plane which would correspond to normal stress and sigma XZ and sigma XY would be the shear stresses. So, they are within the plane.

So, same with all other so you see its surface dependent. So, you have to decide what surface you are talking about then you can determine what forces are acting alright.

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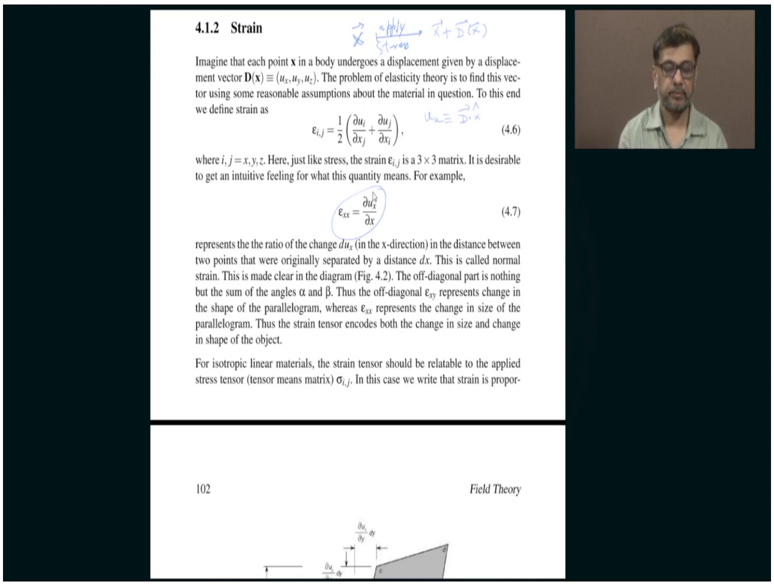

So, now the next important notion or concept is basically strain. Strain is basically the response to stress so; that means, strain is what the elastic body does when stresses are applied to it.

So, usually if you have a rigid body you apply something analogous to a stress which is basically a force a, rigid body will simply accelerate it does not do anything other than that it will simply accelerate; accelerate means it could linearly accelerate or angularly accelerate you know it can spin around and does it can do all sorts of things.

But then it will certainly not change its shape or size or whatever, but; however, an elastic body does exactly that it if you apply forces to an elastic body. The elastic body changes shape and size. So, typically we do not want it to accelerate as a whole or spin around rather we wanted to change shape. So, that would be the perfect kind of elastic object that.

So, it is it would be the antithesis of a rigid body. A rigid body will simply not deform in any way, but it will accelerate either linearly or angularly, but a perfectly elastic body would do the exact opposite that it will simply not accelerate in any way, but it will deform in whatever way you want it to deform. So, that deformation of an elastic body is described by a concept called strain.

So, stress is what you apply, strain is the response ok. So, how do you describe the response of an elastic object to stresses? To describe that we imagine a point x in the body so, we have said that basically what the object does is. So, the body does is it deforms. So, what that means, is that the point which was initially at x now shifts to a different location called x plus D. So, in other words there was a point X in the deformable object. So, apply stress.

So, if you apply stress that point shifts to this location and this D is the deformation of this point X, but it clearly depends. So, different points can deform differently and there is no requirement that all points should deform by the same amount. So, clearly it is a function of which point inside the material you are talking about. So, this displacement is basically given by. So, we use the symbol small letter u to describe the components of this displacement.

So, u subscript x corresponds to the displacement of this point x in the x direction, u subscript y describes the displacement of the point in the y direction and similarly z. Now, what we are going to do is we are going to describe something called the strain tensor. So, in other words just like stress was a 3 by 3 matrix because see what was stress, it was described by something called sigma.

But sigma had components like sigma zz, sigma xz, sigma yz. So, what is sigma zz? Its basically the force acting per unit area at some point perpendicular to the yz xy plane because it is sigma z z. So, it is parallel to the z axis. So, it is perpendicular to the y xy plane. So, the forces acting at that point perpendicular to the xy plane would be sigma zz. So, similarly sigma xz would be perpendicular to the y axis. So, it will be parallel to the xy xz plane ok.

So, bottom line is that it has to be described by a matrix because the force acting depends upon the choice of the surface that you have in mind. So, as a result is more than just a vector. So, it is a vector plus this choice of the surface that you have in mind. So, it becomes kind of vector in three different directions. So, it is therefore, a 3 into 3 object, so which is a matrix or a tensor.

So, similarly a strain also is a tensor for the same reason you know a point can deform in the same direction in which you choose to move. So, if you shift your point from x to x plus D x, but do not shift y and z. So, the point look what I mean by that is that you have a point called xyz you choose to investigate what is going on at the neighboring point. So, the neighboring point could be for example, x plus D x, but then y is still y and z is still z.

So, now at that neighboring point the deformation of your original point can be also in a different direction other than in the x direction. So, you can have this u of x which is basically D dot x hat right. So, this can be a function of x y or z right. So, if you are talking about this, so this is basically the normal strain so; that means, it is the strain is in the same direction in which you are looking so, if you decide to shift your point of consideration from x to x plus D x if the original point deforms in that direction it is called normal strain.

But it can also deform in the perpendicular directions and just like in shear stresses. So, you can have a shear stress applied in perpendicular to a surface. So, here also you can have something in the same direction or in the perpendicular directions. So, the deformation in the perpendicular directions are basically called shear strains. So, you have normal strains and shear strain.

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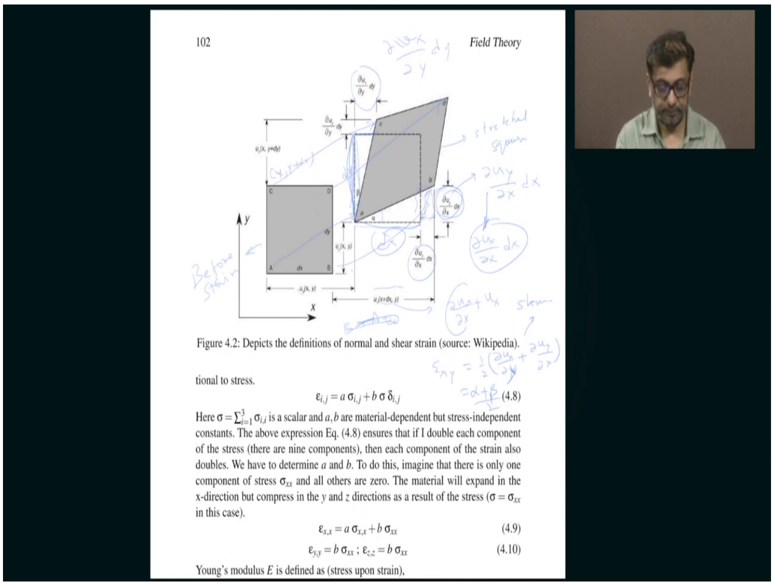

So, it is convenient to you know it is better to visualize this through a diagram like this. So, I think I should spend some time explaining this diagram this figure 4.2. So, what this is basically it tells you that there is a point here. So, this is the original situation this is before strain means when this when the object has not undergone strain.

So, it has not undergone strain it is the shape of this. So, it is just some piece of this elastic object. So, this square A, B, C, D is just some imaginary piece of an elastic object. So, I am specifically considering say in 2 dimensions. So, just imagine forget the 3rd dimension for now, just imagine a two dimensional elastic medium and imagine a small piece of that called the square ABCD so, the interior of the square ABCD.

So, now what we do is we apply some stresses to this. So, in other words we take this elastic object which is sitting in this x y plane see the square ABCD is a small piece of it. So, this entire xy plane is an elastic object. So, imagine you stretch that elastic object. So, what is going to do is that this square will now do this that that the original square was sitting here; it will now not only sit somewhere else because it has been stretched. So, imagine you stretch it like this. So, this square has become stretched.

So, this is a stretched, so stretch square. So, you stretch the square this points big letter A becomes small letter a. So, in other words it gets stretched and it shifts there and then this big letter B shifts to small letter b, big letter D shifts to small letter d and big letter C shifts to small letter c. So, you see this square which was originally a square has now not only shifted like this, but also deformed. So, now, we can understand the various things like this ok.

So, this u subscript s is this distance is the distance between the in the is the distance in the x direction between the new a point and the old A point, so that is u of x that is the definition of u of x ok. But then the distance between the old B point and the new b point right the distance between the new b point which is small letter b and the old B point in the x direction is clearly x plus dx comma y so; that means, you see b is dx away from a in the x direction. You see what is B? B is A plus dx basically.

So, in other words point B is shifted in the x direction from A. So, then this point deforms by a different amount by this much amount. So, this is the deformation of point b, but point b was originally not at x comma y, x comma y a is sitting there ok. So, b is sitting at x plus dx comma y. So, now, you ask yourself how much does that deform, how much does b deform in the x direction?

So, the answer is u x bracket x plus dx comma y not x comma y x comma y would be a how much a would deform. So, similarly in the y direction you can think of the point c which would have deformed by this amount. So, x remains x and, but n y becomes y plus d y right because c is exactly on top of a in the y direction. So, the location of the original location of c is capital C which is x comma y plus d y right. So, y plus d y is the original location of x comma y plus d ok. So, now, the question is when you.

So, this is before you apply any stresses or strains or whatever it is before the material undergoes strain. Now, after it undergoes strain the material deforms and capital C becomes small c, capital C shifts to small c. Now, the question is what is small c? So, small c is both shifts in the x and as well as in the y direction. So, what is u of x? u of x is the amount by which A has shifted ok.

And u x plus dx is the amount by which B has shifted. So, now, if you ask yourself what is the difference between that those two that is basically the normal strain in the x direction, right. So, you see it basically what the normal strain in the x direction it tells you is the amount by which the size of the square has changed see the original size of the square was length of dx.

So, now, it has kind of because it has got stretched the size of the square has now increased in the x direction by this amount ok. So, it has got increased by this amount. So, d u x by d x into d x this much amount. So, original size was dx. So, after stretching the size has increased by this amount. So, that is. So, this is this coefficient is basically called the fraction by which the size of the square has increased is called the normal stress in that particular direction in this case it is the x direction.

So, this is as against the shear strain which corresponds to the normal strain indicates the amount by which the size of that imagined shape changes so; that means, if you have a square when you stretch it the size of the square changes, but not only the size of the square changes the shape of the square also changes. So, it would not remain a square it is not like the square becomes a bigger square.

It could happen but usually it not only becomes a bigger square when you stretch the material it will also not remain a square it will become something like this a trapezoid a parallelogram types. So, you have to understand both means you have to account for both not only the size has changed. So, the size has changed by this amount du by dx into dx that is the amount by which the size of the square has increased, but then we also have to describe the amount by which the shape of the square has changed.

So, the to understand the shape of the square what you do is exactly you calculate as epsilon xy for example. So, if you calculate epsilon xy that is basically a d u x by dy plus d u y by dx one half of that. So, this is basically alpha plus beta. So, this epsilon xy basically is called the shear strain. So, the shear strain is. So, why is this the case? Because you see approximately you can see what is happening this distance.

So, this amount is approximately this much and this amount is dx. So, this distance this angle is just because the angles are small tan theta equals tan alpha is approximately alpha and what is tan alpha? It is this divided by this ok. So, what is this? This is d u y by d x. So, this is d u y by d x into d x ok. So, this divided by dx is this angle. So, this tan of this angle this is this divided by dx. So, that is tan of this angle similarly here tan of beta is this divided by dy this much is dy the square the side of the square is dy its not really square is a rectangle but dy.

So, the this angle is basically this divided by tan beta is this divided by this and what is this? It is d u x by dy into dy ok. So, if I add these two and take the average you get alpha beta by 2 basically right. So, the sum of these two would be alpha plus beta.

So, in other words the shear strain will basically tell you the amount by which the material has changed the shape. See whereas, the normal strain will tell you the amount by which the size of that deformation has increased so; that means, if you imagine as original rectangle when you stretch the material not only the rectangle will become bigger, but it will not remain a rectangle.

So, the amount by which that rectangle will become bigger is called normal strain, the amount by which it changes its shape to something else is called shear strain. So, bottom line is that you have these two concepts; one is called stress, stress is what you apply to the material, but strain is the response strain is what the deformable body does when you apply stresses. So, now, in the next class I am going to tell you exactly how to relate these two concepts.

So, we are going to consider what are called linear deformable material where we assume that the material basically responds linearly; that means, the amount of strain that the material exhibits is going to be proportional directly proportional to the amount of stress that you apply. So, if you double the stress that you apply the strain also doubles. So, those are called linearly deformable material and there is a perfectly good approximation for a vast majority of deformable objects that are of interest to us ok.

So, in the next class I will tell you how to relate these two and that is called the stress strain relations and that will be the starting point of our important discussion of elasticity theory ok. So, I hope you will join me for the next class and I am going to stop here.

Thank you.