Quantum Technology and Quantum Phenomena in Macroscopic Systems

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Lecture – 8
Problem Solving Session-2

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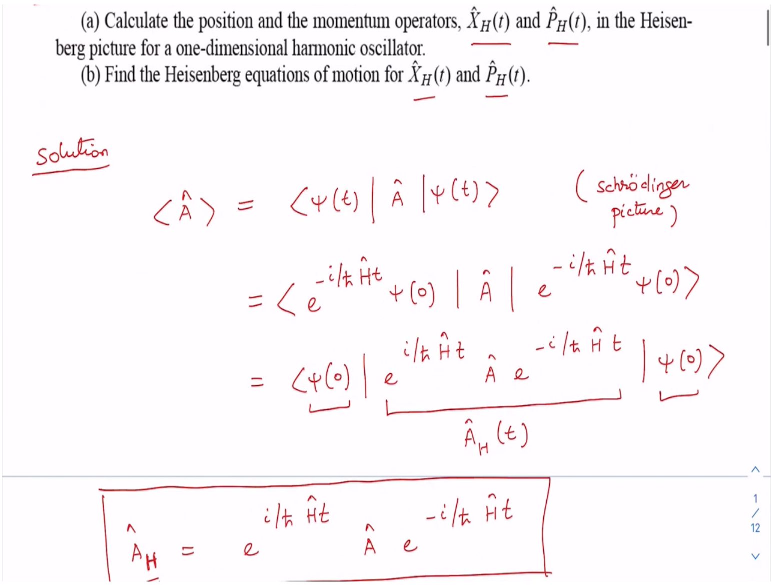

In this problem-solving session two, we are going to solve problems on two level atoms and the Heisenberg representation. As the first problem let us calculate the position and momentum operator X and P in the Heisenberg picture for a one-dimensional harmonic oscillator and in the second part, find the Heisenberg equation of motion for the operator X and momentum P. So, let us do it. But before I do it, let me remind you about the Heisenberg representation.

We know that the expectation value of an operator A with respect to a normalized state vector or wave function psi of t I can write it in this way. So, here psi of t is normalized. So, this is the expectation value that we calculate using the so-called Schrodinger picture. In the Schrodinger picture or representation, the wave function or the state vector is time dependent on the other operator has no time dependency.

This I can write because I know how this wave function evolves under this time evolution operator, that is e to the power minus i by h cross this is the Hamiltonian of the concerned System and this is your psi of 0, A and here you have e to the power minus i by h cross, H of t, psi of 0. This I can write as psi of 0 and I can have here e to the power plus i by h cross H of t A e to the power minus i by h cross H of t psi of 0.

Now you see if I define this as my new operator, where time dependency is now coming into the operator and the state vector or the wave function is now time independent right, now this is the so-called Heisenberg representation of operators. So, A H the operator in the Heisenberg picture is simply e to the power i by h cross H is the Hamiltonian here this H actually stands for Heisenberg.

This is for Heisenberg and you have here A, this is the Schrodinger operator which is or I can simply write it as A and here I have e to the power minus i by h cross H of t. So, with this background or recalling now we can do this solve this problem in this given problem the System is a one-dimensional harmonic oscillator.

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$$= \frac{1}{2m}\left[p, x\right] + 0$$

$$= \frac{1}{2m}\left\{ \hat{p}\,\underline{\underline{\left[\hat{p}, \hat{x}\right]}} + \underline{\underline{\left[\hat{p}, \hat{x}\right]}}\,\hat{p} \right\}$$

$$= \frac{1}{2m}\left\{ -i\hbar\,\hat{p} - i\hbar\,\hat{p} \right\}$$

$$= -\frac{i\hbar}{m}\,\hat{p}$$

So, the Hamiltonian for that one-dimensional harmonic oscillator we know is p square by twice m that is the kinetic energy plus the potential energy half m omega square x square m is the mass of the harmonic oscillator and omega is its angular frequency. So, this is the harmonic oscillator Hamiltonian one dimensional harmonic oscillator Hamiltonian. So, we have to find out what is the position operator in the Heisenberg picture that would be e to the power i by h cross H of t x e to the power minus i by h cross H of t.

Now to simplify this expression we can use the well-known formula Baker-Hausdorff formula e to the power lambda A B e to the power minus lambda A this you know that this would be B plus lambda commutation between A and B, lambda square by 2- factorial. Let me write it this side here, I have here lambda square by 2-factorial A commutation with the commutator A B plus lambda cube by 3-factorial will have A commutation with A, A B and you will have higher order terms in the similar fashion.

So, we have this Heisenberg operator for position would be then, we will have first term would be x then here I will have it as lambda is i t by h cross. I am just taking the Hamiltonian just you have to put the Hamiltonian there and e to the power i by h cross t let me take it as lambda. So, that is what I have here and this is the Hamiltonian H x commutation between H and x then the second term would be 1 by 2 factorial which is simply half, i t by h cross that is your lambda square and here you will have term like H commutation with H x here and so on.

So, let us first of all work out this term and then this term and so on. So, if I work out this commutation H x we will get let me write it the Hamiltonian is p square by twice m plus half m omega square x square and here I have x operator, this will give me 1 by 2 m p square x. Now because x square commutes with x. So, this term the commutation relation for the other one will give me simply 0.

Now this one I can write as 1 by 2 m I can write here it is p, p x plus commutation p x p and we know that p x is equal to minus i h cross, so we will have here 1 by 2 m, from here I will have minus i h cross p, similarly from the other I will have minus i h cross p this is going to give me minus i h cross. So, therefore I will have here it as minus i h cross by m p.

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$$= \hat{x}\left(1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!} - \dots\right) + \frac{\hat{p}}{m\omega}\left(\omega t - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} - \right)$$

$$\boxed{\hat{x}_H = \hat{x}\cos\omega t + \frac{1}{m\omega}\hat{p}\sin\omega t}$$

Similarly, $$\boxed{\hat{p}_H = \hat{p}\cos\omega t - m\omega\hat{x}\sin\omega t}$$

Now what about the other one, this one also let me quickly work it out H, H of x, this would be equal to Hamiltonian is p square by twice m plus half m omega square x square and this already we know this is minus i h cross by m p, again here you see this p square and this term and this term will commute. So, we have to take the commutation between these terms only. So, if I do that, I will have let me take minus i h cross by m from here this side and half m omega square also let me take it out.

So, I will have here x square p and if I do this you will get it as h cross square omega square x. I hope it is very simple or should I do it let me do it quickly. So, what you will have is minus i h cross by 2 omega square mm get cancelled again here I have x, x p plus commutation x p x and x p is equal to i h cross commutation of x p is i h cross. So, therefore I will have here minus i h cross omega square by 2 i h cross twice i h cross x cap and therefore I will have simply h cross square omega square x cap.

So, therefore I have this if I put all the terms here x H of t Heisenberg operator x would be x cap plus 1 by m p t, I think because i t by h cross is there and H of x is equal to this guy. So, therefore you will get the second term as this one and then you will have terms like 1 by 2 factorial t square omega square x and then you will have term like if you do it you will get omega t cube by 3 factorial 1 by m omega p cap.

And if you go further you will get omega t to the power 4 by 4 factorial x cap and so on and therefore, I can now write x cap if I take it common then I have 1 minus omega t square by 2 factorial plus omega t to the power 4 by 4 factorial and I will get a series like this and for if I take p by m omega common then I will get omega t minus omega t cube by 3 factorial plus omega t to the power 5 by 5 factorial and so on.

I think maybe you will get here a minus sign. So, you will get basically a series and you can recognize that the first term here this series is nothing but your cosine series. So, you can write it as x cap cos omega t and the other one you can write it as 1 by m omega p this one is sine series sine omega t all right. So, therefore the Heisenberg representation of the position operator is simply this one.

Similarly, exactly following the way I have done it. So, you can work out Heisenberg representation for the momentum operator and you will get it as p cos omega t, please do verify it yourself, minus m omega x cap sine omega t right. So, this is the Heisenberg representation for the momentum operator.

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$$= \frac{1}{m}\left(\underbrace{\hat{p}\cos\omega t - m\omega\hat{x}\sin\omega t}\right)$$

$$\Rightarrow \boxed{\frac{d\hat{x}_H}{dt} = \frac{1}{m}\hat{p}_H}$$

$$\text{Sly,} \quad \frac{d\hat{p}_H}{dt} = -m\omega^2\hat{x}_H(t)$$

Now let me go to the second part of the problem, here you are asked to find out the Heisenberg equation for the position operator and the momentum operator, that means you have to just work out dx H of dt and you know the Heisenberg equation of motion that would be one by i h cross x H and Hamiltonian here. Now this Hamiltonian is in the Schrodinger picture actually and therefore let me write x is the expression that just now we got, that would be 1 by i h cross here.

And here x H, this one we worked it as x cos omega t, there are other ways also to do this but let me do it this way. I have here p by m omega sine omega t and this commutes with, commutation we have to work out with this one, Hamiltonian is p square by twice m half m omega square x square, we have to just work out this commutation relation. So, this is going to lead us to let us do it you will get.

So, I have first term if I take let me take cos omega t out and here, I have x and p square by twice m I just have to take the cross term because I know x and x here this term commutes and p and p commutes. So, I have to deal with the cross term only and the other term that I have to deal is 1 by i h cross here, 1 by m omega sine omega t and you will have here p, half, I could have taken that also out but anyway, m omega square x square.

So, anyway if you do the mathematics very straight forward you can do it you will finally get it as 1 by m, please do the steps yourself you will get it as p cos omega t minus m omega x cap sine omega t, this is what you are going to get. So, by the way this is not let me put it in this way bracket. So, this is what and what this guy is this already we know that is nothing but the momentum operator in the Heisenberg picture.

So, therefore you have 1 by m p H. So, this is the Heisenberg equation of motion for position operator. And similarly, you can show that d p H of dt, Heisenberg equation for the momentum operator would be minus m omega square x H of t.

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$$\hat{S}_x(t) = e^{\lambda \hat{S}_z}\, \hat{S}_x\, e^{-\lambda \hat{S}_z}$$

$$= \hat{S}_x + \lambda\left[\hat{S}_z, \hat{S}_x\right] + \frac{\lambda^2}{2!}\left[\hat{S}_z, \left[\hat{S}_z, \hat{S}_x\right]\right] + \cdots$$

$$= \hat{S}_x + (i\hbar\lambda)\,\hat{S}_y - \frac{(i\hbar\lambda)^2}{2!}\,\hat{S}_x$$

Let us now work out this problem, the Hamiltonian due to the interaction of a particle of mass m charge q and spin S with a magnetic field pointing along the z axis is, H is equal to minus q B by m c into S z, S z is the z component of the spin vector S. Write the Heisenberg equation of motion for the time dependent spin operators S x, S y and S z. Let us do it, before we do this problem let me remind you some facts about the spin operator S.

And we know that we can write this spin operator or the spin vector S as in terms of the Pauli vector sigma h cross by 2 into sigma, sigma is the Pauli vector and component wise we have S x is equal to x component of the spin vector would be equal to h cross by 2 sigma x, sigma x is the Pauli matrix, x component of the Pauli matrix and you may recall that sigma x is equal to 0 1 1 0.

Similarly, you have S y is equal to h cross by 2 sigma y and sigma y is equal to 0 minus i i 0 and S z is equal to h cross by 2 sigma z and sigma z is equal to 1 0 0 -1. Also, you know the commutation relation between these matrices say sigma x sigma y, you will get 2i sigma z, sigma y sigma z will give you 2i sigma x and we will have, say sigma z sigma x will get 2i sigma y right.

So, using this you can immediately see that the commutation between S x and S y will give me i h cross S z commutation between S y and S z is going to give us i h cross S x. So, you can notice the cyclic order here and we have S z S x is equal to i h cross S y. Let me now come back to the problem because we are asked to find out time dependent of this operator S x of t that means we basically what is asked is the Heisenberg representation of the spin component of the vector S x.

Similarly for the other components, so this would be equal to e to the power i by h cross H of t S x e to the power minus i by h cross H of t, here this Hamiltonian is given as minus q B by m c S z. What I can do, I can write e to the power i by h cross H of t is equal to e to the power minus i by h cross q B by m c S z t, for simplicity purposes let me write it as e to the power lambda S z, where I am taking my lambda is equal to minus i by h cross q B by m c into t.

So therefore, exactly like the previous problem I can have S x of t is equal to e to the power lambda S z S x e to the power minus lambda S z. Now applying the formula that we utilized in the previous problem Baker-Hausdorff formula. We have S x plus lambda commutation between S x and I think it would be S z rather S z S x, then we'll have lambda square by 2 factorial S z commutation of S z and S x and so on.

If, we do it you will get S x plus S z S x commutation will give me i h cross lambda S y and here you will get it as minus i h cross lambda square by 2 factorial you will get it as S x and so on.

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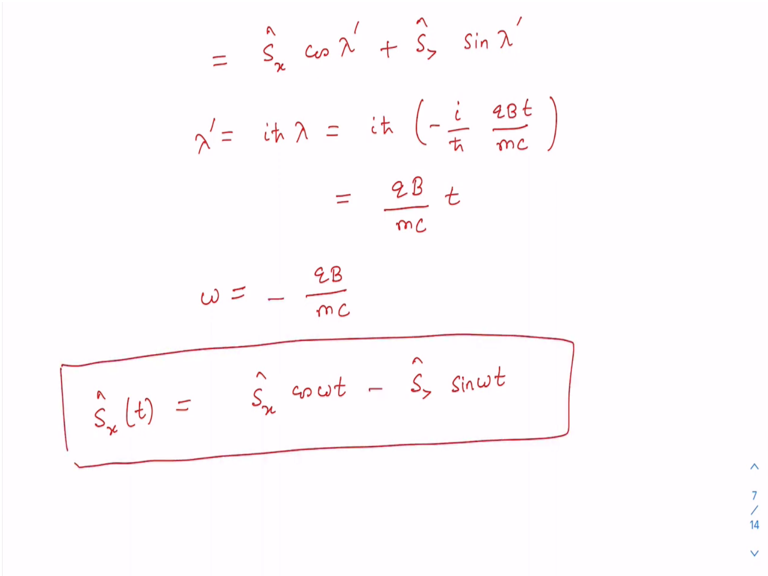

In fact, it is very easy to show that I can write if I take lambda dash is equal to is i h cross lambda then I can write S x of t is equal to S x if I take it common then I have 1 minus lambda dash square by 2 factorial plus lambda dash to the power 4 by 4 factorial and we will have S y would be lambda dash minus lambda dash cube by 3 factorial plus lambda dash to the power 5 by 5 factorial and so on.

So, this series is now well known to you this would be this one is your cos lambda dash and the other one would be sine lambda dash but lambda dash is equal to i h cross lambda which is i h cross lambda is equal to we wrote it as minus i by h cross q B t by m c so this is going to give me q B by m c into t, all right. So, but if I define my frequency omega as minus q B by m c, if I define it as angular frequency omega then we will have we can write S x of t is equal to S x cos omega t plus S y, I think you will have S y sine omega t.

So, this is what we are going to have hopefully I am doing it correctly please verify it yourself, there should not be any missing, let me take it plus then this is cos lambda I will get it like this. I think, yes, I think it is correct, no as per, okay let me define it as minus then I will have it as minus. This is what I am going to have.

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$$= \frac{qB}{mc} S_y(t)$$

$$\boxed{\frac{d\hat{S}_x(t)}{dt} = -\omega \hat{S}_y(t)} \qquad \omega = -\frac{qB}{mc}$$

Similarly, we can obtain:

$$\frac{d\hat{S}_y(t)}{dt} = \omega \hat{S}_x(t)$$

$$\frac{d\hat{S}_z(t)}{dt} = 0$$

Now what about the equation of motion. So, ds x of dt would be equal to the first term is going to give us S x omega sine omega t with minus sign and here I will get S y omega cos omega t. Therefore, I have minus omega S x sine omega t plus S y cos omega t and it will turn out that we can actually verify it later on. You can similarly work out what is S y, you will find out that you will this is nothing but your S y.

So, this is what we are going to get. Of course, you have to, to get this you have to similarly work out S y then only you can see that this is nothing but S y. But let me show you another method which may be more straightforward to work it out. We know the Heisenberg equation of motion for this operator. So, we have 1 by i h cross S x of t the commutation between the operator and the Hamiltonian I have here 1 by i h cross this S x of t I can write it as e to the power i by h cross H of t, S x of 0, e to the power minus i by h cross H of t and this Hamiltonian.

Now because this Hamiltonian commutes with this evolution operator or either way it would be minus i by h cross H of t H, it is very easy to see that this commutes and because of that I can write d S x of dt is equal to 1 by i h cross just look at this expression here I can write it as e to the power i by h cross H of t, S x of 0, this is the Schrodinger representation of the operator of spin component of, x component of the spin operator.

And here I have the Hamiltonian as minus q B by mc, it is S z, S z of 0, e to the power minus i by h cross H of t this I can write. Now, I have 1 by i h cross, let me take this out. So, I have minus q B by m c, e to the power i by h cross H of t, S x of 0, S z of 0, e to the power minus i by h cross H of t. Now commutation between S x and S z is, minus i h cross S y right. So, therefore I will have, let me write it properly, I have 1 by i h cross q B by m c, i h cross and I will have here e to the power i by h cross H of t, S y of 0, e to the power minus i by h cross H of t.

And what this is, this is nothing but the Heisenberg representation of the spin operator y component of the spin operator S y of t. So, I have here q B by m c S y of t. Now if I define my frequency omega as minus q B by m c, I will get it as minus omega S y of t. So, this is what I have d S x of d t is equal to this. So, I think this is more straightforward let me see what we got earlier yes this is what we got.

Similarly, please show we can obtain the Heisenberg equation of motion for y component of the same spin operator and if you do it you will you should get it as omega S x of t, on the other hand if you take for z component of the spin operator it will be 0. So, it will it is not going to evolve in time.

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$$\left.\begin{aligned} H_{11} &= C + D \\ H_{22} &= -C + D \end{aligned}\right\} \Rightarrow \quad D = \frac{H_{11} + H_{22}}{2}, \quad C = \frac{H_{11} - H_{22}}{2}$$

$$\left.\begin{aligned} H_{12} &= A - iB \\ H_{21} &= A + iB \end{aligned}\right\} \quad A = \frac{H_{12} + H_{21}}{2}, \quad B = \frac{H_{21} - H_{12}}{2i} = +\frac{i}{2}\left(H_{12} - H_{21}\right)$$

$$H = \frac{1}{2}\left\{ \left(H_{12} + H_{21}\right)\sigma_x + i\left(H_{12} - H_{21}\right)\sigma_y + \left(H_{11} - H_{22}\right)\sigma_z + \left(H_{11} + H_{22}\right) I \right.$$

Let us now work out this simple problem, a two-level atom has a Hamiltonian H. So, it is given in terms of a 2 by 2 matrix which has the component H 11, H 12, H 21 and H 22. Find the appropriate expansion coefficients to write this completely in terms of the three Pauli spin matrices plus the unit matrix. Let us do it, so we can write this Hamiltonian H 11, H 12, H 21, H 22 in terms of the Pauli matrices sigma x, sigma y, sigma z and identity matrix it would be 2 by 2 matrix here.

So, coefficients are, say A B C and D. So, this is what we have. Now if I know what is sigma x sigma y sigma z and sigma I. So, let me write it the first term would be sigma x is 0 1 1 0. So, I will have 0 A, A 0, sigma y I know 0, minus i, i, 0. So, here I have 0 I will have minus i B, i B, 0 and the third term would be c 0 0 -c and the last term would be because sigma is this identity matrix.

So, here I will have D 0 0 D. So, if I add all of them then I will get C + D, A – i B, A + i B and I will get -C +D. So, now if I compare it term by term then you will have H 11 would be equal to C + D, H 22 would be equal to -C+ D, H 12 would be equal to A – i B and H 21 would be equal to A + i B and from these two terms immediately you see that I will get D is equal to H 11 + H 22 divided by 2, I just have to add these two terms and then you will immediately get it.

Similarly, you will get C is equal to if you subtract them you will get H 11 - H 22 divided by 2, again from here you will get A is equal to if you sum them up you will get H 12 + H 21 divided by 2 and B you will get it as H 21 minus H 12 by 2i or I can also write it as minus i by 2, if I take it up there or if I take plus inside I will have H 12 – H 21. So, this is what I will get now if I put everything.

So, I will get H is equal to a half of if I take half out then I will get H 12 + H 21 sigma x + i H 12 - H 21 sigma y and I will have H 11 - H 22 sigma z and H 11 + H 22 sigma I that is identity matrix or if you are not comfortable with sigma you can simply write identity matrix I. So, this is the solution.

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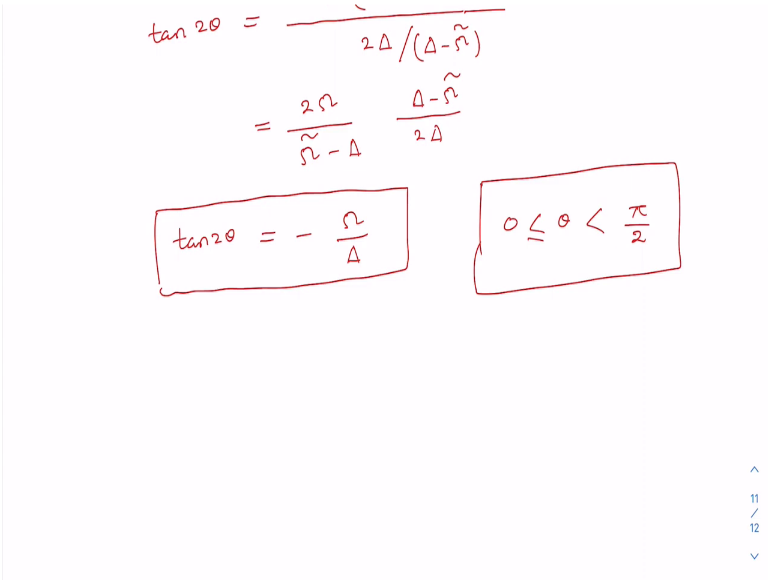

Let us now work out this problem, you have to prove this relation, in fact this relation we have obtained and we discussed in the context of dress state in lecture 5 and I asked you to show it, but here let me do it and explain the things little bit more clearly compared to the lecture. In the lecture if you recall we got this relation tan theta, theta is the Stueckelberg angle, tan theta is equal to omega that is the Rabi frequency divided by the omega tilde minus delta, omega tilde is the generalized Rabi frequency which is equal to square root of omega square plus delta square, delta is the detuning parameter.

Then I asked you to use this trigonometric relation to prove this relation here and tan 2 theta is equal to 2 tan theta divided by 1 minus tan square theta. Let me first work out what is 1 minus tan square theta. 1 minus tan square theta is equal to 1 minus tan theta is omega divided by omega tilde minus delta whole square, let me simplify it. I will have in the denominator omega tilde minus delta whole square.

Here I will have omega tilde minus delta whole square if I open it up, I will get omega tilde square plus delta square minus twice omega tilde delta minus omega square and I know that omega tilde square is omega square plus delta square then I have here delta square minus twice omega tilde delta minus omega square divided by omega tilde minus delta whole square.

So, from here I get 2 delta square minus twice delta omega tilde divided by omega tilde minus delta whole square and if I take 2 delta common I will get delta minus omega tilde divided by omega tilde minus delta whole square and from here you see that I will get because this is square I will get 2 delta divided by delta minus omega tilde. So, this is my 1 minus tan square theta.

Now let me work out what is tan 2 theta. Tan 2 theta is equal to 2 tan theta and tan theta is omega divided by omega tilde minus delta and 1 minus tan square theta already we worked out and that is 2 delta divided by delta minus omega tilde which I can write as twice omega divided by omega tilde minus delta and here I will have delta minus omega tilde divided by 2 delta.

So, this will lead me to minus omega by delta. So, I have proved it but let me make this limit of the angle that is this theta has to lie between 0 and pi by 2. Let me explain it little bit one minute. So, I will have it as this let me explain this limit what about this limit.

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case 2

$$\Delta \gg \Omega \Rightarrow \frac{\Omega}{\Delta} \ll 1$$

$$\tan\theta = \frac{\Omega}{\tilde{\Omega} - \Delta}\,; \qquad \tilde{\Omega} = \sqrt{\Omega^2 + \Delta^2} \rightarrow \tilde{\Omega} \approx \Delta$$

$$\approx \frac{\Omega}{\Delta - \Delta}$$

$$\rightarrow \infty$$

$$\Rightarrow \theta \approx \frac{\pi}{2}$$

$$\boxed{0 \leq \theta < \frac{\pi}{2}}$$

First of all recall that in terms of the Stueckelberg angle I can write my dress state ket plus as sine theta ket g plus cos theta ket e and minus ket as this plus ket plus and plus ket and minus ket are the dress states and here I will have here cos theta ket g minus sine theta ket e and also we have tan two theta already we know that is omega tilde omega by delta or tan theta is equal to we have it as omega by omega tilde by delta either of this expression is going to be useful when I am analyzing it.

Let me consider one case where this detuning parameter is much smaller than say minus omega, omega is the Rabi frequency and we assume that the Rabi frequency is positive say omega is greater than 0 by convention then let us see what we will get? We will get what about the angle if this is the case if delta the tuning parameter is much less than minus of omega.

Now tan theta is equal to already I wrote omega divided by omega tilde minus delta and because omega tilde is equal to omega square plus delta square. So, because I have your delta is much smaller than omega. So, I therefore can write that omega tilde is nearly equal to omega. So, I have here omega divided by omega minus delta which I can actually write as 1 divided by 1 minus or plus minus delta by omega now from here you can see that I have minus delta by omega is much greater than 1.

So, therefore because of the fact as minus delta by omega is much greater than one. So, I can consider that tan theta is approaching 0 that means that angle theta is nearly equal to 0. So, this is one of the limits of the angle that we have that it is, it is one bound. And another one, let me consider the other extreme in the second case, let us say I have this detuning parameter is much larger than the Rabi frequency.

So, these actually imply that omega divided by delta is much less than 1. We will see what it leads us to ten theta is equal to now I have omega now let me write omega tilde minus delta. So, omega tilde is equal to omega square plus delta square and because of this I can write omega tilde would be nearly equal to delta, right because it is much smaller than omega Rabi frequency is much smaller than the detuning parameter.

So, I have here omega divided by delta minus delta. So, it tends to infinity it means. So, this implies that my theta approaches the angle pi by 2. So, hence we have the angle is lying between 0 and pi by 2. So, this is the meaning of these bounds.