Dynamics of Classical and Quantum Fields: An Introduction
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Electromagnetic Fields
Lecture - 11
Green's Theorem and Green's Functions

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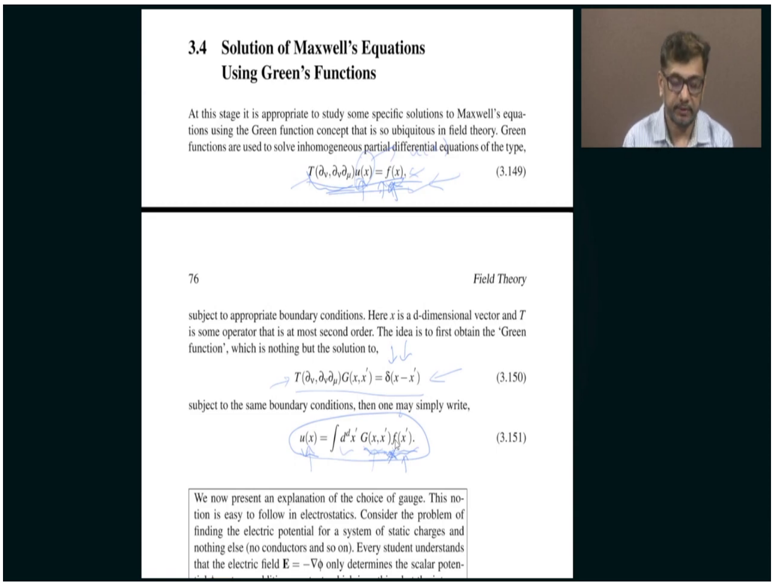

Ok. So, in today's class, I am going to discuss another topic and that is the Solution of Maxwell's Equation using Green's functions. So, you see this technique of Green's function appears repeatedly in many applications in physics. So, it is worthwhile to know what Green's functions are.

So, specifically Green's functions basically always allow you to know the solution at some other point, if you know the solution at some given point. So, basically it is like a propagator. So, it also appears in some other contexts later on when we discuss multi-particle systems, where the number of particles is not conserved. So, but the defining characteristics of Green's function is that they obey a certain equation which is very generic, and that equation is basically of the form that is described here which is 3.150.

So, T is some operator which can, I have chosen it to depend upon a maximum of 2 spatial space time derivatives. So, it could involve time or only space. So, bottom line is that usually in applications in physics, the operators that we consider are at most second order.

So, the idea is that we have to learn how to solve these types of equations. But more generally what will happen is that in applications you will find that the solution that we seek for some say this is this could be for example, the potential of some charge distribution or it could be basically the 4 vector potential.

That is the scalar and the vector potential combined of some electromagnetic field. And f could be some source term, right. So, you could have sources of the electromagnetic field. And this would correspond to the Dalembertian operator in that case. So, typically, that would be the case in electrodynamics. In electrostatic this would be just the Laplacian, and the right hand side would be either the charge density if you are talking about electrostatics or the current density if you are talking about magneto statics.

So, bottom line is that see the Green's function technique basically allows you to know the answer to this question. That is it allows you to find the answer for u of x. But, then to find the u of x it is easier many times to solve a generic equation, which is called the Green's function equation.

So, the idea here is that see the sources can keep changing. So, you see the point is that you can replace the given set of sources by some other set of sources. It is very inconvenient to repeatedly keep solving this equation, this 3.149, again and again just because you have changed the source to something else.

So, what is the more convenient is to solve basically the solution for a point source but then located at some arbitrary point. So, that means, you imagine a point source that that is located at some point called x dash. And then you find out your you know the answer you are looking for, for that point source. But the claim is that because these equations are going to be linear in the unknown which is u u of x. Because it is linear you can always add up all the sources.

So, if you have a charge distribution rather than a point charge, then you can just go ahead and construct the charge distribution as basically the sum of lots of point charges with appropriate weights. So, as a result, your solution will also be a summation of the answers for the corresponding answers for the point charge with the appropriate weight. So, that is exactly what this is.

So, 3.151 basically tells you that the answer you are looking for u of x is basically the summation or in this particular case the integration. So, this is f of x is your weight. So, that, so this would be the correspond to the weight of the delta function. So, this is your Green's function for the point charge.

So, the answer for the charge distribution f is given by the answer for the point charge multiplied by the charge distribution, summed over all the locations where the charge distributions are found which is x dash. So, this is basically the power of the Green's function technique. That is you do not have to repeatedly solve for your unknown which is u of x, every time you change your source which is f of x, ok.

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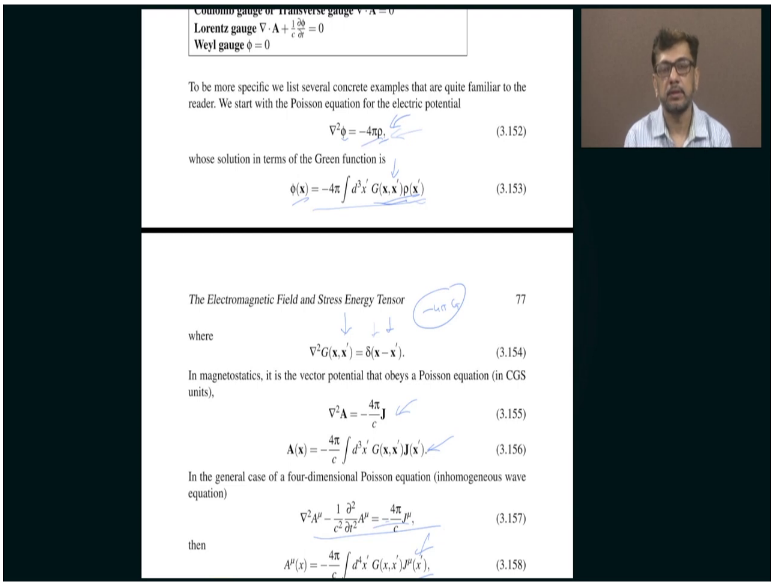

So, having said that let me go ahead and give you some more specific examples. So, that was very general. So, a more specific example would be say in electrostatics it would correspond to solving say the Poisson equation.

So, you have the Poisson equation which is del squared phi equals minus 4 pi rho, and the solution to this is clearly based upon what we just discussed. It is phi of x is minus 4 pi integral over all the locations where the sources are present, which is rho of x dash times the Green's function of the.

So, the remember what the Green's function is, it is the it is the solution of the potential when there is a point source located as x dash. So, it is the solution of the potential at position x, when there is a point source at x dash. So, that is exactly what this is. So, similarly this is what I just told you is electrostatics.

But then you can also do something very analogous for the case of magnetostatics. So, instead of the scalar potential in magnetostatics, you have to replace by vector potential, and instead of charge density you have current density. So, it is pretty much the same thing, mathematically there is no difference, ok.

So, you can also combine these two and you can talk about electrodynamics, where you have both scalar and vector potentials together and they influence each other. In that case, you should be talking about the 4 vector; you know the 4 vector potential. So, that means, time component would correspond to the scalar potential and space components would correspond to the vector potential.

So, put together it could correspond to a 4 vector. And J mu is basically the 4 vector current. So, that would correspond to charge density and current density. And a 0 would correspond to scalar potential and a 1, a 2, a 3 would correspond to the components of the vector potential.

So, like I told you, just like you can write down the solution for the Poisson equation in terms of the Green's function, even for this wave equation with a source, you can still do the same thing, except now this Green's function obeys this sort of wave equation, but with a point source. So, there is a point source at x dash.

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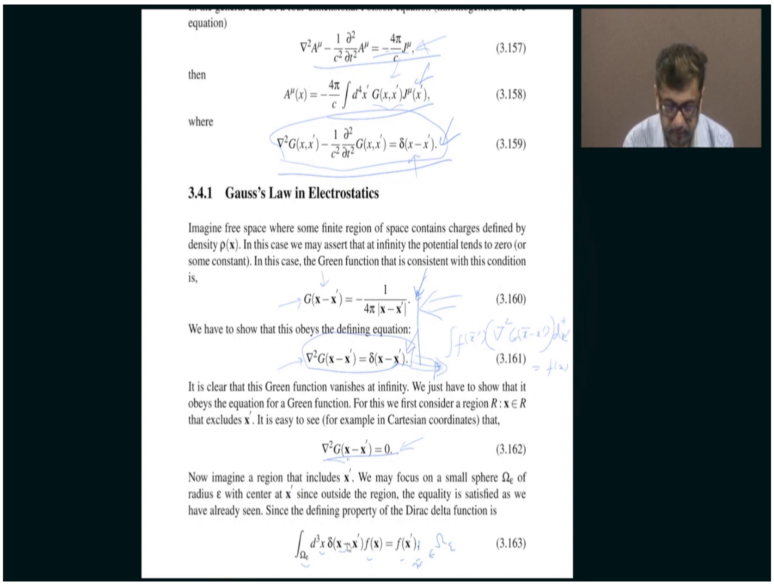

So, the bottom line is that in all these cases the prescription is that you first solve for the Green's function for a point source and then you use it to construct your solution for any other source.

So, now, let me come to something very basic which seems quite obvious, but I think it is worth pointing out nevertheless. And that is suppose you are in you are talking about electrostatics. So, therefore, the operator in question is the del squared operator, ok. So, there is the Laplacian. So, the question is what would be the Green's function of the Laplacian. So, this is; so, in other words the answer is G, where G obeys del squared G equals Dirac delta.

So, we all know what that is right because we know what is the potential. So, what is the physical meaning of this? Basically, G is proportional to this electric potential produced by a point charge sitting at x dash. So, the electric field produced at position x, when a point charge is sitting at x dash.

So, now, we all know what the answer is and that is basically this. So, what I am going to do is its 1 by x with the appropriate pre-factor. So, now I have to convince you that that appropriate pre-factor is indeed what I have written there which is minus 1 by 4 pi. And so, that is not entirely obvious because you see Dirac delta is something a very peculiar object.

So, usually what happens is that if you are not careful, if you blindly go ahead and differentiate this with respect to x. For example, you del squared take del squared of both sides of this, you simply get 0 on the right hand side. Because usually you will subconsciously think x is different from x dash and then everything will cancel out and you will get 0.

But, so therefore, this result is of course, correct whenever x is not equal to x dash. But when x is approaching x dash this is not correct, so we have to be careful. So, the question is how do you do this carefully, how do you do del squared G carefully. So, the answer is the following.

So, you see we do not actually. So, in other words, what we do is we note that this is just a short hand for writing something which is more mathematically rigorous. See what this really means mathematically is that this is f of x dash right, del squared G of x minus x dash right, d cubed x dash. So, and that is basically equal to f of x. That is what this means, strictly speaking, or you mean I had done, I have done it the other way.

So, ok let me do it the other way.

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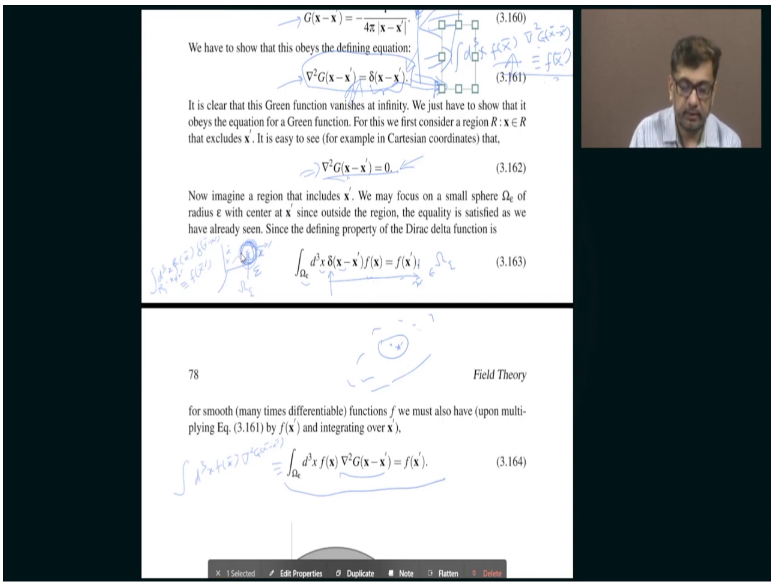

So, basically what I have done is that this equation that is 3.161 is a shorthand for writing this, ok d cubed x f of x right del squared G of x minus x dash equals f of x dash. So, that is what this means, ok. So, that is the meaning of this, ok. So, that is the meaning of this.

So, the question is how do you make sense out of. So, in other words, I have to really prove this rather. So, you see proving this is identical to proving this. Proving this is not convenient mainly because you know this Dirac delta has a very precise mathematical meaning which actually really means this. So, in other words, any object called delta which obeys this for any f is called the Dirac delta. So, we really should be proving this identity, ok.

So, now, to prove this identity what we do is; so, I told you that you know if you blindly take del squared G you will get 0, but then there is a hidden assumption that x dash is not equal to x. So, now, you imagine some region also, you are if you see here you are supposed to integrate over all space. So, imagine you have a coordinate system here, ok.

So, let me write this coordinate system. So, imagine you have a coordinate system and this is your x dash. And you imagine you have separated out a small sphere out. So, there is a small sphere of radius epsilon around x dash and the rest. So, bottom line is that while doing this you are supposed to integrate over all space right. So, you are supposed to integrate over all space, is not it.

So, this is what we want to prove. So, you are supposed to integrate over all space, but then I am going to split this up into two regions, one is region 1 where the x is not equal to x dash, right. So, in other words, so that region is outside the sphere. So, it is outside the small sphere where x can never be equal to x dash. But then there is another region which I call omega epsilon region which is inside this and that region allows for x to become a arbitrarily close to x dash.

So, now in the first region where x dash is always inside the sphere. So, it is not all the x points are outside the sphere and x dash is inside the sphere, so x and x dash will never touch each other. So, because they will never touch each other, del squared G is anyway

0. So, I do not have to consider that, so that is anyway 0. So, this is basically same as saying this.

So, I just want to impress upon you that doing this is same as doing this, right. So, instead of integrating over all space I simply integrate over this, over this small sphere of radius epsilon, ok. So, that is sufficient because outside the sphere anyway it is 0. We just verified that by brute-force by just taking del squared.

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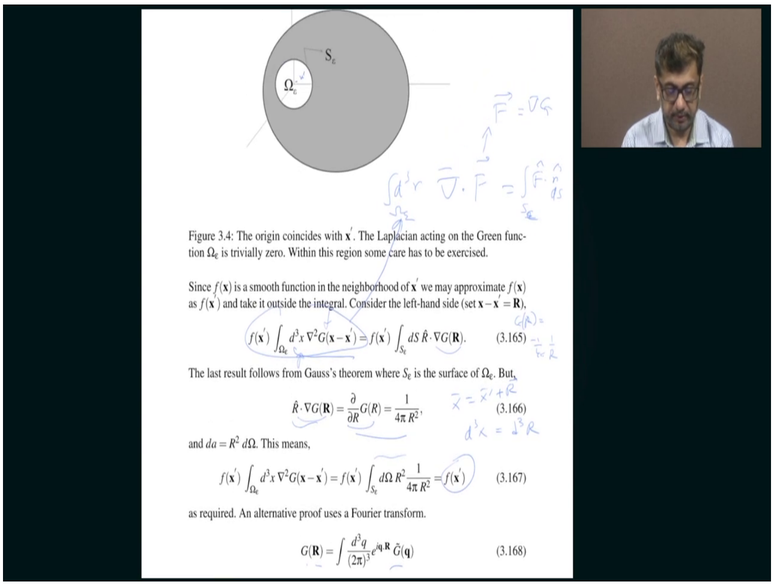

So, now the question is how do you verify this. So, once you verify this you are done. So, that means, by definition del squared G is Dirac delta. So, if you can show this then it is obvious that del squared G is Dirac delta.

So, how do you show this? So, for showing this first we assume that f of x is very smooth. So, if f of x is smooth, then I can there is something called the mean value theorem of integral calculus which tells you that you can replace. So, if you have a smooth function under the integrating between some limits, there will always be some point within those limits, where the you know you can pull out that, you can substitute the value of x for that particular intermediate value and pull it out and it will still be correct.

So, that is the mean value theorem of integral calculus. And because the those see x, x is arbitrarily close to x dash because it is inside this omega epsilon I will simply replace x by x dash for inside f of x because you see f of x is smooth, right. So, only in f of x, I can do that because f of x is smooth. So, if f of x smooth means infinitely many times differentiable.

So, I can always replace x by x dash and pull it out of the integration. And then you get this result. So, the, so in other word, this integration the left hand side is basically same as this because I have just approximated f of x. It is not really an approximation; it becomes exact as epsilon tends to 0.

So, now all I have to do is I have to integrate this over omega epsilon. But fortunately see the reason why I chose a small sphere is because spheres are nice to integrate with. I could have chosen a cube or something more complicated, but that would have made my life unnecessarily complicated. So, I simply chose a sphere because that is the most convenient thing to do.

So, now if I use the sphere then I can use my Gauss's theorem and I can replace you know the volume integral of del squared is normal component of the surface integral, right. So, basically you are rather, so is this is something like divergence of some other quantity, right, so some other quantity F. So, del cube R over omega epsilon is basically the normal component of F, right over that surface.

So, what is F? F itself is, F itself is del of G, where G is your Green's function. So, then its del dot del G that is del squared G, right. So, that is what. So, imagine del of G is F vector, so then you get basically this is nothing but volume integral of divergence of F. So, divergence volume integral of divergence of s from Gauss's theorem is surface integral of the normal component. So, if you do that then you see, all you have to do is find the gradient of, so that is your F of capital R is x minus x dash.

So, you just have to find; so, you shift your coordinate. So, that x dash is a constant. So, I can shift, I can write as small letter x as x dash plus R and then my d cubed x will be basically d cubed R because x dash is constant when I am integrating. So, then I will simply be able to do this. So, then this is nothing but the they just the scalar derivative know because G of R is just minus 1 by 4 pi. So, then G of R is nothing but.

Student: G of r.

Minus 1 by 4 pi into 1 by R, that is all. So, then it is d by dR of G of R. So, then you simply just go ahead and integrate because now you know how to integrate. So, the sphere is basically it has fixed radius. So, it is d omega into R squared. R is fixed, so you just see its R squared into d omega into 1 by 4 pi R squared because that is what R hat dot grad G is, it is 1 by 4 pi R squared. But then, R squared, R squared will cancel integral d omega is also 4 pi, so 4 pi 4 pi will also cancel and you get f of x dash.

So, bottom line is that that completes the proof in other words what we have succeeded. So, you might be wondering what it is I am doing. So, basically I am just trying to show you that if I take del squared of this, I really will get Dirac delta not 0, because you see if you do not do this carefully if you take del squared of 3.160 on both sides in a very naive and not so careful way, you will simply get 0, you will not get Dirac delta.

So, I have carefully showed you that if I take del squared of G, where G is defined to be minus 1 by 4 pi naught anything else, minus 1 by 4 pi into x minus x dash magnitude, if I take del squared of this G I in fact, get Dirac delta x minus x dash not 0, and that is not that easy to prove. It is easy to prove its 0, but then 0 is the wrong answer when x is can be equal to x dash. If x can never be equal to x dash then 0 is the right answer.

But then there are many situations where x is as close to x dash as you want so, in which case you have to be careful. So, that is why you have to carefully prove that it is actually Dirac delta of x minus x dash and not anything else so, to prove that you have to do all this, ok.

So, so therefore, it is only with that minus 1 by 4 pi pre-factor it is because with any other pre-factor del squared G is still 0, when x is not equal to x dash. But then that minus 1 by 4 pi is very crucial if you want to get Dirac delta and not some you know minus 1 by 4 pi into Dirac delta or some constant into Dirac delta. So, it is really minus 1 by 4 pi and not anything else, ok.

So, but then this is of course, a very mathematically strict way of doing things. But you can do it more easily, but with less rigor by using Fourier transforms.

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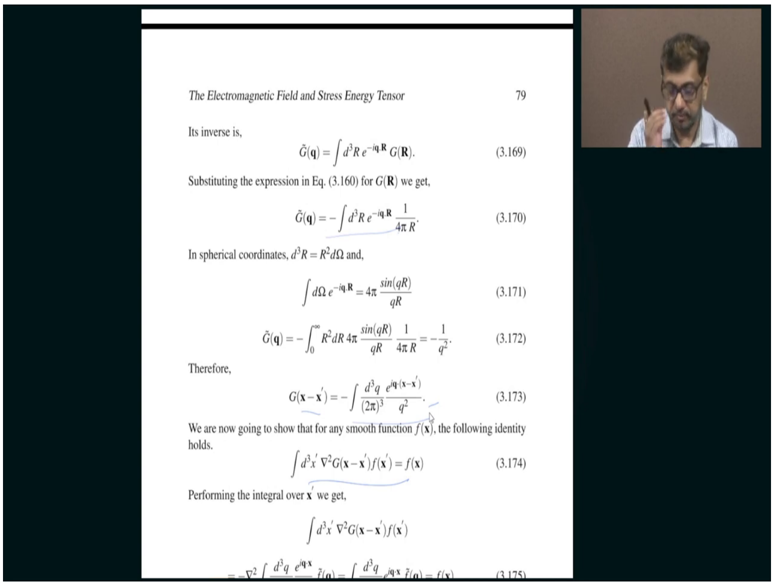

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Its inverse is,

$$\tilde{G}(\mathbf{q}) = \int d^3R\, e^{-i\mathbf{q}.\mathbf{R}}\, G(\mathbf{R}). \quad (3.169)$$

Substituting the expression in Eq. (3.160) for $G(\mathbf{R})$ we get,

$$\tilde{G}(\mathbf{q}) = -\int d^3R\, e^{-i\mathbf{q}.\mathbf{R}}\, \frac{1}{4\pi R}. \quad (3.170)$$

In spherical coordinates, $d^3R = R^2 d\Omega$ and,

$$\int d\Omega\, e^{-i\mathbf{q}.\mathbf{R}} = 4\pi\, \frac{sin(qR)}{qR} \quad (3.171)$$

$$\tilde{G}(\mathbf{q}) = -\int_0^\infty R^2 dR\, 4\pi\, \frac{sin(qR)}{qR}\, \frac{1}{4\pi R} = -\frac{1}{q^2}. \quad (3.172)$$

Therefore,

$$G(\mathbf{x}-\mathbf{x}^{'}) = -\int \frac{d^3q}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}^{'})}}{q^2}. \quad (3.173)$$

We are now going to show that for any smooth function $f(\mathbf{x})$, the following identity holds.

$$\int d^3x^{'}\, \nabla^2 G(\mathbf{x}-\mathbf{x}^{'}) f(\mathbf{x}^{'}) = f(\mathbf{x}) \quad (3.174)$$

Performing the integral over $\mathbf{x}^{'}$ we get,

$$\int d^3x^{'}\, \nabla^2 G(\mathbf{x}-\mathbf{x}^{'}) f(\mathbf{x}^{'})$$

So, basically you Fourier analyze your G of R and you simply invert the Fourier transform and so I will allow you to read this. So, you just do it. So, the reason why it is not so rigorous is because it has a 1 by q squared and you are just integrating over q without. And so, this these types of integrals are not mathematically well defined because your integrant basically, yeah.

So, I mean so there are all these issues that you have to be careful.

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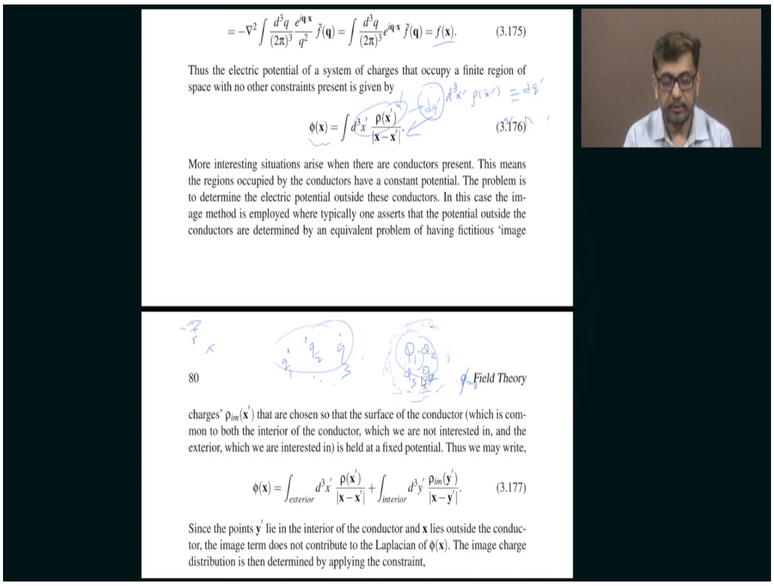

So, but then this will also allow you to prove the same thing that integral of d x dash del squared G of x minus x dash is f of x, ok. So, this is less rigorous, but quicker and, but whatever it is.

Bottom line is that having done all that you can now convince yourself that the potential function at some other point. So, if you have a charge distribution in some finite region, so that is basically the potential of the charge distribution some other point is basically the charges. So, d cubed x dash into rho of x dashes basically your d q dash so, the charge at that location x dash.

So, you just replace that by this and then you add up over all your charges. So, that is what that is. So, that is your potential, ok. So, this is basically your d q x, d q dash. d cubed x dash into rho x dashes. So, rho is charge density. So, if you multiply by small volume you get a small charge called d q dash.

So, d q dash by x minus x dash is your electric potential produced by d q dash at location x dash and then you add up over all the locations, ok. So, then you get the potential function at some point x. So, now, you can also use this Green's function to do something more interesting. So, till now what we did is fairly straightforward.

We know you just have a charge distribution in some finite region you want to know what is the electric potential somewhere else. But what would be more interesting is, many times what happens is that in many problems in electrostatics the charge distributions are not directly given, they are indirectly given.

So, by that I mean typically, you will be told that there are some charges which are explicitly known, but there are other charges which are not explicitly known for example, they are known indirectly in what way? You just specify that there is some conductor sitting in some space. So, there is a conductor of a certain shape, may be like a spherical conductor.

So, what you are told therefore is that that conductor for example is grounded. So, that means, you are told that that region or space has a constant potential equal to 0. So, where the conductor is located the potential is constant and 0, and that whole thing region is sphere. So, that indirectly implies that there are charge distributions sitting on the conductor which are not specified, ok.

But then what is specified is, it is you are told that the potential function is 0. So, now you want to find the electric potential somewhere else. So, typically your; so, this is the problem description you could have bunch of charges sitting here, ok dot dot dot, but you can also have bunches of charges sitting on the surface which you are not told what they are. But you are told that this is grounded, so that means, this potential is potential or phi equals 0. So, on the surface it is 0.

So, now, you want to know what is the potential somewhere outside this conductor. So, then; so the question is how do you answer such questions? So, to answer such questions you there is a very powerful technique and that is called the image method. So, what you do is basically you replace this problem by; so, the point is the difficulty here is that you do not have any means of modeling the conductor because we do not know what charges are sitting here.

So, what you do is that you just say that as far as a point outside is concerned, it only cares about the fact that the potential here is 0. So, you see the moment you. Firstly, there is a theorem which says that the solution of a Poisson equation is basically unique, right.

So, the point is if you find a solution it is also d solution. So, the point is that where to find a solution all we have to do is you have to simulate this conductor.

So, in other words, what we do is we replace this conductor and we pretend that there are some charges here. So, we replace this conductor by some charges which are called image charges, ok. So, we will put a bunch of charges here. However, many we require. But then we say that this now this conductor is not there, ok. So, now, we are going to say this conductor is not there. So, instead of the conductor there are these charges, ok instead of, not in addition to, instead of, ok. So, the conductor is simply not there.

But this method will be wrong if you also try to answer what happens inside the conductor because inside the conductor answer is already given, the potential function is 0, right. Inside the conductor and on the surface it is 0, you do not need an answer inside the conductor. We just want to know what it is outside.

So, to know what is outside what we do is we write a bunch of charges in such a way that all these outside given charges and the charges that we have imagined put together will conspire in such a way that the they will make sure that the potential on the surface is actually is 0. And those are called the image charges.

So, you can always in fact, you can convince yourself that it is always possible to find image charges. So they can be complicated, but they will always be, they will always exist. You can always find them.

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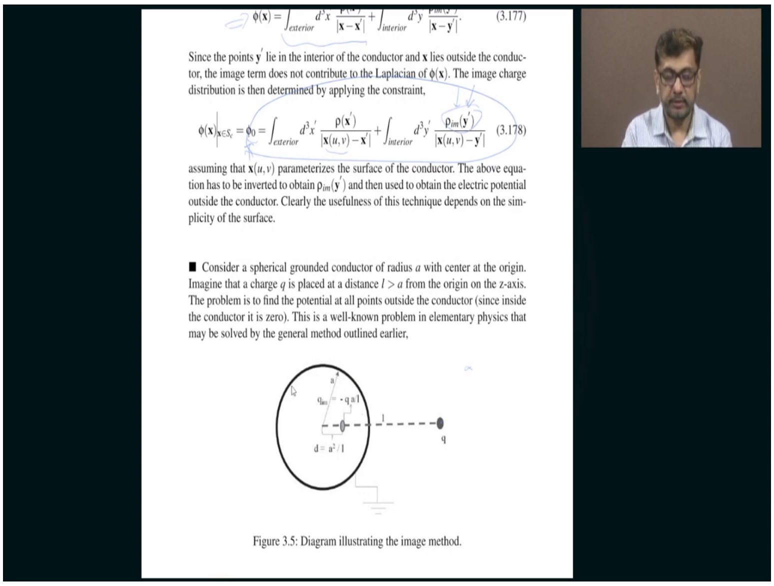

So, that is the bottom line. So, this is just a mathematical description of that procedure. So, what we do is that we assume that exterior to the conductor, there are known charges, these are known charges.

So, somebody has told you what those charges outside are. But then somebody also says that there are there is a conductor and this on the surface of the conductor the potential function is say phi naught, need not be 0. It can be, you can put a battery there and make it phi naught.

So, if that is the case then what we do is, so we replace the conductor by these are called image charges, im stands for image. So, basically we replace by image charges and then and this is x of u, v, is the parameterization of the surface of the conductor. So, you parameterize the surface of the conductor. And then the bottom line is that you solve for this. So, this equation indirectly specifies what this is.

So, if you invert this equation. So, if you invert 3.178, because left hand side is known which is 50 which is a constant, you can invert this which might be very hard. I told you it is not at all easy. And you could invert this and find rho of im which is the charge density of the image charge, ok. So, this is the general prescription. So, you can work out standard questions like this.

So, if you have a sphere which is grounded and there is a charge q outside, what is the electric potential; somewhere there. So, for that you have to it is sufficient to introduce one image charge, so you replace the conductor by this image charge and this already existing charge and you can convince yourself that these two put together will ensure that the potential on all the points on this circle are 0, ok.

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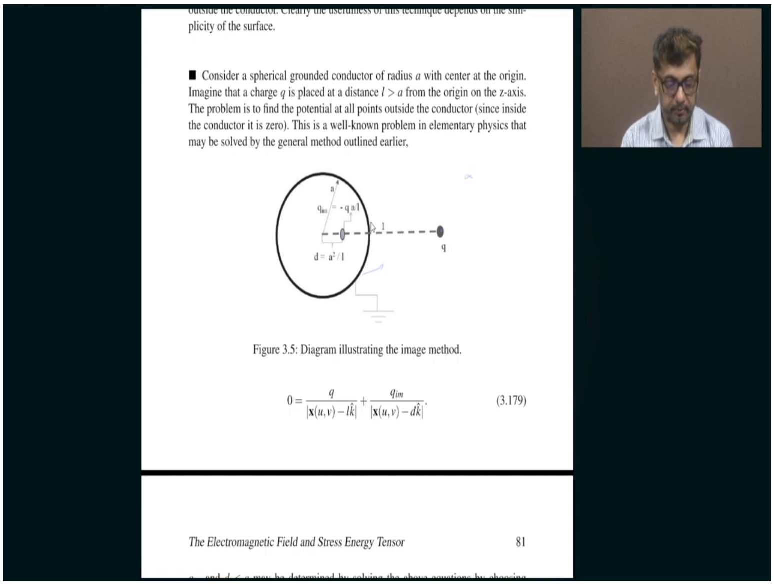

So, that will be the case if you ensure that the charge is sitting at a squared by l, where a is the radius of the sphere and l is the distance from the center of the sphere to this other charge an outside charge, ok. And then you make sure the image charge is negative compared to the outside charge, but the magnitude is different. So, it is magnitude is a by l times the charge that is sitting outside.

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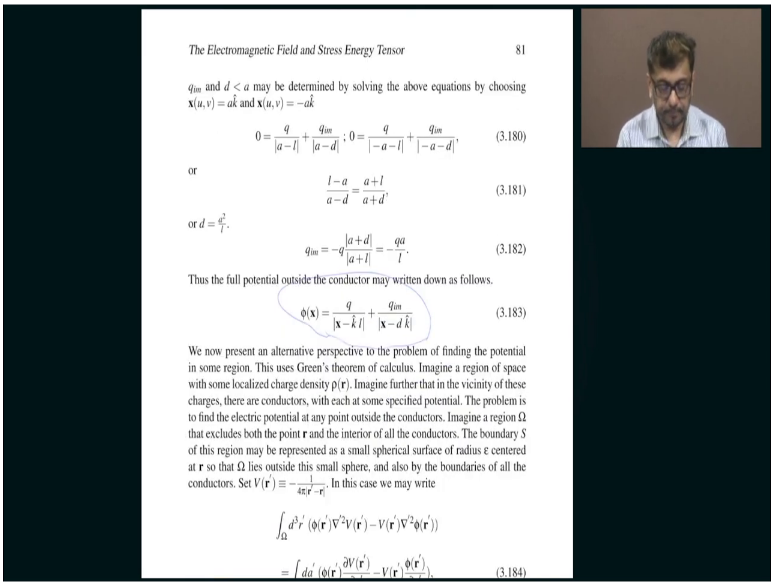

So, if these two conditions are met. This is going to produce the right answer. So, these two charges put together will produce a potential outside somewhere to be this. So, outside it will be this, ok. So, this is the image method.

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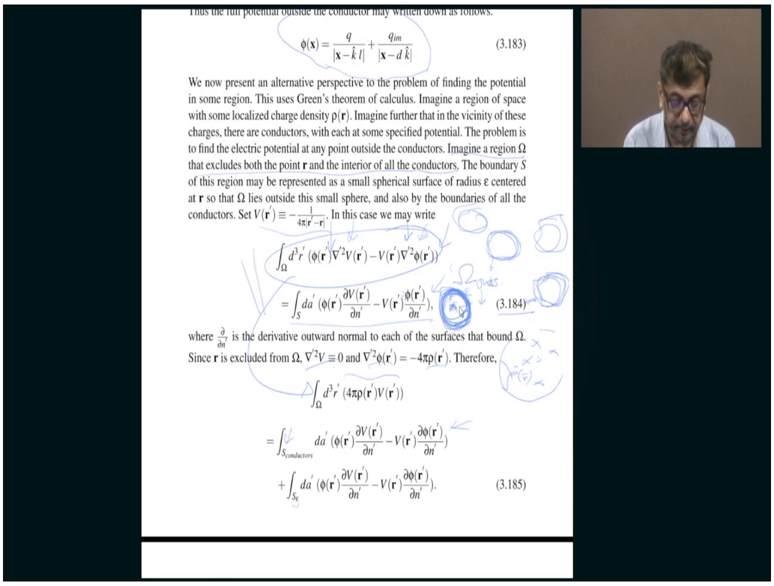

So, now what I am going to do is I am going to discuss another technique, ok. I am going to discuss a more general version of this image method, ok.

So, what I am going to do is I am going to discuss a more general version of this image method which basically uses what is called Green's theorem. So, the Green's theorem is going to be a very powerful tool in our analysis. So, basically, so what I am going to discuss now is just the is just the mathematically formal way of describing the image method.

So, the bottom line is this, that we use this theorem this is the this is a mathematics theorem. So, if phi and V are some functions, and suppose you want to calculate something like this. So, you might as well calculate this instead. So, that is what the theorem says.

If you want to calculate the volume integral of phi into del square del squared V minus V into del squared phi that is same as calculating the surface integral, where the surface is the surface bounding this omega, right, of phi into the normal gradient of V minus V into normal gradient of I missed a del there, normal gradient of phi, ok. So, now what I am going to do is, so I am going to imagine right imagine a region omega that excludes a point r and the interior of all the conductors. So, for this I have to draw pictures.

So, imagine you have a bunch of conductors, ok. So, you have a bunch of conductors here and there are bunch of charges here, ok outside somewhere. So, these are actual charges somebody has put them there, ok. So, there are all bunch of charges somewhere. It could be all over the place, but it has to be in some finite region. And even the conductors have to be in some finite region. So, these are all conductors at various potentials.

So, the bottom line is that. So, imagine that the omega that you are looking at is this omega outside. Outside means the outside the conductors and outside the; so you first you write down your point of interest, you are interested in this point, ok. So, you are interested in this point. So, it is also outside this. So, that means, your omega is here. So, it is outside this point of interest and it is also outside the conductors, ok. So, that is what this is.

So, imagine omega excludes both the points r, and it excludes the interior of the conductors, ok. So, the boundary of this region. So, the, so this is the region. So, if the boundary of this region will be two disjoint pieces, one is a small spherical surface of radius epsilon, this one. So, this is a small spherical surface of radius epsilon centered at r, ok. So, that omega is outside this sphere. And it is also the boundaries are the these are the boundaries. So, there are many many boundaries of this omega out.

So, it is this, this is one boundary, this is one boundary, this is one boundary. But notice that this is not a conductor. This is our small imaginary epsilon sphere. These are all conductors. All potentials at all these points are constant, but here they need not be constant, ok.

So, now, I am going to since this is valid, this Green's theorem is valid. So, this 3.184 is called Green's theorem, ok. So, this Green's theorem is valid for any phi and V. So, specifically I am going to select V to be minus 1 by 4 pi mod of r dash minus r, because I know that that corresponds to a point charge.

So, now, notice that since r dash, so the r dash is, so the point inside omega, but notice that omega excludes r, right. So, there is no chance that omega; that means, r dash and r can never be close to each other, because r dash is outside that sphere small sphere of radius epsilon, where r is located at the center, whereas, yeah r is at the center of that small sphere. So, r dash is outside that small sphere.

So, there is no chance that r dash and r will come very close because they will always be minimum distance epsilon from each other. So, if that is the case then del squared V is 0 because they will never come close to each other. So, del squared V is 0. But then del squared phi is of course, minus 4 pi rho because that is what we expect. In region omega, we expect del squared phi to be obeying Poisson equation, ok.

So, that is the bottom line. So, you see the this is 0 and this is your Poisson equation, right. So, this is what that is, ok. So, therefore, this left hand side will basically become this because this is 0 and this is minus 4 pi rho, ok. So, that is what that is. But then this will, so this will now split up into many portions.

One is the, so I told you this s. s is what? It is the surface of the boundary of this omega outside. So, omega outside has many many boundaries, they all in the shape of spheres.

At least they are oval shaped. This is a perfect sphere which is basically of radius epsilon, tiny sphere of radius epsilon which is very tiny, it tends to 0. But these are actual huge conductors. But these need not be spheres; they can be some irregular shape also. But bottom line is that you have all these boundaries.

So, you have these conductor boundaries which I have called s conductors which correspond to the shapes of the different conductors which have separate separated out like this. But then there is this s of epsilon, which is the boundary of the small sphere which is sitting with center at r.

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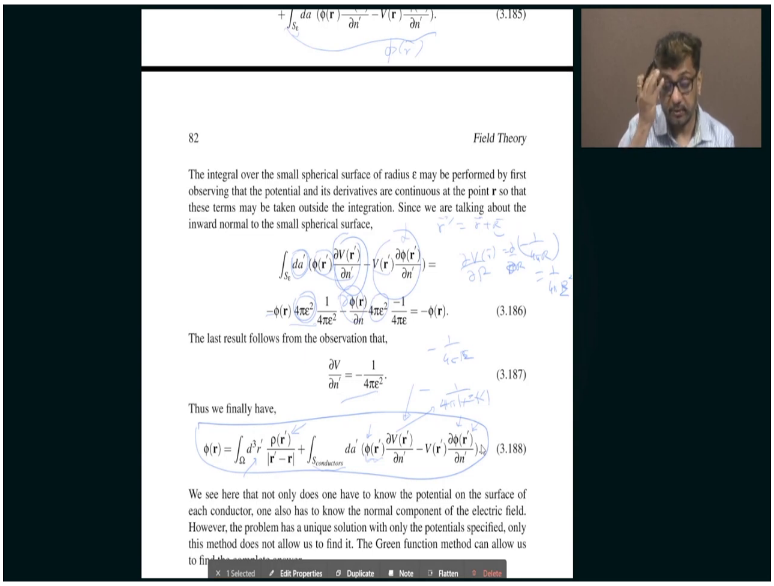

So, we can actually evaluate this at least. This is of course, difficult because you have to know what those conductors are. So, if no further information is given, you cannot proceed beyond this. So, this s conductors can be the basically is the you know boundary of all those lots of conductors which are sitting somewhere. So, you cannot simplify further if you do not know what they are. So, we leave that as it is, ok. So, this 3.185 s conductor’s integral, we cannot simplify further.

But the next one we can simplify further. And how do you simplify it? See, you simplified by noting that first of all you again as usual you make this thing that r dash, you write as r plus r, ok. So, and your r, r is basically this this r is on the surface of that small sphere of radius epsilon.

So, if that is a small sphere, then what is da? It is basically 4 pi epsilon naught, I mean 4 pi epsilon squared. Epsilon is your small radius, ok. So, epsilon is the radius of the small sphere. And area is basically; so, there is no question of integration because. So, you basically again use mean value theorem because phi is smooth. So, you approximate it by phi of r.

And then you see this; what is V? V is basically minus 1 by 4 pi r. So, the dV by dn is basically dV by dR. So, so if you do dV by dR, you will get plus 1 by 4 pi R squared, but R is epsilon R is the radius means, R is that thing that is sitting on the surface of the sphere vector. So, it is 1 by 4 pi epsilon squared because of this, and it is da is also 4 pi r squared, r means epsilon.

So, it is 4 pi epsilon squared and this is 1 by 4 pi epsilon square. So, and they will cancel out, ok. So, they will cancel out because and there is a minus sign because basically you are talking about the inward normal to the spherical surface because notice that the normal component is outs, look you have to you have to look at the outward normal to the volume, which is inward normal to this sphere, ok. The volume in question is this one.

So, the outward normal to the volume is inside the conductor like this, inside this sphere like this. So, the outward volume, outward normal to the volume that you are interested in volume is the intermediate spaces between the conductors and the sphere. So, the outward volume to that is the inward normal into the small sphere. So, because of that there is a minus sign, ok, right.

So, having done that you can easily convince yourself. And this other term is negligently small because this is of course, of some constant value. But whereas, this is minus 1 by, so basically this is minus 1 by 4 pi R, R is epsilon and, but then this da is 4 pi epsilon squared. So, epsilon squared by epsilon tends to 0. So, this term does not contribute as epsilon tends to 0, only this contributes.

So, bottom line is after all that effort this whole thing becomes phi of r, ok. So, that is the point. So, this becomes phi of r. And this one was already that. So, then you can take that out and then finally, you can write this. This is a very beautiful formula. So, what this says is that if you have a whole bunch of conductors and you have a whole bunch of charges described by this charge distribution, the potential at any point r is basically given by the usual coulomb potential due to the charges.

But it, there are also contributions from the charges sitting on the surfaces of the conductors which are not given explicitly, but if you know the so this V of course, continues to be keep in mind what that is. This is 4 pi x minus x dash with a minus sign, so that be.

But then if somebody tells, so this you have to integrate over the surface of the conductor. So, somebody has to tell you what the phi's are at the surface and somebody has to also tell you what are the shapes of the conductors. So, if somebody tells you the shapes of the conductor, so and you they tell you what is the potential, not only they tell you the potential on the surface of the conductor, they should also tell you the gradient.

So, that means, they have to tell you the normal component of the gradient. So, basically they have to tell you what is the electric field, the electric potential and the electric field on the surface because the electric potential is that, electric field is a derivative so, the function and its derivative, both have to be specified, ok, not just the function.

See the value of the function at some point does not tell you what the derivative of the function is at that point, right. So, you have to specify both. You have to specify the potential function on the surface and the gradient of the potential function also on the surface. So, knowing one does not imply knowing the other, both are independent, ok. So, having specified both then you can go ahead and find the.

So, this is a very general method called Green's function method, uses using Green's theorem. So, this Green's function because you see this V is the Green's function of the Laplacian which is what you get in electrostatics. So, this Green's function method is very powerful because it tells you the potential at any point when the whole bunch of charges rho of r sitting somewhere, whole bunch of conductors doing their own thing, and then you want to find the potential somewhere outside the conductors.

Inside the conductor is obvious because it is whatever that potential is, is if the potential on the surface of the conductor is some phi naught, inside also it is phi naught. So, that is not interesting. It is only outside all those conductors we have to know, right. So, that is given by this answer.

So, it is remarkable that such a general problem statement has such a closed answer, see. So, if this is a general problem statement, a whole bunch of conductors doing their own thing, whole bunch of charge distributions, charges sitting somewhere, doing their own thing. I want to find the potential somewhere. The answer is 3.188. So, remarkable that you can actually write down the answer like that.

But of course, the catch is that doing that integral over this conductor surfaces can be a pain because those conductors can be of you know some irregular shape and all that, ok.

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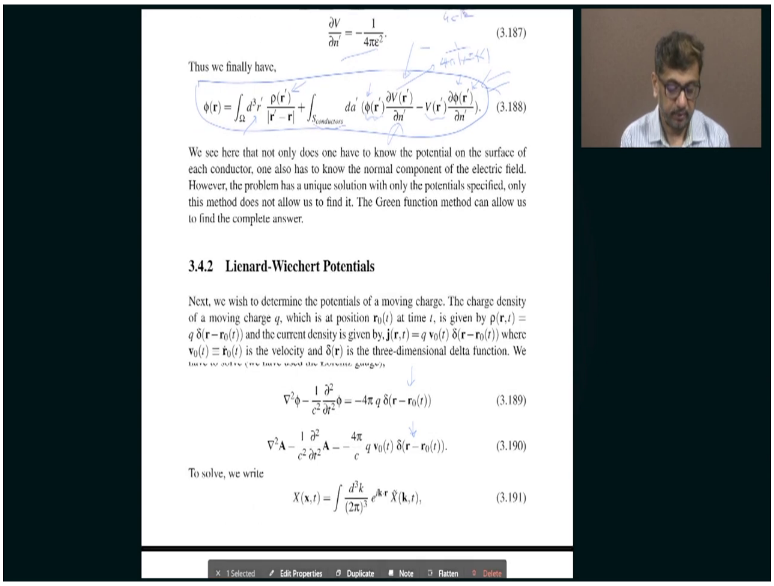

So, I am going to skip the next section which is basically the solution of the wave equation when you have point sources right, due to moving charges. So, that means, if you have a point electrical, electric charge that is moving in some arbitrary way. So, it could be relativisticly moving, means it can be moving close to speed of light and so on so forth.

So, being able to find the electric field produced by moving charge, moving in some complicated general way. So, that is an interesting problem, but it is also kind of a peculiar question which is of limited interest. It does not have a very general application. It is interesting for its own sake, not because it really leads to any larger insights.

So, as a result, I am not going to spend too much time on that. I am going to skip this all together. So, those of you are interested can look it up, ok. So, it is rather lengthy that derivation and all that.

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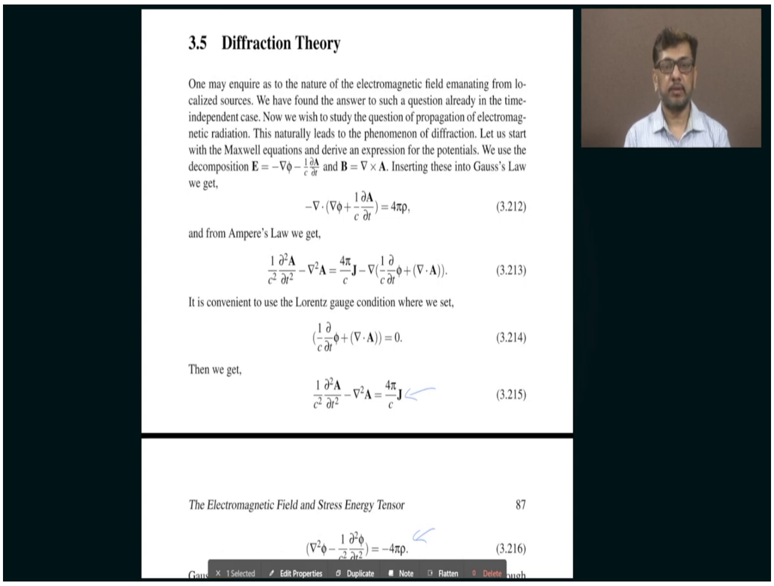

So, what is more generally applicable and important is basically diffraction theory. So, diffraction theory is very similar to this Green's function electrostatic problem, except that instead of Laplacian you were doing Dalembertian that is the wave equation. We are not going to be solving the Poisson equation which is basically the Laplace equation with a source.

See, what is Poisson equation? It is a Laplace equation with a source. So, similarly in diffraction theory what we are going to be solving is the wave equation with a source, ok. So, I want to do proper justice to this subject. So, I am going to stop here. And in the next class we will start discussing the rigorous theory of diffraction. So, diffraction of electromagnetic waves, specifically light, I mean what we normally think of diffraction of light.

So, you see bottom line is that in many optics textbooks, diffraction is presented from a historical view point where you know you have this Huygens experiment, Young's description, and so, there is then it leads up to Fraunhofer's theory and so on. So, there is all kinds of (Refer Time: 45:22) theory of, so there are whole bunch of historical developments which are presented and they all seem very haphazard and unrelated.

So, whereas, my treatment is going to be very reductionist, in the sense that I am going to think of diffraction as a natural and immediate consequence of electromagnetic wave propagation. Because that is of course, the distilled final answer to that question, the age old question of you know what is the nature of light, and how does it behave in the presence of matter and so on. So, this is the question that bothered the great thinkers of antiquity starting from Newton, and you know his contemporaries like Huygens and so on.

So, bottom line is that the electromagnetic description of diffraction theory is really the final answer. I mean this is the final word on the subject. Because that tells you the correct way in which a light has to be described, you know as a wave. So, I am going to stop here. And in the next class I will start Diffraction Theory, ok.

Thank you.