Dynamics of Classical and Quantum Fields: An Introduction

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Electromagnetic Fields
Lecture - 10
Stress-Energy (Energy-Momentum) Tensor

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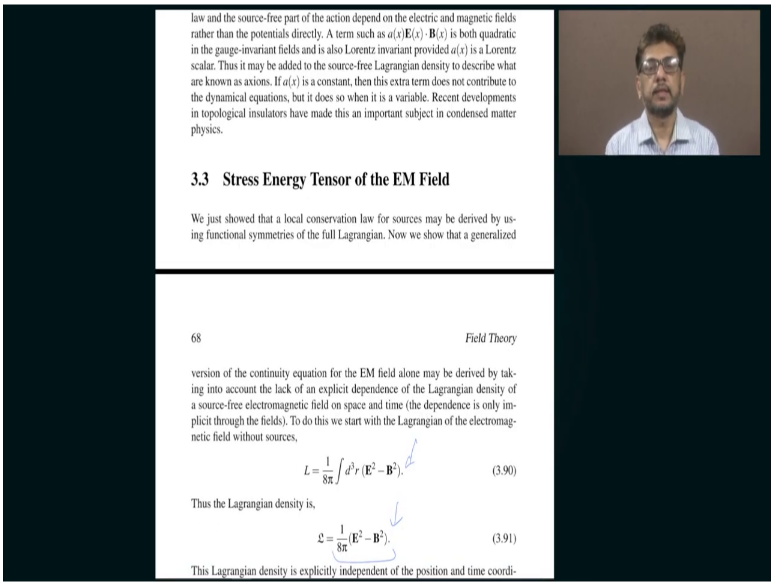

So in today's class I will be discussing something called the Energy Momentum Tensor or the Stress Energy Tensor of the electromagnetic field.

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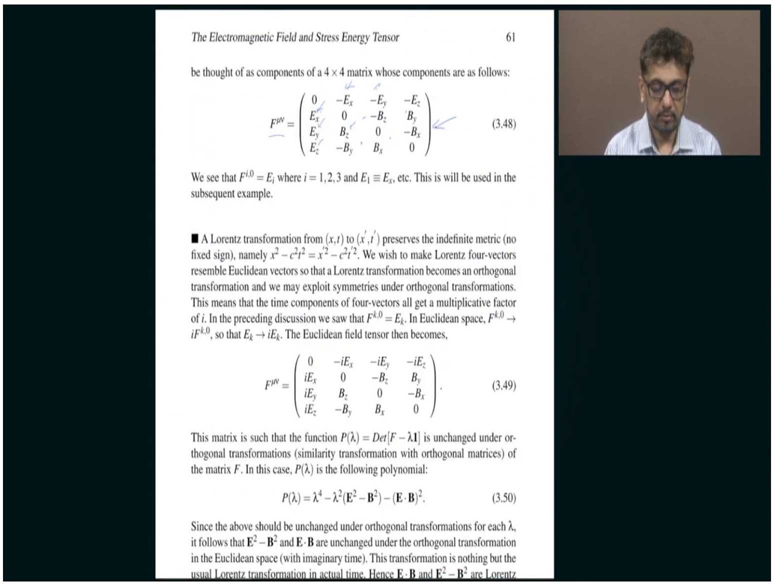

So, if you recall in the last class I had pointed out that there is a two component object which is called a tensor which is conserved quantity so; that means, its made of the components its an anti-symmetric tensor; that means, the diagonal elements are all 0 the off diagonal elements are the components of the three components of the electric and three components of the magnetic fields.

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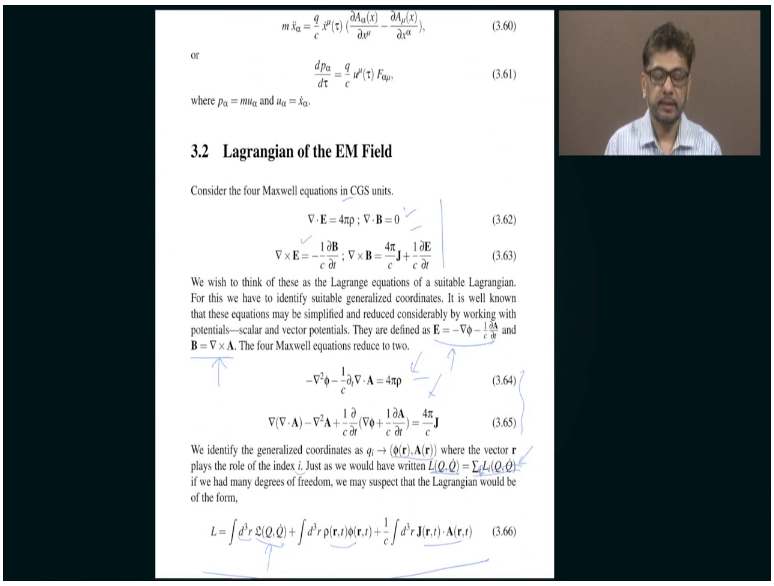

So, the point is that this is a conserved quantity in the sense that its 4 vector divergence is 0 with respect to any one of the indices.

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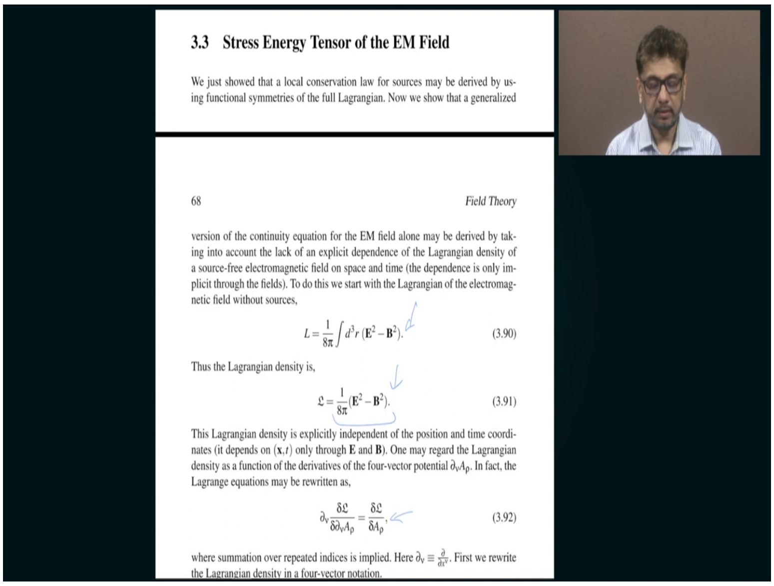

So, the bottom line is that what I want to do now, is I want to convince you that there is an conserved quantity in consistence consistent with Noether's theorem there is a conserved quantity which is also a two component tensor; that means, its basically a tensor of rank 2. So, the question is how do you show that?

So, to show that we start with this Lagrangian of the source less electromagnetic field; that means, imagine that there is a there is an electromagnetic field where there are no sources or that is possible and basically the Lagrangian is given by the integral of the Lagrangian density and the Lagrangian density is just the square of the electric minus square of the magnetic field. So; that means, the difference of the squares of the electric and magnetic field. So, keep in mind I am working in CGS unit. So, E and B have same dimensions.

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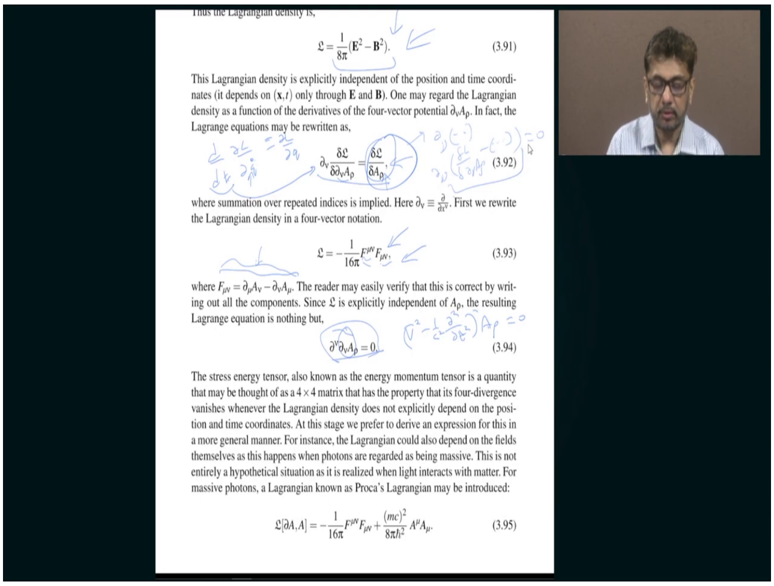

Thus the Lagrangian density is,

$$\mathfrak{L} = \frac{1}{8\pi}(\mathbf{E}^2 - \mathbf{B}^2). \quad (3.91)$$

This Lagrangian density is explicitly independent of the position and time coordinates (it depends on $(\mathbf{x}, t)$ only through $\mathbf{E}$ and $\mathbf{B}$). One may regard the Lagrangian density as a function of the derivatives of the four-vector potential $\partial_\nu A_\rho$. In fact, the Lagrange equations may be rewritten as,

$$\partial_\nu \frac{\delta \mathfrak{L}}{\delta \partial_\nu A_\rho} = \frac{\delta \mathfrak{L}}{\delta A_\rho}, \quad (3.92)$$

where summation over repeated indices is implied. Here $\partial_\nu \equiv \frac{\partial}{\partial x^\nu}$. First we rewrite the Lagrangian density in a four-vector notation.

$$\mathfrak{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}, \quad (3.93)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The reader may easily verify that this is correct by writing out all the components. Since $\mathfrak{L}$ is explicitly independent of A_ρ, the resulting Lagrange equation is nothing but,

$$\partial^\nu \partial_\nu A_\rho = 0. \quad (3.94)$$

The stress energy tensor, also known as the energy momentum tensor is a quantity that may be thought of as a 4×4 matrix that has the property that its four-divergence vanishes whenever the Lagrangian density does not explicitly depend on the position and time coordinates. At this stage we prefer to derive an expression for this in a more general manner. For instance, the Lagrangian could also depend on the fields themselves as this happens when photons are regarded as being massive. This is not entirely a hypothetical situation as it is realized when light interacts with matter. For massive photons, a Lagrangian known as Proca's Lagrangian may be introduced:

$$\mathfrak{L}[\partial A, A] = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{(mc)^2}{8\pi\hbar^2} A^\mu A_\mu. \quad (3.95)$$

Now, bottom line is that you can also express this Lagrangian in terms of the field tensor. So, this is what I had displayed earlier the 4 by 4 matrix and the anti-symmetric 4 by 4 matrix. Now, in terms of the field tensors you see the Lagrangian is purely a function of the derivatives of the potentials as it were. Now you can see that because. So, the Euler Lagrange equations therefore, can actually be even written as in the 4 vector notation like this.

So, think of the way we would have written it in the context of point particles we would have written it as d by d t of d l by d q dot equals d l by d q. So, basically this is taking on the role of del nu. So, it becomes generalized to include spatial coordinates as well and then this is del nu. So, you can prove that this is this because we see in special relativity space and time indices are on an equal footing. So, you cannot really make a distinction like that.

So, you should be able to accommodate a spatial as well as time indices. So, when you do that you get this, but then keep in mind that this F mu nu does not depend upon the vector potentials themselves it depends on the derivatives the derivative with respect to space time coordinates. So, because of that the for the electromagnetic field in empty space this the right hand side is always 0 ok.

So, now the left hand side you can convince yourself is basically this one. So, in our del L by del you know if you differentiate with respect to one of them. So, you will just get the other one and then because of anti-symmetry its this is the result. So, so in other words the four divergence of any component of the vector potential is 0. So, bottom line is that this is fine, but except that I would have preferred a more general situation where the right hand side is not 0; that means, I want to take into account a situation where the Lagrangian density is a function of the vector potential not only its derivatives.

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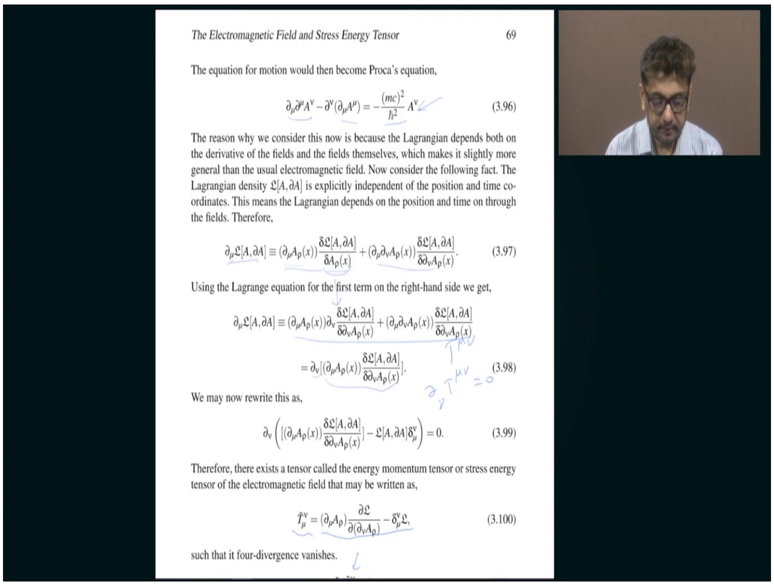

So, for that we have to introduce a vector potential dependent Lagrangian density. So, when you do that you get this relation and so, you will see that in addition to your usual terms like this you also get these terms ok. So, this is called Pocas Lagrangian ok. So, this is just to point out that more general situations are possible.

Now, I am going to show to you that the most general type of Lagrangian such as this which includes you know dependence on the vector potentials themselves not just the derivatives ok actually even the more general ones like this lead to conservation laws. So, the question is how do you show that? So, to show that you just what you do is basically you prove that this itself can be written as some 4 divergence.

So, as a result when you take this. So, if you can write this as del nu of something then I can take this to the other side and then this becomes del nu and del l by del nu a rho minus dot dot dot equals 0. So, this becomes your conserved quantity. So, that is what I am going to do now. So, let us first evaluate that the gradient or basically the derivative of the Lagrangian density with respect to one of the space time coordinates.

So, in that case it by chain rule you can show that this is equal to this. So, it becomes. So, you see the Lagrangian density is a function of the vector potential and its first derivative. So, the you know in the Lagrangian formalism the Lagrangian density is a function of the vector potential and its first derivative. So, because of that you keep in mind that in the Lagrangian formalism the coordinate and its you know time derivative are considered independent q and q dot are independent variables.

So, but then you see q dot. So, keep in mind that I told you that in special relativity dot means the time derivative. So, the time derivative has to be generalized to include the spatial components as well because in special relativity space and time gets mixed up by Lorentz transformation. So, there is no precise notion of time or space it is just space time put together as a precise notion.

So, as a result the spatial derivative of the Lagrangian density can be written using chain rule as follows and you see if you use Lagranges equation you can rewrite this term in this way. So, this is the Euler Lagrange equations. So, now, when you do that you see miraculously the right hand side of this just becomes the derivative or the gradient the 4 gradient of another quantity ok.

So, now bottom line is that this means that I can rewrite. So, I can basically rewrite this equation ok. So, I can also rewrite this equation in this way ok. So, if you expand this out you get this result. So, you can try this out.

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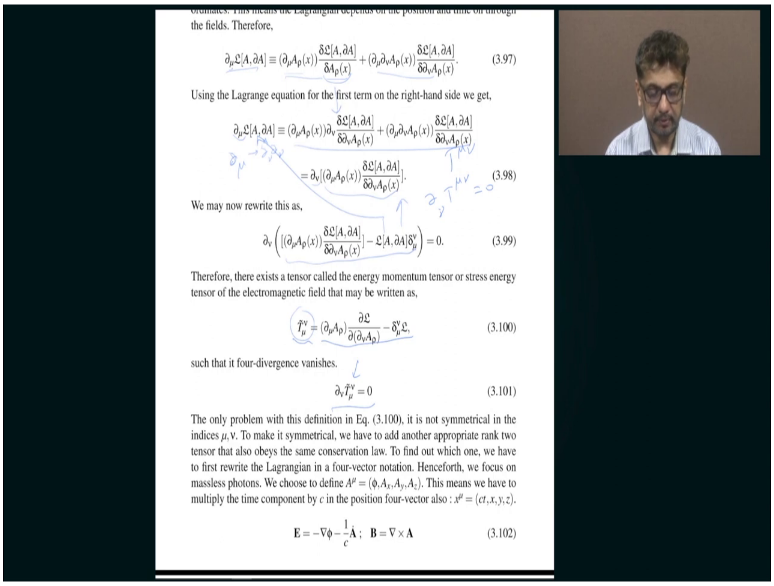

So, if I expand this out I get this result because that is what this is right. So, basically this I can rewrite as del mu. So, del mu I can rewrite as del nu delta mu nu ok. So, that is what I have done here. So, this term comes from here. So, when I do that and I put everything to one side I get this equals 0.

So, what this means is basically saying that there is a conserved quantity called T tilde which is now a two component object and its 4 divergence vanishes. So, the thing about the not so, nice thing about this is that its not symmetrical. So, we want it to be symmetrical because you know the anti-symmetrical portions do not convey much meaning they are just a burden.

So, you will we will probably show that and some of the exercises that by the anti-symmetric part does not convey much meaning. So, we want a symmetric version of this. So, what we do is that we add a term like this.

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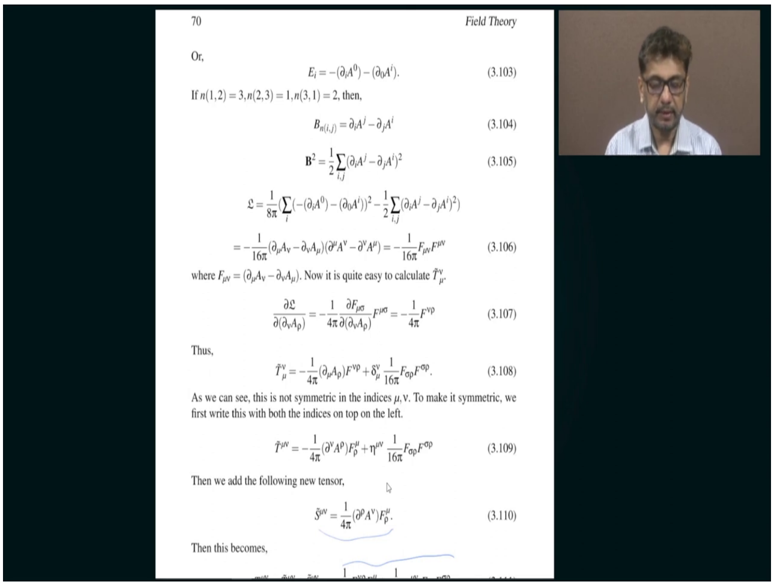

So, the rest of this is just making comparison with the traditional electromagnetic fields in the Maxwell kind of language rather than the 4 vector language. So, now what we do is that you see this T tilde is not symmetric; however, this S tilde can be added to this which makes it symmetric.

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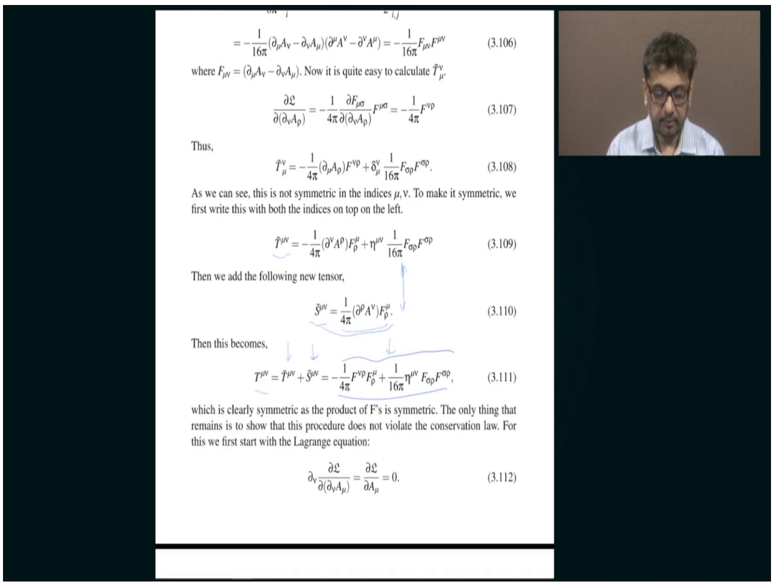

So, what we are going to do is that we are going to define a new object called T mu nu which is basically this T tilde which we derived by simply combining Euler Lagrange equation with you know chain rule. So, you see that is what we got T tilde and we showed that see we showed that its conserved, but then we can also add the S tilde; S tilde is basically a new two component object which when added to this T tilde still has to maintain conservation laws.

So; that means, we have to make sure that T mu nu is still conserved in the sense that its 4 divergence is 0. So, in other words because T tilde already is conserved we have to make sure that S tilde is also conserved, but then on top of that we have to choose a S tilde such that T tilde plus S tilde is symmetric. So, you see this choice ensures that it is symmetric. So, if you add this to this if you add these two you get T mu nu which is clearly symmetric ok.

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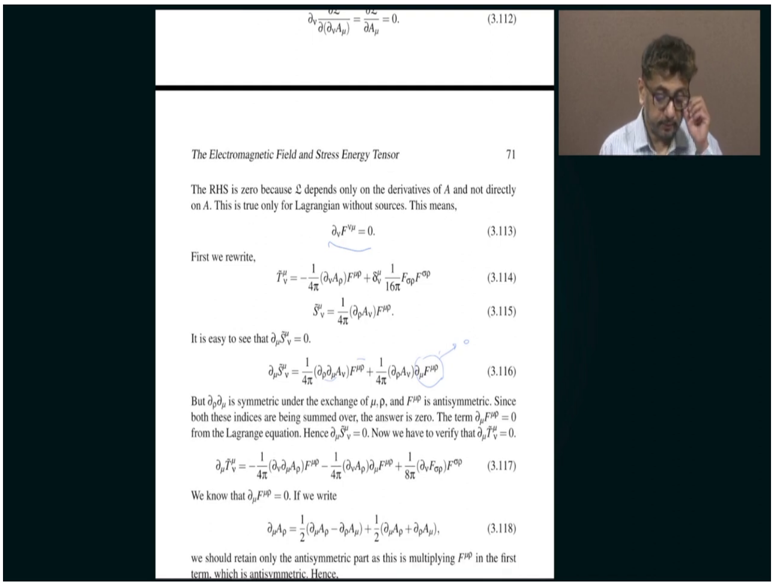

So, but then we have to show that S tilde is conserved. So, to show that S tilde is conserved you just take the derivative of S tilde with respect to say del mu and you will see that it is conserved. So, because these are mu F mu nu is anti-symmetric under interchange of mu and rho F mu rho is anti-symmetric under interchange of mu and rho, but then this is just the product of del rho and del mu which is clearly symmetric.

So, when you mix a anti symmetric with symmetric term you get 0 ok. So, similarly this is 0 because this is conserved F mu nu is conserved ok. So, as a result S tilde is clearly conserved.

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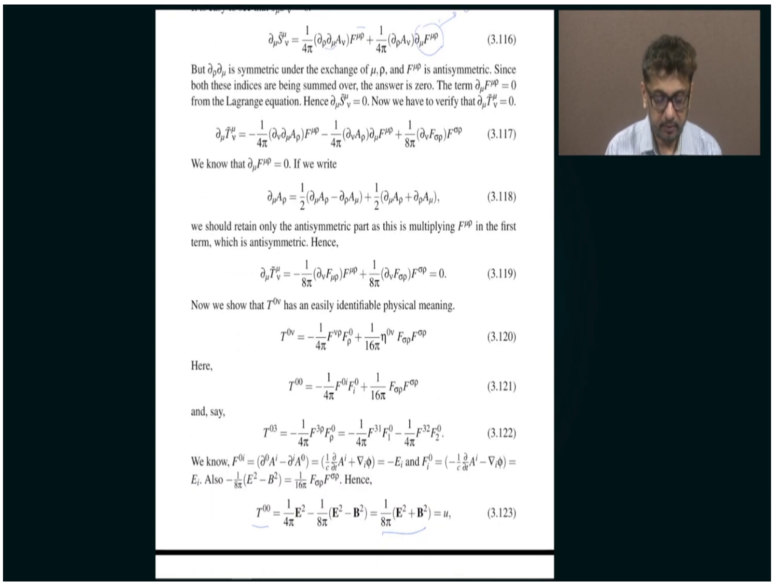

$$\partial_\mu \tilde{S}^\mu_\nu = \frac{1}{4\pi}(\partial_\rho \partial_\mu A_\nu)F^{\mu\rho} + \frac{1}{4\pi}(\partial_\rho A_\nu)\partial_\mu F^{\mu\rho} \quad (3.116)$$

But $\partial_\rho \partial_\mu$ is symmetric under the exchange of μ, ρ, and $F^{\mu\rho}$ is antisymmetric. Since both these indices are being summed over, the answer is zero. The term $\partial_\mu F^{\mu\rho} = 0$ from the Lagrange equation. Hence $\partial_\mu \tilde{S}^\mu_\nu = 0$. Now we have to verify that $\partial_\mu \tilde{T}^\mu_\nu = 0$.

$$\partial_\mu \tilde{T}^\mu_\nu = -\frac{1}{4\pi}(\partial_\nu \partial_\mu A_\rho)F^{\mu\rho} - \frac{1}{4\pi}(\partial_\nu A_\rho)\partial_\mu F^{\mu\rho} + \frac{1}{8\pi}(\partial_\nu F_{\sigma\rho})F^{\sigma\rho} \quad (3.117)$$

We know that $\partial_\mu F^{\mu\rho} = 0$. If we write

$$\partial_\mu A_\rho = \frac{1}{2}(\partial_\mu A_\rho - \partial_\rho A_\mu) + \frac{1}{2}(\partial_\mu A_\rho + \partial_\rho A_\mu), \quad (3.118)$$

we should retain only the antisymmetric part as this is multiplying $F^{\mu\rho}$ in the first term, which is antisymmetric. Hence,

$$\partial_\mu \tilde{T}^\mu_\nu = -\frac{1}{8\pi}(\partial_\nu F_{\mu\rho})F^{\mu\rho} + \frac{1}{8\pi}(\partial_\nu F_{\sigma\rho})F^{\sigma\rho} = 0. \quad (3.119)$$

Now we show that $T^{0\nu}$ has an easily identifiable physical meaning.

$$T^{0\nu} = -\frac{1}{4\pi}F^{\nu\rho}F^0_\rho + \frac{1}{16\pi}\eta^{0\nu} F_{\sigma\rho}F^{\sigma\rho} \quad (3.120)$$

Here,

$$T^{00} = -\frac{1}{4\pi}F^{0i}F^0_i + \frac{1}{16\pi} F_{\sigma\rho}F^{\sigma\rho} \quad (3.121)$$

and, say,

$$T^{03} = -\frac{1}{4\pi}F^{3\rho}F^0_\rho = -\frac{1}{4\pi}F^{31}F^0_1 - \frac{1}{4\pi}F^{32}F^0_2. \quad (3.122)$$

We know, $F^{0i} = (\partial^0 A^i - \partial^i A^0) = (\frac{1}{c}\frac{\partial}{\partial t}A^i + \nabla_i \phi) = -E_i$ and $F^0_i = (-\frac{1}{c}\frac{\partial}{\partial t}A^i - \nabla_i \phi) = E_i$. Also $-\frac{1}{8\pi}(E^2 - B^2) = \frac{1}{16\pi} F_{\sigma\rho}F^{\sigma\rho}$. Hence,

$$T^{00} = \frac{1}{4\pi}\mathbf{E}^2 - \frac{1}{8\pi}(\mathbf{E}^2 - \mathbf{B}^2) = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) = u, \quad (3.123)$$

So, we can construct a T mu nu which is conserved which has this property ok. So, in other words it is firstly, symmetric an secondly, its also conserved. So, now, we can go ahead and ask ourselves what is the actual meaning. So, we have constructed a symmetric object. So, it is completely unrelated to F mu nu clearly because F mu nu is anti-symmetric fully anti-symmetric where S T mu nu is symmetric.

So, that is the big difference the similarity is that both are conserved T mu nu is conserved F mu nu is conserved in the sense the 4 divergence of both are 0. But then these two are not related at all because one is anti-symmetric, the other is symmetric. So, the question is we know what is the physical meaning of F mu nu I just displayed that earlier its a 4 by 4 matrix with diagonal element 0 the other components of the matrix are basically the components of the electric and magnetic fields.

So, now, the question is in a similar way we want to understand what are the components of T mu nu. So, for that let us work out specifically T 0 0 ok. So, we can start by say T 0 nu and then we work out T 0 0, T 0 1 T 0 2 and that will be useful. So, you see when I try to work out T 0 nu its going to come out like this and specifically T 0 0 is going to be like this whereas, T 0 3 for example, is going to be like this, but then what is F 0 i.

So, i you see the Latin indices correspond to the spatial coordinates the Greek indices could be either spatial or time coordinates. So, it includes time if I write Latin indices like i j a h it implies that its only the spatial coordinates excluding time. So, if that is the case then F 0 i is clearly minus E i and the covariant version of that is plus E i. So, and also the this is by definition the Lagrangian density. So, it is basically minus 1 by 8 pi yeah. So, its with a minus sign. So, A squared minus B squared.

Now, if you combine these two. So, T 0 0 is this and you subtract this out you see you will get this result and this is the energy density of the electromagnetic field ok. So, that is the physical meaning of T 0 0.

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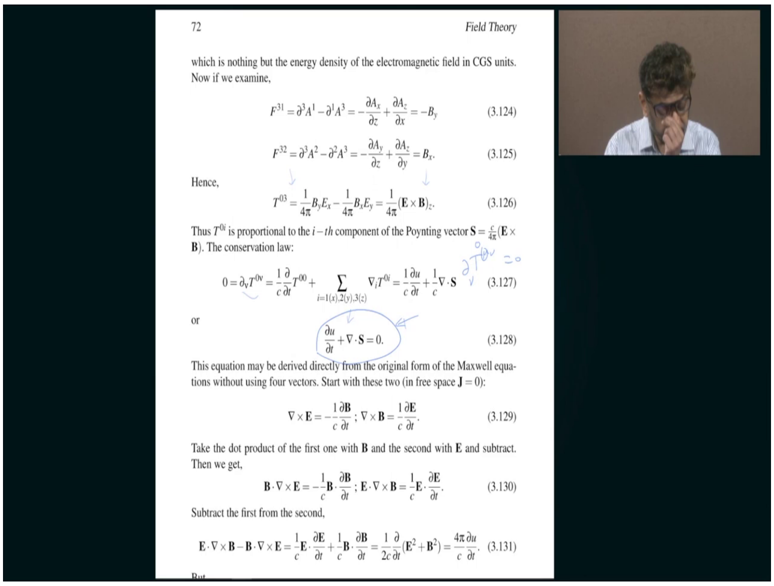

So, similarly you can work out what is the physical meaning of T 0 1, T 0 2, T 0 3, but specifically if you look at T 0 3, it comes out as the z component of E cross B. So, that is basically the z component of the pointing flux. So, the pointing vector. So, that you know from electromagnetic theory corresponds to the momentum carried by the electromagnetic field. So, the momentum density as it were. So, this is the energy density, this is the momentum carried by the electromagnetic field. So, bottom line is that you have these two ideas ok.

So, now, you can show that because you see if you look at some region of space, if energy in that region of space is increasing its because momentum is flowing into to that region or if the energy is decreasing it because momentum is flowing out of that region. So, that is the energy conservation. So, that the 4 divergence of the energy momentum tensor equals 0 implies that actually implies that conservation law. So, if you explicitly work this out what this means is basically this result this d u by d t plus divergence of S equals 0 ok.

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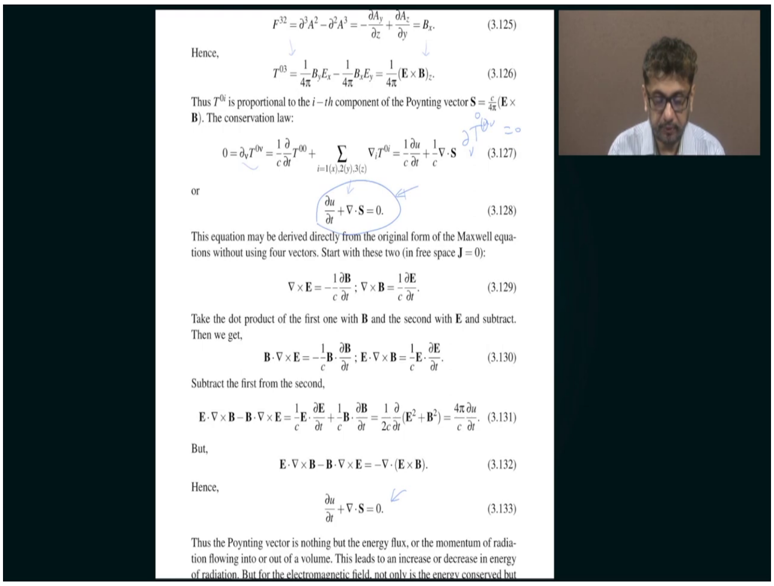

So, now this of course, could also have been derived directly from Maxwell's equations you know using just the vectorial notation which I am not going to go through. So, bottom line is that this is for T 0 nu. So, therefore, for T 0 nu there is a conservation law of this sort.

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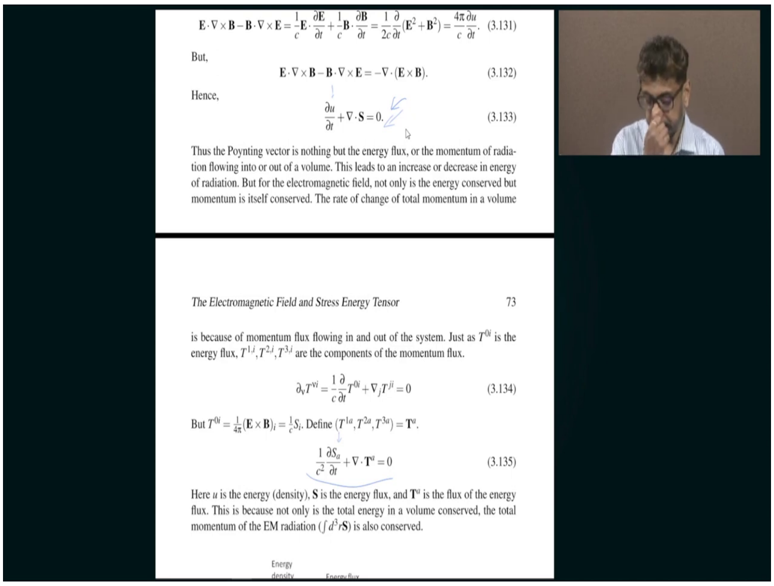

But then you could also do it for T T i nu that or T nu i basically is that you do not have to look at the zeroth component you can look at the other ones. Now if you look at the other ones also you get a similar conservation law ok. So, now, you see what was earlier u in the d u by d t now becomes the pointing vector in there.

So; that means, not only is energy conserved in a region. So, the momentum is also conserved the total momentum in a certain region is also conserved if the momentum S is a kind of energy flux. So, because from here you can see that S represents a kind of energy flux. So, now, if the energy flux is changing its due to some other kind of flux which is the flux of the energy flux. So, which is that meaning of the remaining components of the energy momentum tensor ok.

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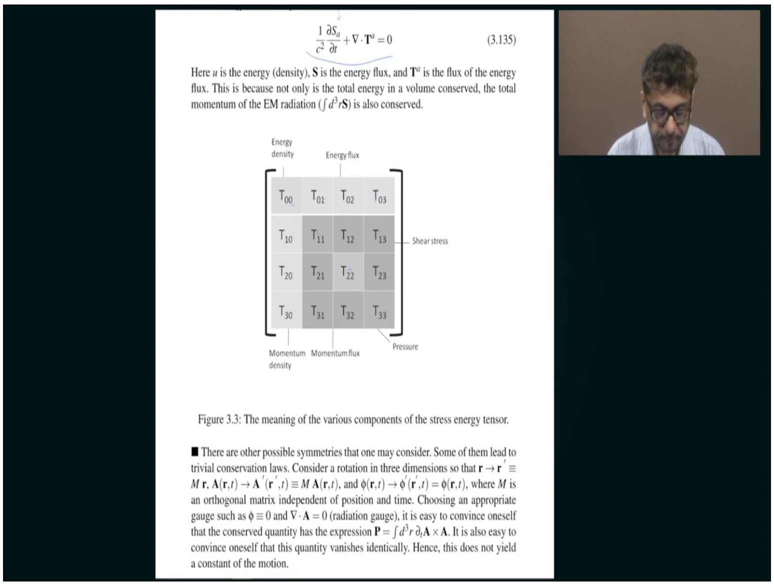

So, that is less intuitive and harder to visualize intuitively, but nevertheless you can still write down a matrix which is symmetric and its 4 by 4 and you can easily identify if not all many of the components. So, you see the first column corresponds to the energy density at the top left and then you have the momentum density in the remaining rows.

Then similarly the diagonal components from T 11 T 2 they correspond to radiation pressure ok. So, the remaining components are basically identifiable as shear stress and energy and momentum flux. So, bottom line is that put together all these components are lead to their appropriate conservation laws and basically put together are responsible for the energy momentum content of the electromagnetic field ok.

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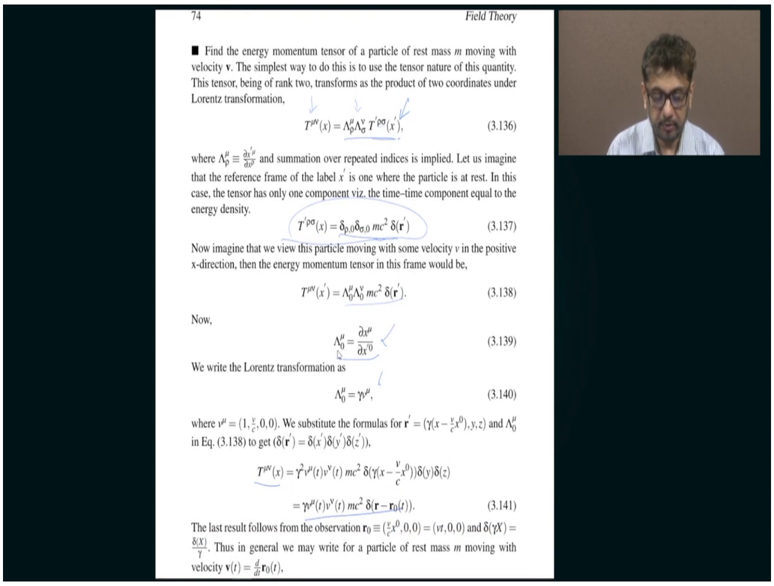

So, now I am going to show to you some examples that will convince you about this especially the 4 vector notations and tensor notation and so on so forth. So, I have given you some examples. So, let us start with some simple example like Noether's type of example. So, imagine that you have a vector potential A and you replace it by a transformed vector potential where you simply rotate that A by some amount ok, but that phi is a scalar. So, it does not get rotated.

Now, so, if you work in this gauge for example, where phi is 0 and divergence of A is 0 then you can easily convince yourself that this type of rotation leaves the Maxwell equations invariant. So, the Lagrangian invariant so; that means, there must be a conserved quantity and that conserved quantity is basically going to be this ok. So, you can convince yourself that is what it is ok.

So, bottom line is that yeah. So, this is a trivial example because even though it looks like. So, I have given you this example to convince yourself that there are many situations in which you get a conserved quantity which is actually trivial. So, if you work this out you will find that this appears to be a conserved quantity and indeed it is simply because its also identically 0 ok.

So, I gave you this example just to point out that not all conserved quantities are interesting for example, if you get a 0 as your answer its certainly conserved, but its not interesting ok. So, now, let us go to some really interesting examples. So, now, I want to find the energy momentum tensor for example, of a point particle. So, see if you have a point particle if its at rest, then its energy momentum tensor is clearly only energy there is no momentum nothing else certainly none of the other components are going to be there. So, its going to just be energy density at the location of that point.

So; that means, the. So, if T dash is the reference frame in which the particle is at rest clearly this is the energy momentum tensor. It is just m c squared which is the energy times the Dirac delta function at r dash which means assuming the particle is at the origin. So, m c squared Dirac delta at r dash is basically the energy momentum tensor of the point particle. So, now, the question is I want to find out what this energy momentum tensor is if you transform to a moving frame ok.

So, if you transform to a moving frame you get this t mu nu. So, now, you know that from your the fact that these are tensors under Lorentz transformation its going to transform like this ok. So, when it transforms like this its clear that since T dash is Kronecker delta of rho equals 0 and sigma equals 0 this is how its going to transform and we know what are these lambdas which are basically the matrix involving space time rotations which is basically the Lorentz transformation.

So, and we can clearly work this out as gamma into v mu. So, v mu is 1 v by c 0 0. So, that is for boosts along the x direction which is customary ok. So, if you work this out you will see that the t mu nu can be written like this ok. So, where its going to involve the 4 velocity of the particles in that new frame. So, the general frame and gamma is the dilatation factor and this is r naught is the location of the particle at any given time.

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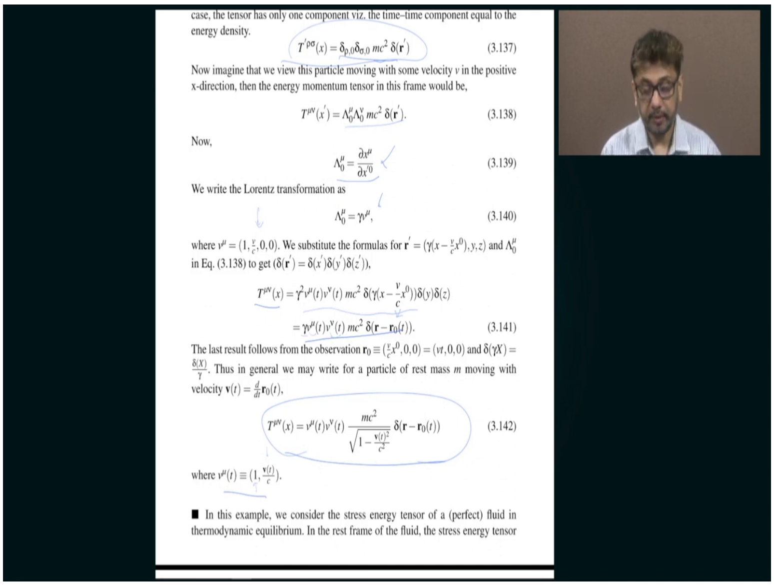

So, in general you can write this. So, this is what its going to be m c squared is the rest energy and this is the instantaneous velocity of the particle and this 4 vector velocity is just one for if that is time component and its v by c if it is space component right. So, this is your energy momentum tensor of a point particle which is quite interesting because its nice to know. So, you see in the case of point particle also the energy momentum tensor has all kinds of off diagonal components all those shear components and those which ones I displayed right.

So, the momentum flux and energy flux all kinds of off diagonal things are there in this. So, yeah. So, its not something we could have guessed. So, that is why we have to derive it ok we could not have guess this. We could have guess the T dash which is basically energy momentum tensor when the particle was at rest, but not when it was in motion ok.

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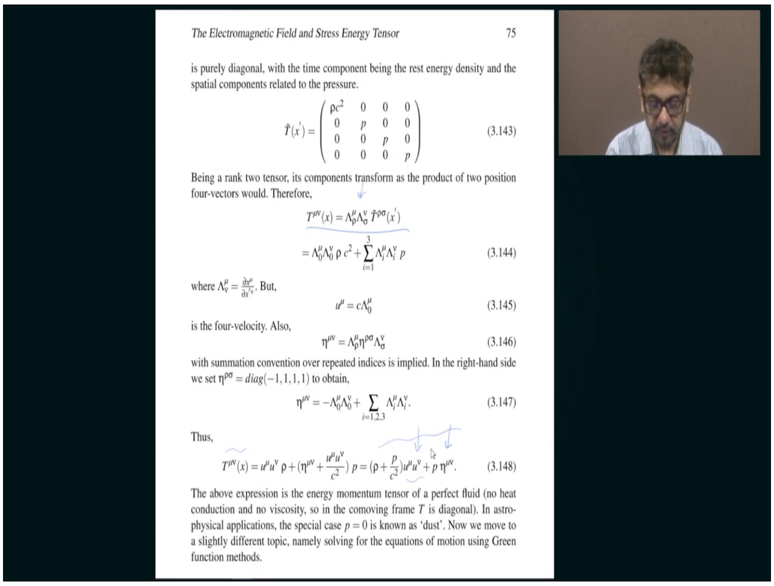

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is purely diagonal, with the time component being the rest energy density and the spatial components related to the pressure.

$$\tilde{T}(x') = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \quad (3.143)$$

Being a rank two tensor, its components transform as the product of two position four-vectors would. Therefore,

$$T^{\mu\nu}(x) = \Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}\,\tilde{T}^{\rho\sigma}(x') = \Lambda^{\mu}_{0}\Lambda^{\nu}_{0}\,\rho\,c^2 + \sum_{i=1}^{3}\Lambda^{\mu}_{i}\Lambda^{\nu}_{i}\,p \quad (3.144)$$

where $\Lambda^{\mu}_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}}$. But,

$$u^{\mu} = c\Lambda^{\mu}_{0} \quad (3.145)$$

is the four-velocity. Also,

$$\eta^{\mu\nu} = \Lambda^{\mu}_{\rho}\eta^{\rho\sigma}\Lambda^{\nu}_{\sigma} \quad (3.146)$$

with summation convention over repeated indices is implied. In the right-hand side we set $\eta^{\rho\sigma} = diag(-1,1,1,1)$ to obtain,

$$\eta^{\mu\nu} = -\Lambda^{\mu}_{0}\Lambda^{\nu}_{0} + \sum_{i=1,2,3}\Lambda^{\mu}_{i}\Lambda^{\nu}_{i}. \quad (3.147)$$

Thus,

$$T^{\mu\nu}(x) = u^{\mu}u^{\nu}\,\rho + (\eta^{\mu\nu} + \frac{u^{\mu}u^{\nu}}{c^2})\,p = (\rho + \frac{p}{c^2})u^{\mu}u^{\nu} + p\,\eta^{\mu\nu}. \quad (3.148)$$

The above expression is the energy momentum tensor of a perfect fluid (no heat conduction and no viscosity, so in the comoving frame T is diagonal). In astrophysical applications, the special case $p = 0$ is known as 'dust'. Now we move to a slightly different topic, namely solving for the equations of motion using Green function methods.

So, similarly we can asks a similar question what is the energy momentum tensor of a fluid? So, if a perfect fluid. So, imagine you have a perfect fluid in the rest frame you are in the rest frame of the perfect fluid in which case its energy moment. So, it has pressure it exerts pressure the fluid could exert pressure, but it also certainly has a mass density.

So, you have the energy density and the pressure. So, these are the components of the energy momentum tensor when the. So, when the fluid is at rest. Now the question is similarly suppose you are moving relative to the fluid what is the energy momentum tensor? So, as usual you do a Lorentz transformation and you can convince yourself that. So, the energy momentum tensor is actually given by the as usual the 4 velocities and the pressures and the densities ok. So, this is your Minkowski metric.

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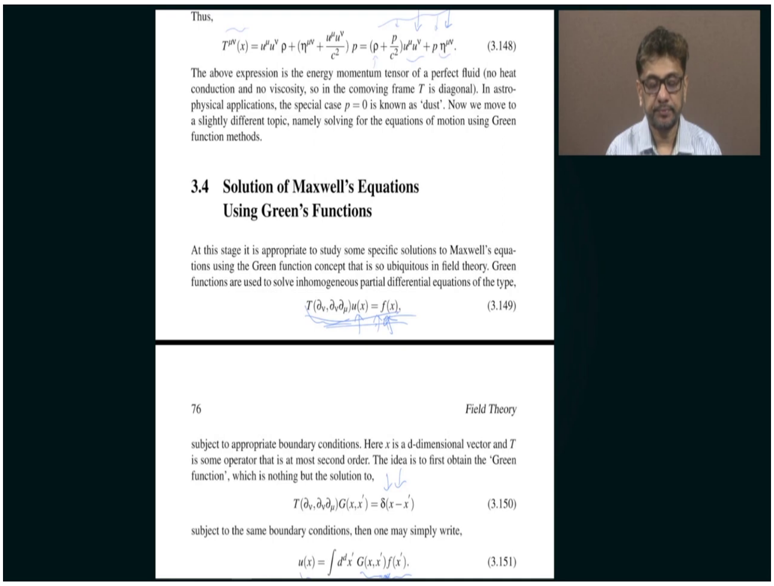

So, this is the energy momentum tensor of the perfect fluid. So, in the next class I will explain to you how to solve Maxwell's equation. So, this till now I have explained to you the content of the energy momentum tensor of the electromagnetic field. So, which is a nice concept. So, in the next class I will tell you specifically how to find the solutions of Maxwell's equation that is analogous to you know calculating the trajectory of a point particle see after all Maxwell's equations are basically Euler Lagrange equations of some suitable Lagrangian.

So, solving Euler Lagrange equation is basically finding trajectory if that is was a point particle. So, this is basically also like finding a trajectory except now that the coordinates of a point particle, but the they are actually fields themselves the coordinates are the fields themselves. So, the trajectory we are looking at is basically how the fields themselves change with time ok. So, that is what we are going to do in the next class. So, I hope you will join me for the next class.

Thank you.