Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture-44
Quantum Optomechanics Squeezed States

Hello, welcome to lecture 33 of the course. In the last class we discussed normal mode splitting in the context of quantum optomechanics. In this lecture we are going to discuss squeezed states of mechanical oscillator, also we are going to conclude the course.

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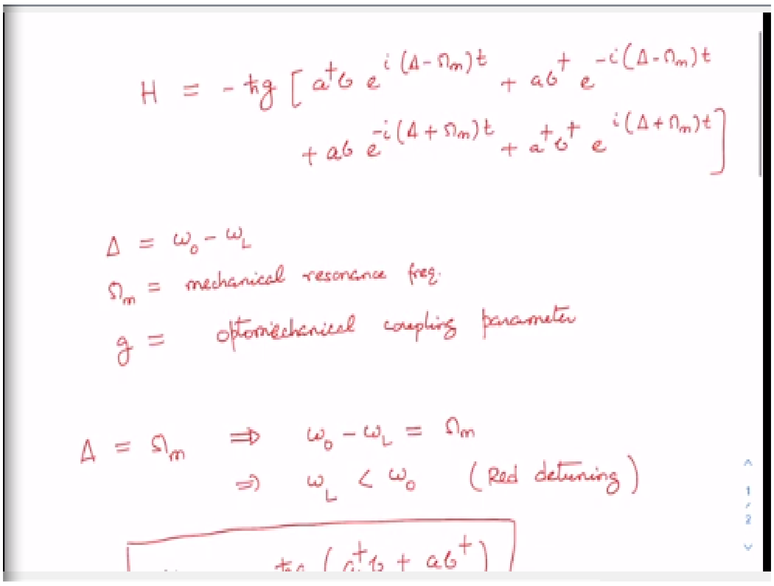

So, let us begin. In the last class we have written down the linearized Hamiltonian in the absence of noise and damping as follows that was - h cross g a dagger b e to the power delta - omega m t + ab dagger e to the power -i delta - omega m t + ab e to the power -i delta + omega m t + a dagger b dagger e to the power i delta + omega m t. And here let me remind you again that delta is the detuning parameter which is defined as the cavity resonance frequency minus the laser frequency.

Omega m is the mechanical resonance frequency, and g is the optomechanical coupling parameter for the linearized Hamiltonian. We discussed the situation when we have set delta = + omega m that means the detuning parameter we set at resonance frequency of the mechanical oscillator. This actually refers to the case that delta is greater than 0 or in fact if you analyze it because delta = omega 0 - omega L. So, this is now equal to omega m. So, you can clearly see that omega L is less than omega 0.

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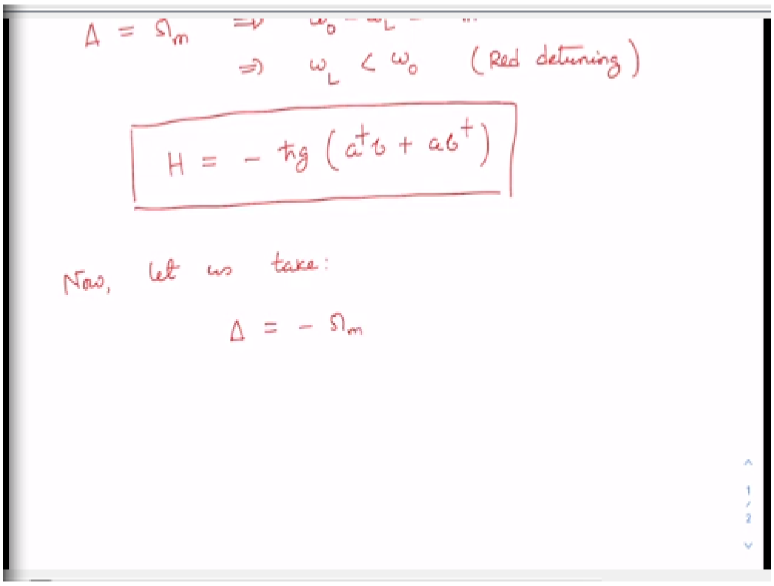

And this is the situation or condition or regime called the red detuning. This regime we have discussed in the last class and under rotating wave approximation in this regime we get our Hamiltonian as - h cross g a dagger b + ab dagger. Now as you know a refers to the annihilation operator corresponding to the optical mode and b refers to the annihilation operator corresponding to the mechanical mode.

So, this Hamiltonian physically state that one can transfer information from the optical mode to the mechanical mode and vice versa subject to appropriate conditions. Now let us discuss the other case, now let us take or let us set detuning parameter delta at - omega m.

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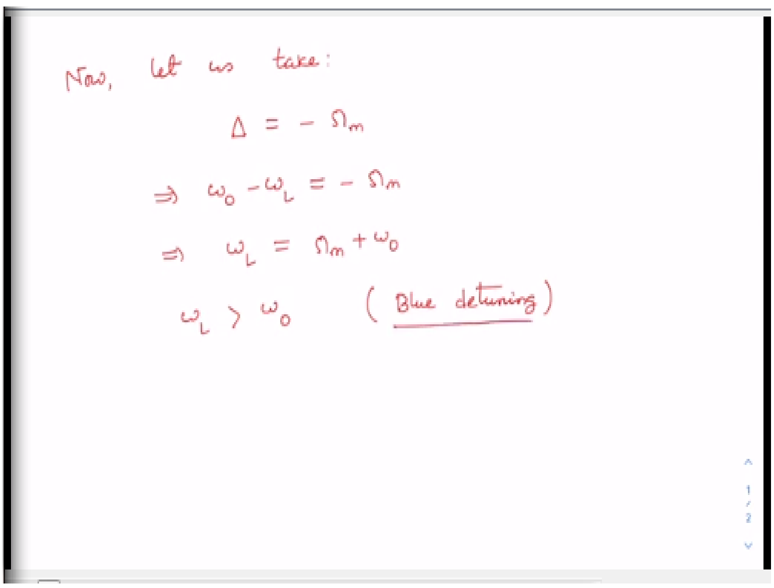

So, this will immediately tell you because delta = omega 0 - omega L and now this is equal to - omega m. So, you can clearly see that omega L the laser frequency = omega m + omega 0. So, that means your laser frequency is greater than the resonance frequency of the cavity and this is the regime of blue detuning. So, this is the regime of blue detuning.

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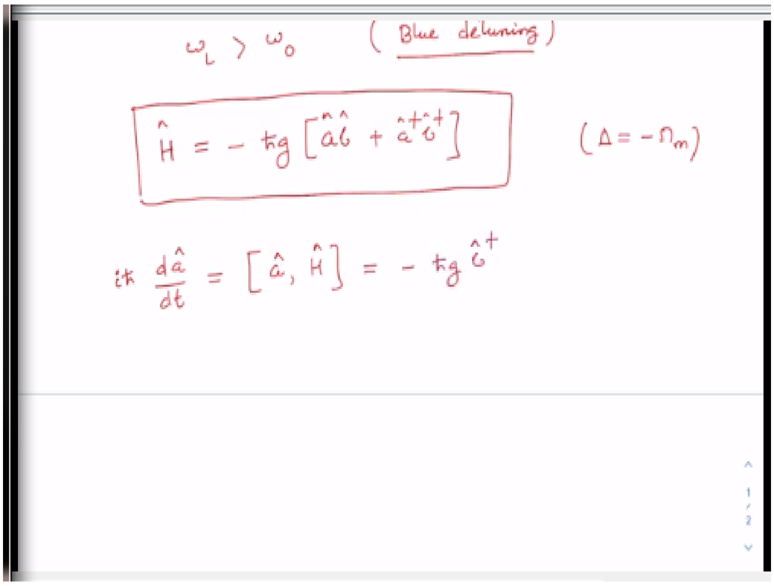

Now we are going to analyze the Hamiltonian in this regime and in this case the Hamiltonian after removing the highly oscillating terms that means I am talking about this Hamiltonian if I put delta = - omega m and then we can remove the highly oscillating term then our Hamiltonian will take this form that is - h cross g ab + a dagger b dagger. And this is what we get in the blue detuning regime or when I set delta = - omega m.

We are now going to analyze this Hamiltonian and to do that let us first write down the Heisenberg equation of motion for the mode a and b and that is easy to do, we have done it several times in the course. So, Heisenberg equation for the mode a ih cross del a del t and that would be equal to a commutation a H. Actually I am not putting the hat sign, then you will have here this will give you - h cross g b dagger.

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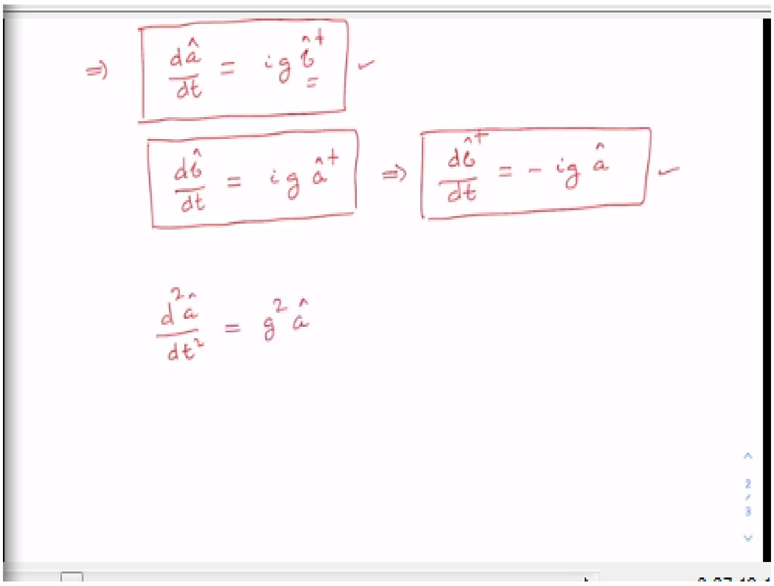

And from here therefore you can write da dt = i g b dagger and similarly you can get the other equation for the mode b that would be db dt = i g a dagger. Now let us solve this coupled equation and that is easy to do, what I will do? I can just take the help because here I have b dagger. So, from here I can write db dagger dt = - ig a and you know how to uncouple this set of equations. So, if I use this equation here and from here I can get the equation d 2 a the second order differential equation for a that would be d 2 a dt 2 = g square a.

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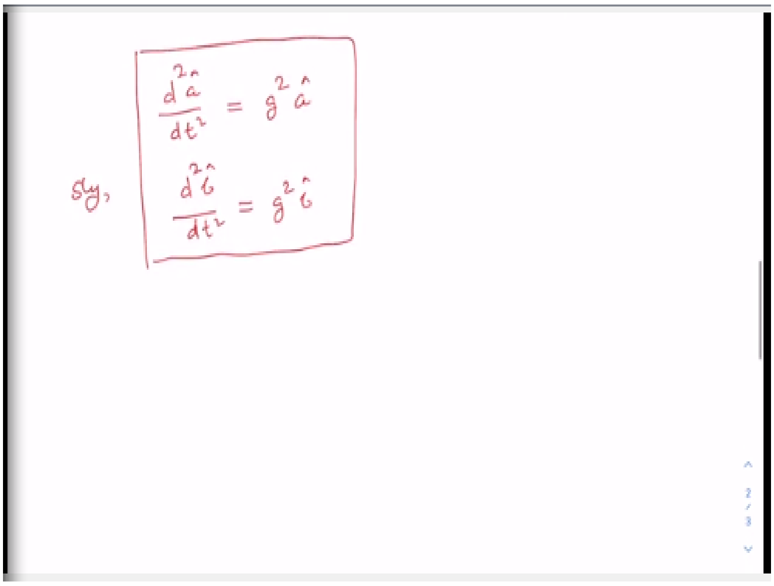

And similarly I can get the other equation for b that would be d 2 b dt 2 = g square b. So, the solution I can write as general solution for these 2 equations.

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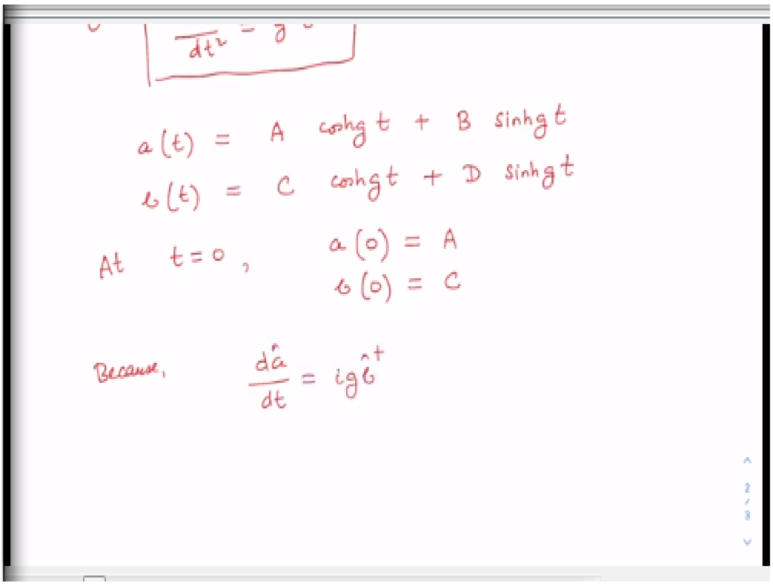

The first equation will have the solution a of t is equal to say a cos hyperbolic gt + B sine hyperbolic gt, I have to find out the constant a and b, we will do that from initial condition. Similarly for b of t I can write C cos hyperbolic gt + D sine hyperbolic gt. Now let us take at t = 0, say you can immediately get it that that would be a of 0 would be A and b of 0 would be C. And because we have da dt = igb dagger, this equation have already got.

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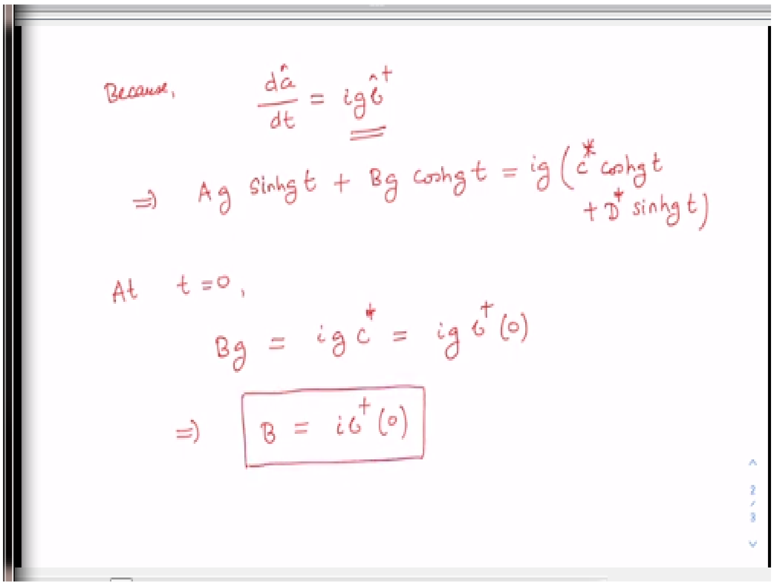

And from this now we have already taken a da dt if I take, so I will get Ag sine hyperbolic gt + Bg cos hyperbolic gt. These are very trivial algebra as you can see then I have ig b dagger I also have B here. So, A and B I have here, so b dagger would be you will have c dagger or rather I will say c is a complex quantity, let me write it like this and I have cos hyperbolic gt and then I have complex conjugate of D, then sine hyperbolic gt.

Now at t = 0 I have Bg = ig c star that would be equal to ig it would be b dagger 0. So, from this equation, so therefore I have my constant B = ib dagger 0. So, I got my constant B, I got constant A and constant C and you can find out in a similar way the constant D as well.

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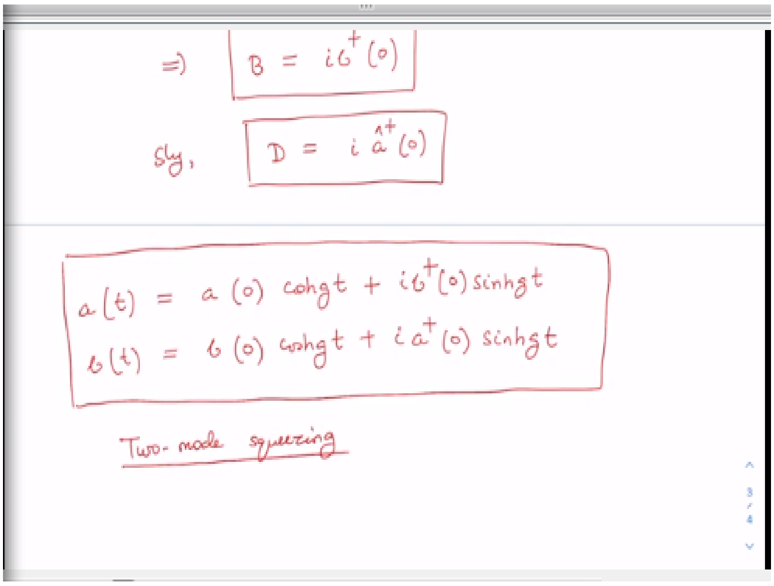

So, D would turn out to be i a dagger 0. So, I got all my 4 constants. So, therefore I can write the full solution as follows. So, a of t would be equal to a of 0 cos hyperbolic gt + ib dagger 0 sine hyperbolic gt and b of t would be equal to b of 0 cos hyperbolic gt + i a dagger 0 sine hyperbolic gt. These 2 equations lead us to a phenomena called 2 mode squeezing, as the name suggests 2 mode squeezing, however it does not mean that individual modes are getting squeezed.

Rather you will find that their relative coordinates or normal coordinates are getting squeezed or anti-squeezed. Two mode squeezing is relatively a difficult concept and let me explain it a little bit.

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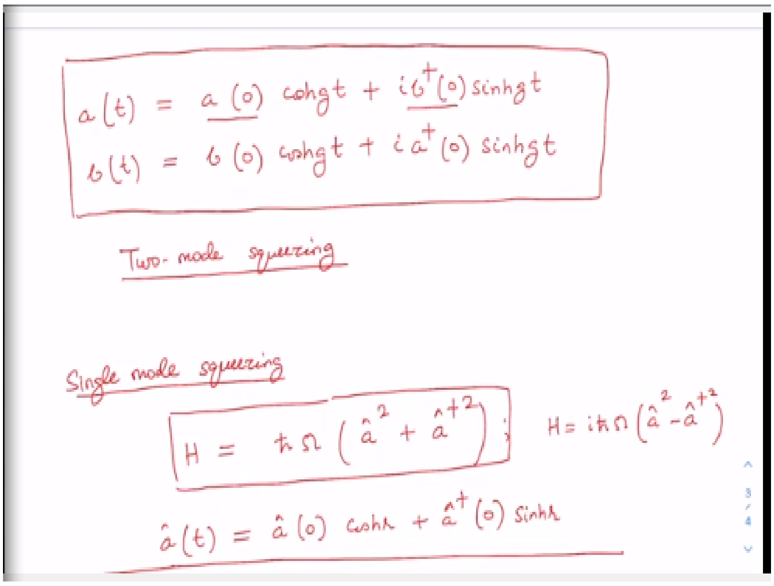

In module 1 you may recall that in the case of single mode squeezing we discussed this particular Hamiltonian or a Hamiltonian of this form which had terms like say a square + a dagger square or maybe you have seen the Hamiltonian of this type where we wrote it as ih cross omega a square - a dagger square, both the Hamiltonians are equivalent because it has to be Hermitian, so either of the form is okay.

So, let us say we have this Hamiltonian. Then if you write down the Heisenberg equation of motion and solve it you get an equation of motion then you can show that the solution is going to be a of t = a of 0 cos hyperbolic r, where r is the squeezing parameter, a dagger 0 sine hyperbolic r. We discovered that this eventually get squeezing in one quadrature of the annihilation operator a at the cost of amplification of the other quadrature.

Now in the present case the scenario is little bit different because if you look at the solution here; as you can see from here that unlike the single mode case here the solutions involve both the oscillators. So, you have here a of 0 is there and as well as contribution from the other oscillator is also there. So, individual modes are not getting squeezed or amplified but something else is getting squeezed.

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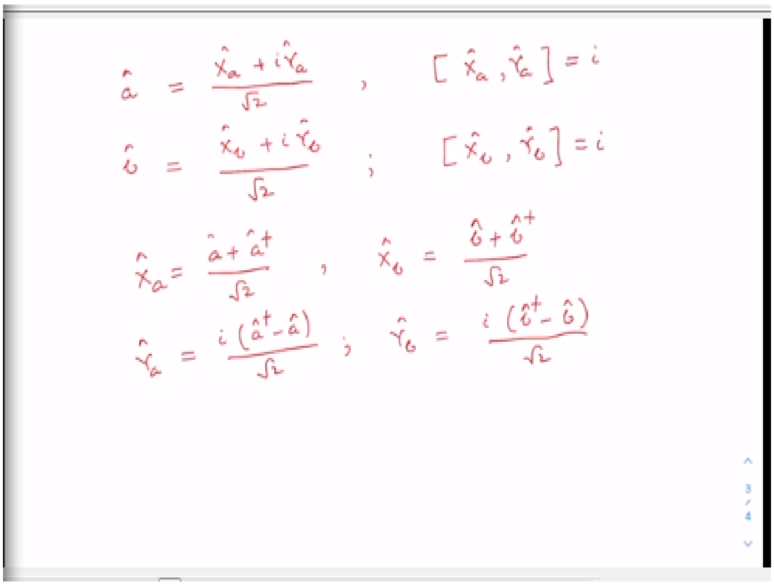

In lecture 32 if you recall we have written down this annihilation operator a and b in terms of their quadrature as follows we wrote a = X a + iY a divided by root 2 and b operator we wrote as X b + i Y b divided by root 2. Such that with the condition that X a, Y a satisfy the commutation relation and X b, Y b satisfy this commutation relation. So, that ultimately a a dagger = 1 and b b dagger = 1.

From it we can write X a = a + a dagger by root 2, all these are operators and X b = b + b dagger by root 2; while Y a quadrature Y a = i into a dagger - a by root 2 and Y b = i into b - b by root 2. In terms of this X, Y quadratures of the 2 oscillators we define the normal coordinates of the combine oscillator as follows.

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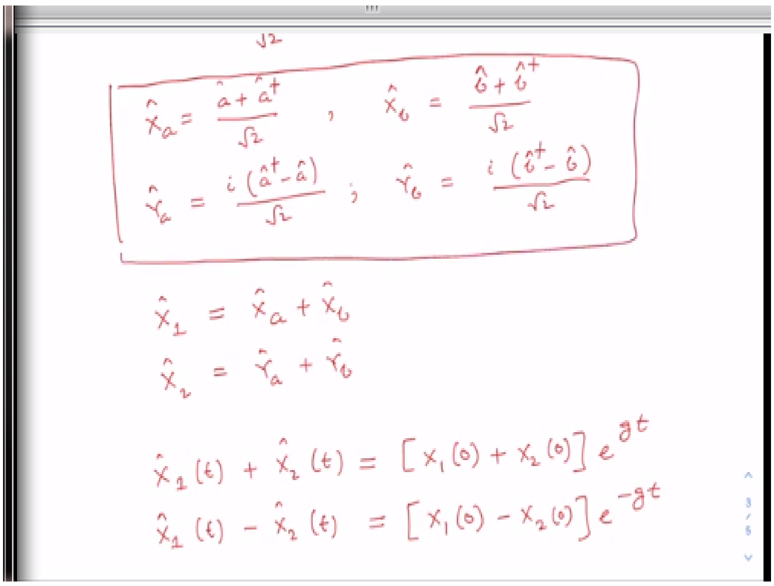

We have defined normal coordinate of the 2 combine oscillators X 1 = X a + X b and X 2 = Y a + Y b. Now it is a trivial exercise, very easy algebra you can do. To show that the sum and difference of these 2 normal coordinates say X 1, the sum X 1 of t + X 2 of t or is difference X 1 of t – X 2 of t, this sum and difference of this normal coordinates, the sum gets actually amplified. You can show, you can prove it very easily X 1 of 0 + X 2 of 0 e to the power gt, so the sum of the normal coordinates are getting amplified.

While the difference in the normal coordinate that is getting squeezed, so you can prove this very trivially. Let me just give you an idea how to do that? You just need to do a bit of straightforward algebra. Because you have X 1 of t + X 2 of t, for example you can write it as X a + X b + Y a + Y b as per our definition.

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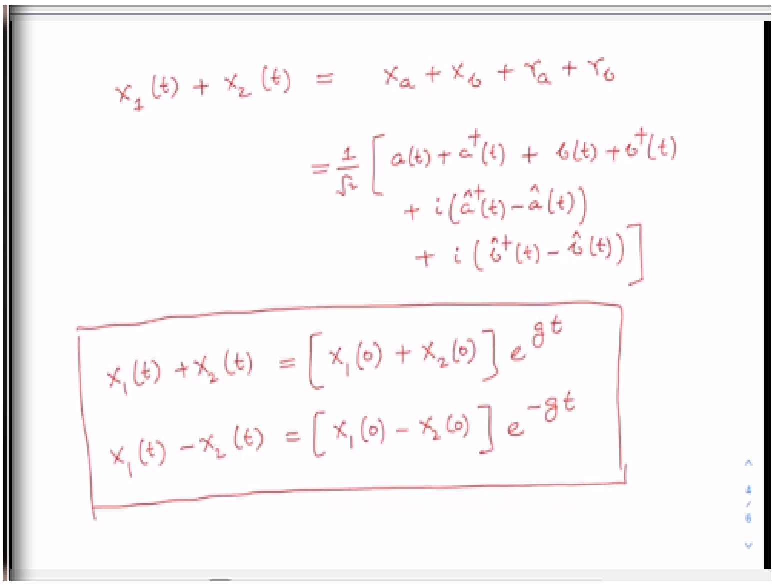

And then X a we can write in terms of already we have this here all this we can write in terms of their corresponding creation and annihilation operators. We just have to put it there, so you will get 1 by root 2 X a would be a of t + a dagger of t and X b = b of t + b dagger of t, divided by root 2 that anyway I am taking it out and Y a = i into a dagger of t - a of t and Y b is i, b dagger of t - b of t.

Now we know the solution a of t and b of t and you can also get the corresponding a dagger of t and b dagger of t, you just need to put these solutions what I am talking about is these solutions, you just have to put it there, do the bit of straight forward algebra. Then you will be able to finally show that X 1 of t + X 2 of t, this sum of the normal coordinates as time goes on these coordinates get amplified, you will get this.

And for the other case the difference and would be equal to X 1 of 0 - X 2 of 0 e to the power - gt. And this is what we mean by 2 modes squeezing when we are discussing squeezing of the combined oscillator, the optical oscillator and the mechanical oscillator. Let me point out some notable features of 2 mode squeeze states quickly.

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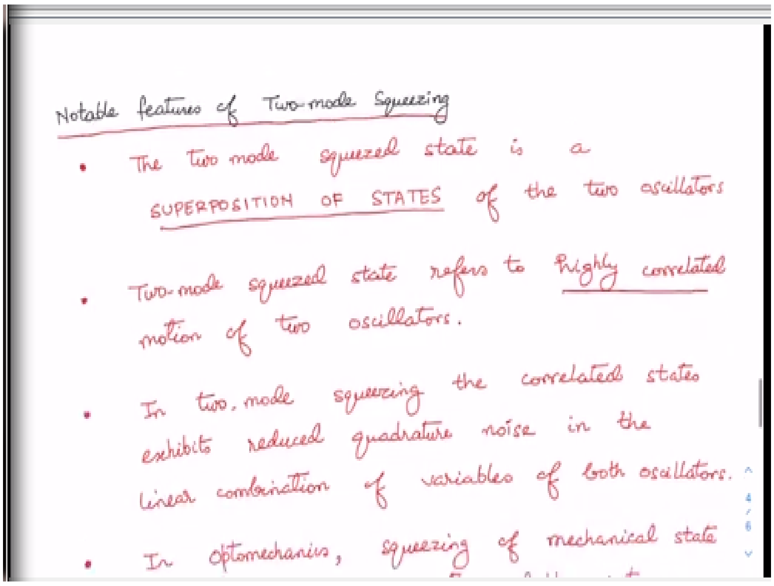

Two-mode squeeze state is a superposition of states of the 2 oscillators. Here we are having the mechanical oscillator and the optical oscillator in optomechanics. Two-mode squeeze states refer to a highly correlated motion of the 2 oscillators. In fact two-mode squeeze states which we are not discussing in this course are intricately related to the phenomenon of optomechanical entanglement.

If generally we have two-mode squeezing then there is a possibility that we will be able to find there is some kind of entanglement is happening between the optical as well as in the mechanical oscillator.

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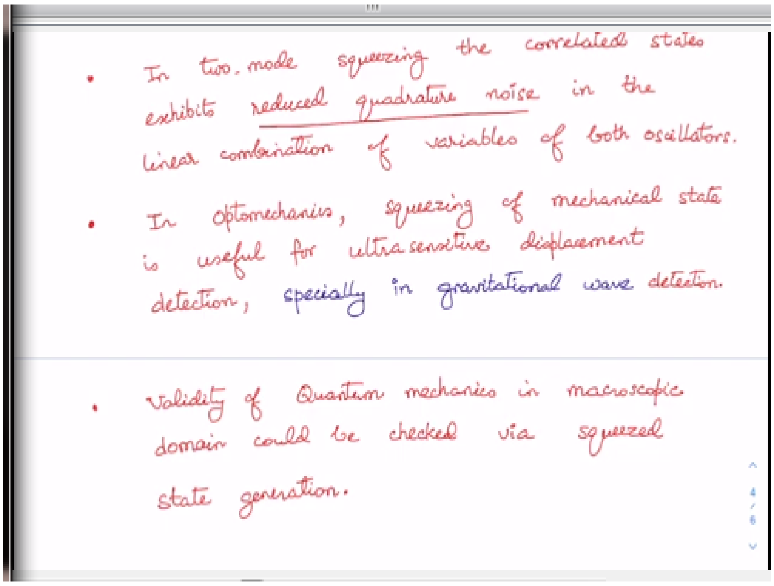

Two-mode squeezing its correlated state exhibit to reduce quadrature noise which is very useful for practical applications. For example in optomechanics squeezing of mechanical state is useful for ultra sensitive displacement detection and this is particularly useful for gravitational wave detection because as you know the gravitational wave is a very weak signal.

And if a gravitational wave has to perturb a movable oscillator, then we should not have any kind of other noise, other noise has to be very much low, so that we can get the influence of the gravitational wave. And in this case squeeze states are particularly useful. And also, if we want to study the validity of quantum mechanics in macroscopic domain then generating squeezed state is useful and helpful. In fact let me now talk about how squeeze state can be generated in the context of optomechanics.

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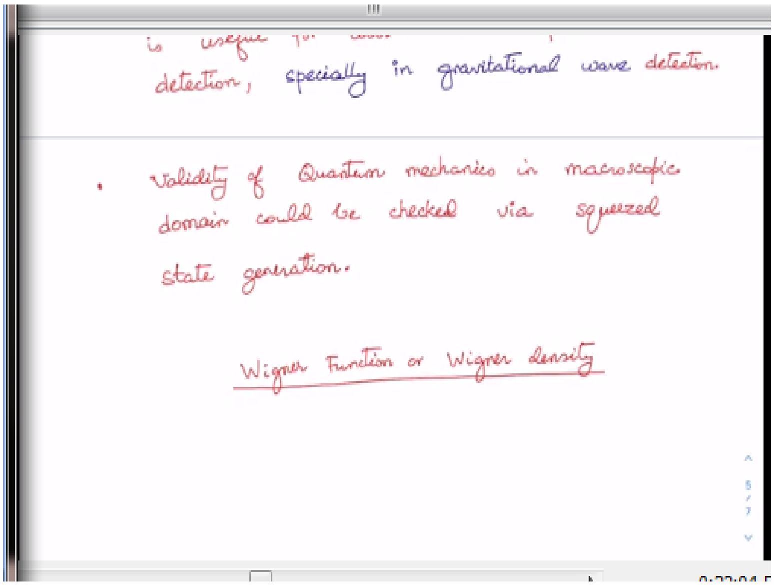

But before I do that let me briefly introduce you to a useful mathematical tool which is often used to discuss non-classical states like squeezed states and this mathematical tool is called Wigner function. Also, it is sometime or mostly it is known as Wigner density, so I will just give a brief overview of what this quantity or this function is.

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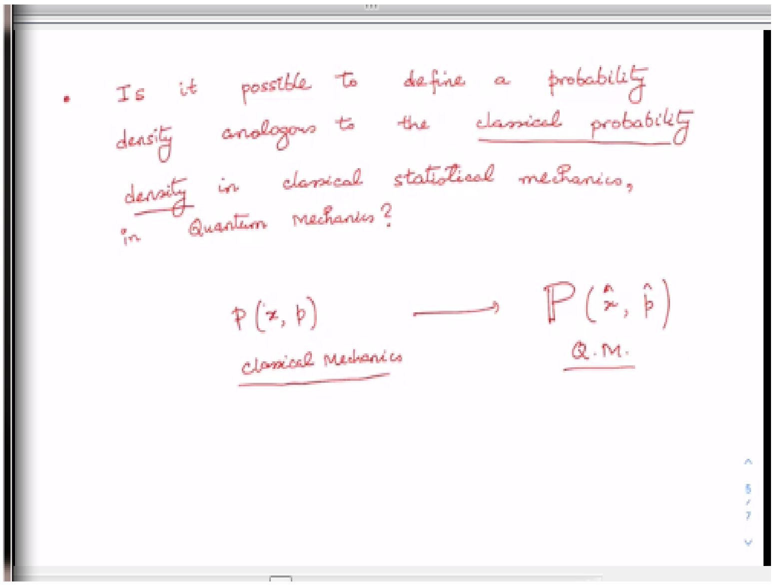

The idea of Wigner density or Wigner function originated from this question. Is it possible to define a probability distribution or probability density analogous to the classical probability density of classical statistical mechanics in quantum mechanics? So, this is actually very desirable because if we want to describe classical limit at high temperature then this is going to be very useful.

Because you know that when we go to the very high temperature regime, then all the quantum mechanical principle are basically we may actually do away with the quantum mechanics and we can use classical mechanics in those regimes. Now at first sight it appears to be hopeless idea because you know that in classical mechanics the probability distribution is a function of the position in momentum.

And in the so-called quantum mechanics if we want to do that, let me just explain it this way. If we want to have say some probability distribution function let me denote it by say P. And it is going to be a function of this x and say p because we are now quantum mechanics. But in quantum mechanics we know that it is not possible to measure position and momentum simultaneously, we cannot define them. So, at first it appears to be a very hopeless idea.

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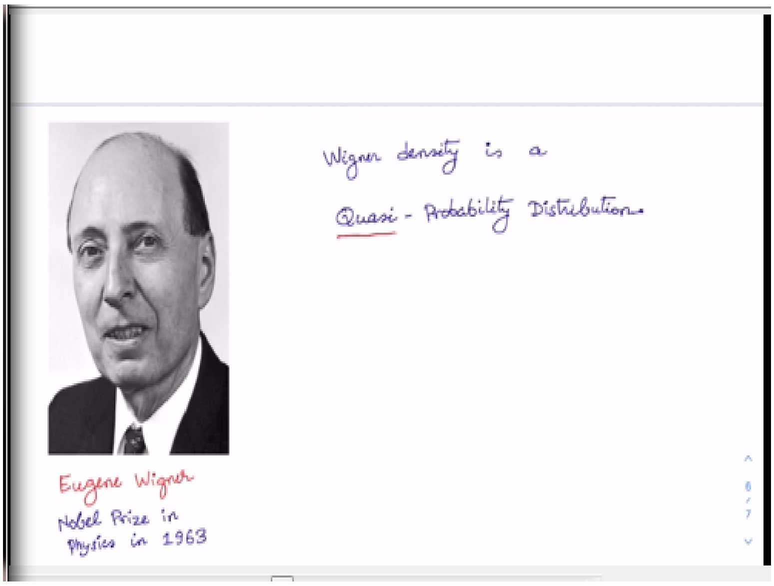

But Eugene Wigner, this gentleman who got Nobel Prize in physics in 1963, he was a Hungarian physicist. In 1932 he came out with a technique which gives us quasi-probability distribution, this is not exactly the probability distribution that you get in classical physics but it is quasi. I will explain what it mean, this method turns out to be quite powerful. And now I am going to give you a brief idea only.

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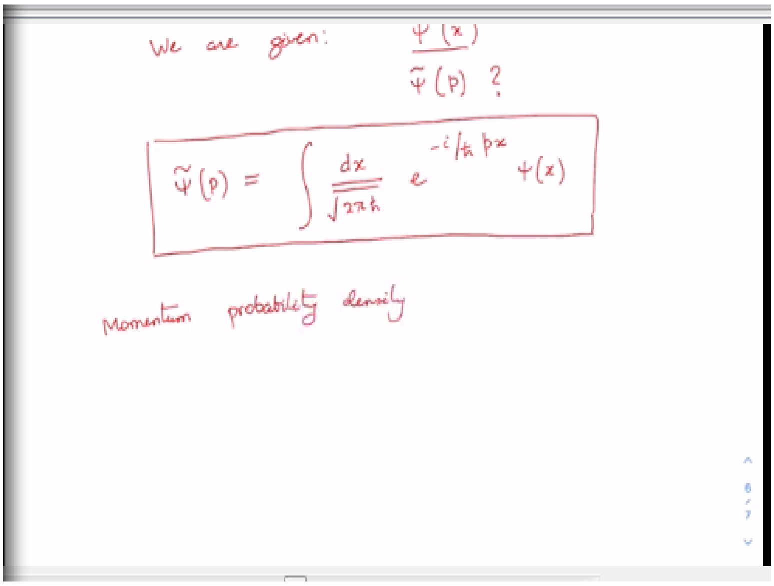

$$\tilde{\psi}(p) = \int \frac{dx}{\sqrt{2\pi\hbar}}\, e^{-i/\hbar\, px}\, \psi(x)$$

Say let us say we are given the wave function psi of x in position space. And we want to know the wave function in the momentum space, let me denote it by say psi tilde of p, so what is this guy? And all of us know that this can be done mathematically by taking the Fourier transform of this wave function psi of x. So, psi tilde of p would be equal to, so this is the Fourier transformation. And I am sure all of you are familiar with this e to the power - i by h cross p of x psi of x.

So, if you take the Fourier transformation, mathematically speaking then you will be able to get the momentum wave function from the position wave function or if you take the inverse Fourier transform you will get the other one. That means you will be able to given the momentum wave function, you will be able to get the position wave function. And the momentum probability density if you ask, we can find that also. The momentum probability density would be simply the modulus of psi tilde p square.

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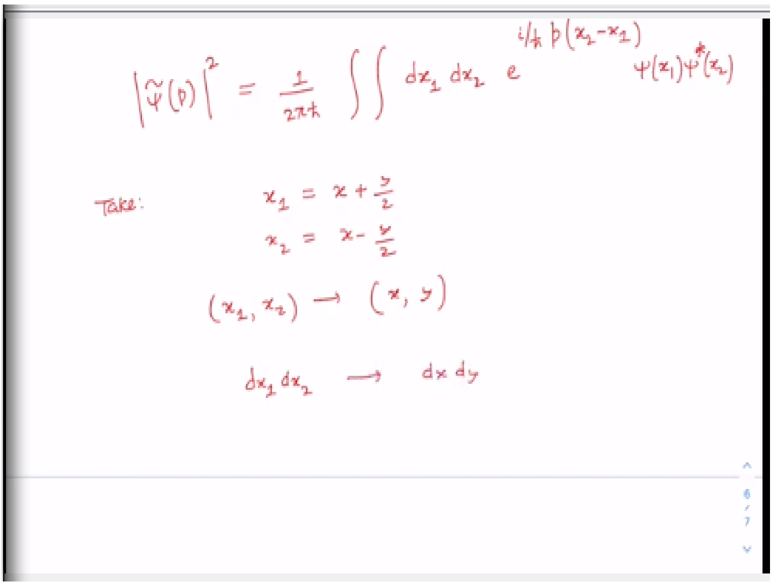

Let me get the expression for that using this particular expression here. So, that would be 1 by 2 pi h cross, you have to take the multiply psi tilde p with the complex conjugate. Then it will give rise to a double integration, so let me define 2 different variables x 1, x 2. So, in one case I have to take the complex conjugate, so I will get i by h cross p x 2 - x 1 and you will have psi of x 1, psi star of x 2, I think you can easily make it out, it is very easy.

We are getting it from this particular expression. Now this could be written in a more simplified form by change of variable. So, let me take say x 1 = x + y by 2 and x 2 = x - y by 2, that means I am going from this set of variables x 1, x 2 to a new set of variables x and y. And if I do that you will see that Jacobian turns out to be the determinant of the Jacobian of transformation is equal to 1. So, therefore I can write this dx d2, this product would be simply the product of dx dy.

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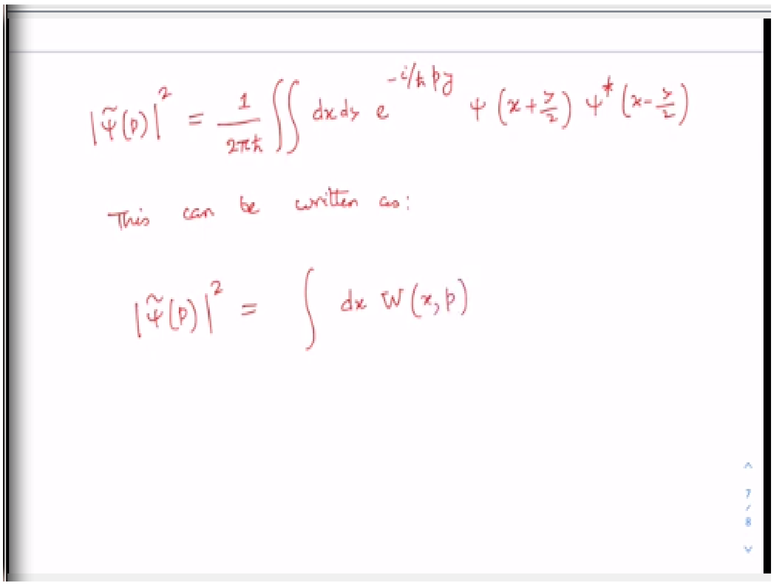

And therefore, I can now write my momentum probability density in the new variable as psi tilde p mod square = 1 by 2 pi h cross and you will have here dx dy. And I am not writing both the integral, ok let me write double integral e to the power -i by h cross. So, x 2 - x 1 you can see that this would be simply it will give you y x 2 **-** - y you will get, so you will have - i h cross py here.

And here you have psi of x + y by 2 and psi star x - y by 2. This can be actually written as psi tilde p mod square = integration dx. And I now defined a new function W of x of p and this is the so-called Wigner distribution function.

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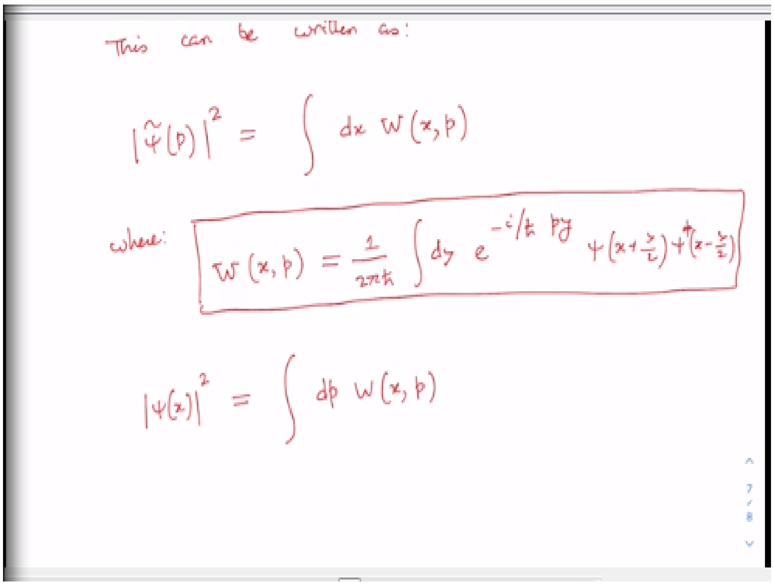

And here I can read out that W of x, p, this is the Wigner distribution function = 1 by 2 pi h cross dy e to the power - i by h cross py psi x + y by 2 psi star x - y by 2. So, this is what the

Wigner distribution function is. In fact, given the Wigner distribution function we can obtain the position probability distribution function as follows. If you are given the Wigner distribution function, the position probability distribution function would be simply the integral taken over the momentum space, that would dp W x of p.

Now if our quantum system is not a pure state because you know that if we have a quantum system which is pure, we can define a wave function. But if our system is not a pure state, it is in a mixed state that means it is an incoherent classical mixture of different wave functions is with different probability.

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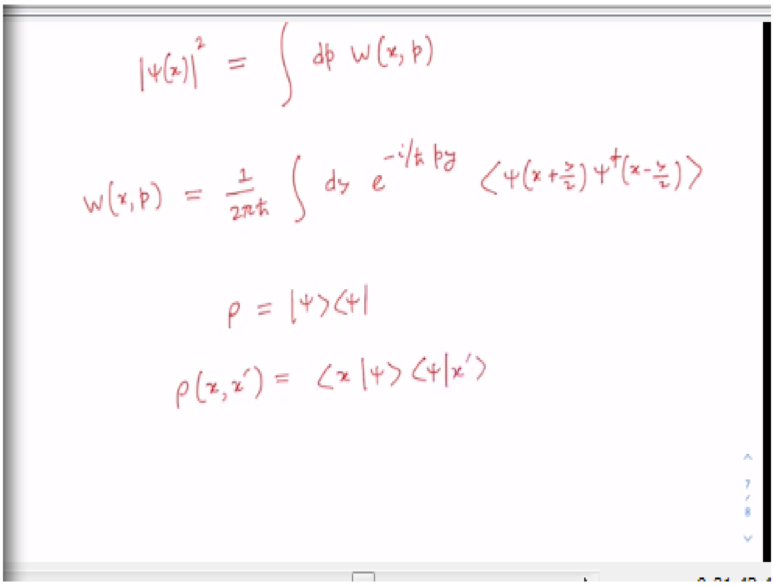

Then this Wigner distribution function we can write as an ensemble average, so let me write it like this. You have W of x, p = 1 by 2 pi h cross dy e to the power -i by h cross py. Now here I have to take the ensemble average because we have a coherent classical mixture of different wave functions, each with different probability. So, we have to write it in this form.

And if you recall that this I can further simplify, write it in a more familiar form. If you recall we define our density matrix as say this is ket psi bra psi and if I can write it in the position space the density matrix as say x, x dash I can write it in this representation, so I will have this.

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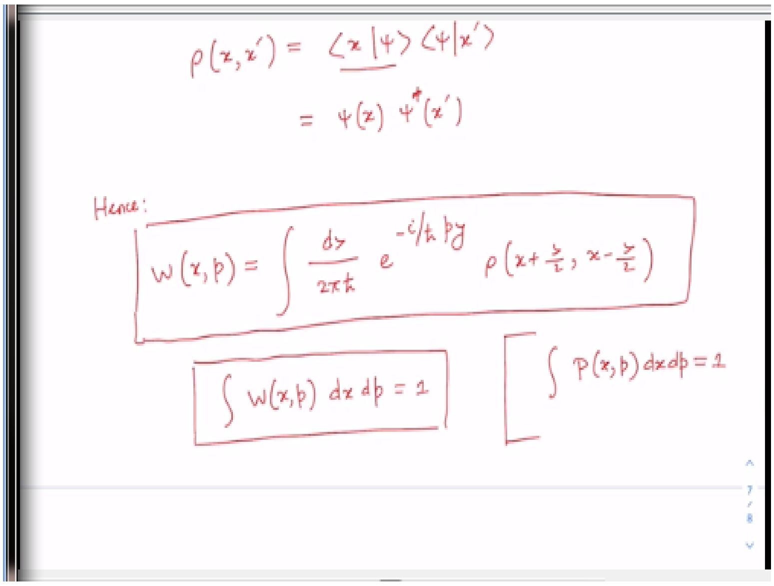

And this guy is nothing but the position wave function and this is the complex conjugate of the position wave function. So, therefore hence we can write the Wigner distribution function as follows W x, p = integration dy 2 pi h cross e to the power -i h cross py rho x + y by 2 x - y by 2. Now it is actually a straightforward mathematics, to show that you can easily prove that this Wigner distribution function is normalized dx dp.

This is just like in classical statistical mechanics the probability distribution function you have dx dp = 1 in classical statistical mechanics. The total probability has to be equal to 1, similar here you are getting kind of analogous expression for the Wigner distribution function also.

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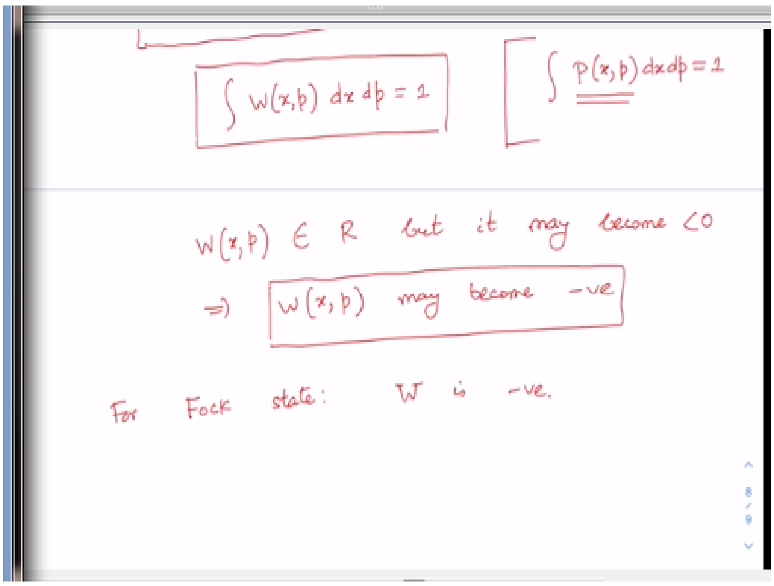

But there is a difference, the difference is that Wigner distribution function W of x, p is always a real quantity, it is always real but it may become less than 0. That means it can become negative implies though it is a real quantity may become negative. So, that means this is something which is completely different from the classical case. Because in classical statistical physics the probability distribution function this p, this quantity is always a positive quantity.

It can never be a negative quantity, probability can never be negative but the Wigner function can become negative. And whenever we have this Wigner distribution function as negative no counterpart is found in classical physics. And those kind of states for which Wigner function becomes negative are called non-classical states to give you an example the so-called Fock state or the number state.

For Fock state the Wigner distribution function is found to be negative and we do not have any classical counterpart of Fock state or the number state. I will not go into the details of the calculations but let me show you some pictorial representation of quantum states using Wigner density.

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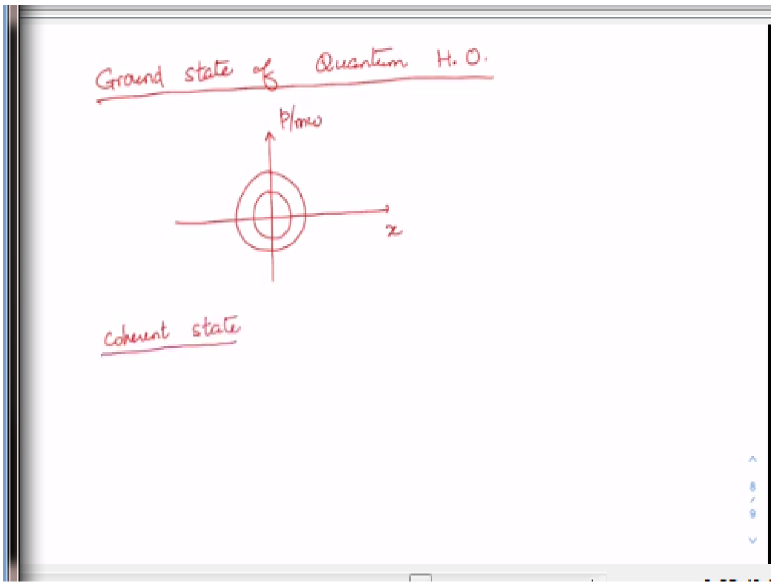

For example the ground state of quantum harmonic oscillator. If I talk about the ground state of quantum harmonic oscillator it is Wigner distribution function can be represented this way. You have here say x axis in the phase space and here the momentum axis and the ground state of harmonic oscillator are Gaussian and they have contour line states are simply circles, so they are completely circles. And what about the coherent states of harmonic oscillator?

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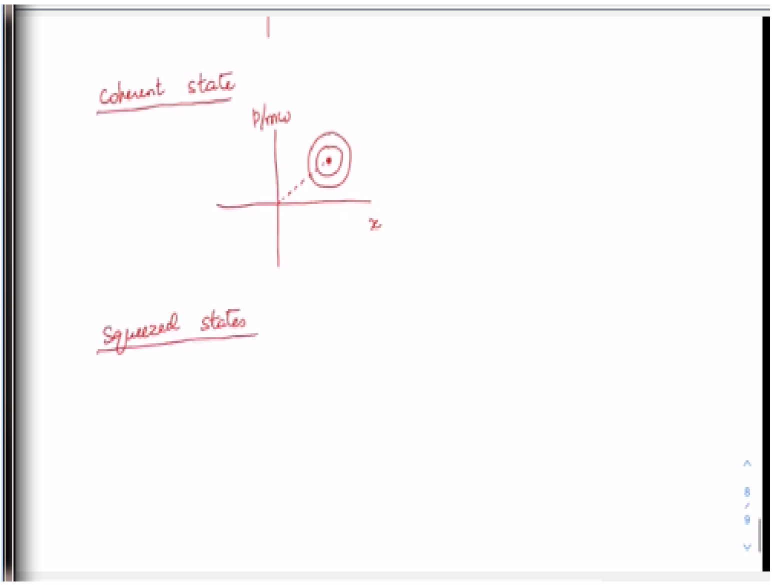

As you know the coherent states are displaced ground state of harmonic oscillator, so they are also easy to get. So, these are Wigner distribution plot, so you have here p by m omega because it is a displaced ground state. So, coherent state suppose it is getting displaced from the origin and you have again this contour lines are circular but only it is displaced and now what about the squeeze states?

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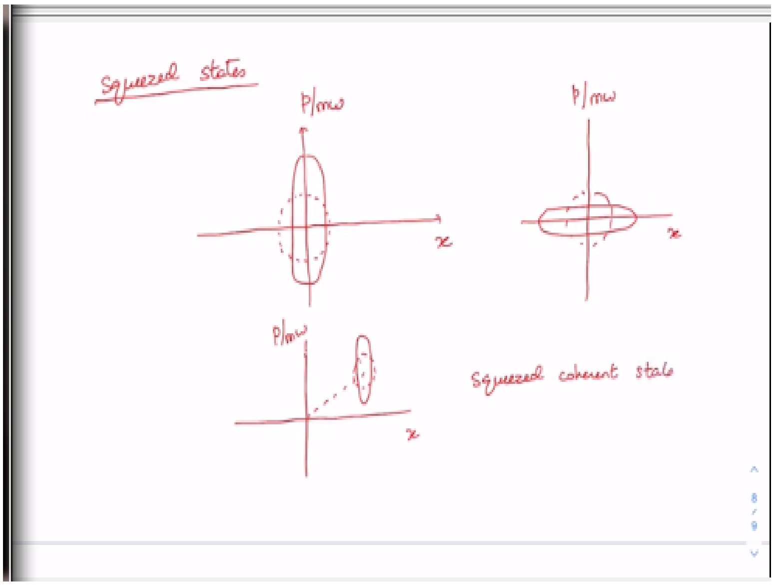

Squeeze states are the states where one of the quadrature are getting squeezed while the other or other one is getting amplified. So, here you will have say this is x axis and this is in p by m omega the Wigner distribution function. Suppose originally we have this original state, this is the ground state of the harmonic oscillator. And now we squeeze it, if we squeeze it say x quadrature is getting squeezed while the p quadrature is getting amplified.

So, then the Wigner distribution plot function if you calculate it, you will get a plot of this type. Here you see the x quadrature is getting squeezed while the momentum quadrature is getting amplified. Or you can also have a situation where you can have the momentum quadrature getting squeezed and the x quadrature is getting amplified. So, these are what we will get you can actually do the detail calculations.

Calculations are little bit involved but when you draw it this kind of a phase plot this is very explanatory. Also you can have squeezed coherent state, you will see what I mean by squeeze coherent state is this. You have say these are your axis coherent state axis is it is displaced ground state and then you now squeeze it, say you squeeze the x quadrature then your momentum's quadrature is getting amplified at the cost of squeezing of the x quadrature, these are called squeezed coherent state.

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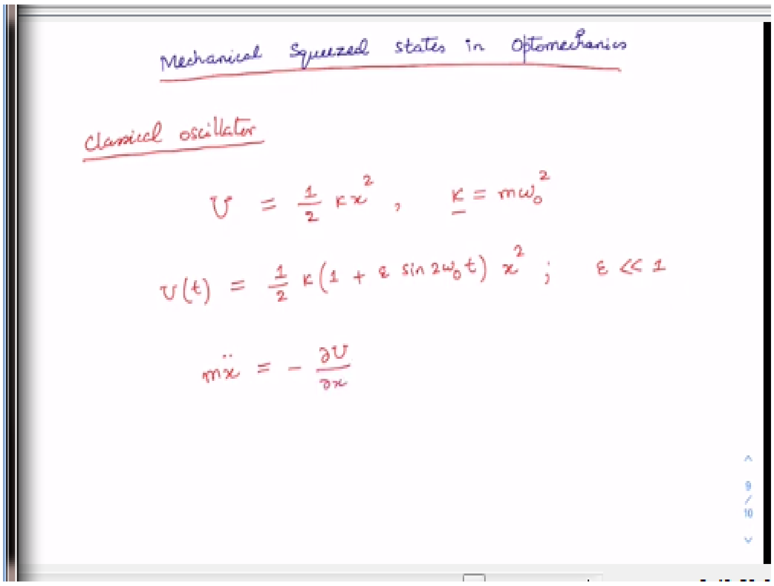

Now let me give you a brief idea about how mechanical squeeze states could be generated in optomechanics. A good insight could be obtained if we discuss classical squeezing in classical mechanical harmonic oscillator. As you know that in classical mechanical oscillator, let me say classical oscillator the potential energy function for the usual harmonic oscillator is given as U = half kx square, where k is the spring constant.

And you know that k = m into omega 0 square, omega 0 is the natural frequency of the mechanical oscillator. Let us modulate the spring constant k of the oscillator. We do that by modulating the frequency by a small amount at twice the natural frequency. So, if I modulate the spring constant then I will do a time dependent modulation. So, I will have this potential energy function as half k is the spring constant, I modulate by this amount 1 + epsilon.

It is a small quantity epsilon into sine twice omega 0 t and we have then x square also. Here epsilon has to be a very small quantity, ok. So, if I introduce this kind of a frequency modulation or spring constant modulation then you will find that we will get kind of squeezing even in the classical context. Now if I use this potential energy function to write the equation of motion, so we have as mx double dot = -del U del x.

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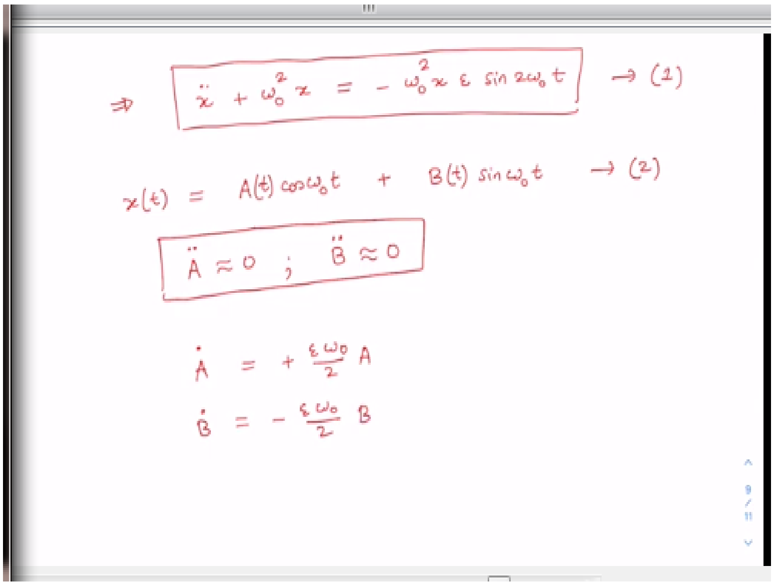

This will give us the equation of motion for the modulated harmonic oscillator, that would be x double dot + omega 0 square x = -omega 0 square x epsilon sine twice omega 0 t. Let me say this is equation number 1; we are going to use it. Now because epsilon is this quantity is very small, we can guess a solution of this equation 1. So, let us say x of t the solution is of this type where we have A cos omega 0 t + B sine omega 0 t.

Now if this modulation is not there then A and B are going to be constant but because we have modulated the harmonic oscillator. So, these coefficients A and B these are no longer constant but they become some time dependent constant like this. But the time variation is taken to be very slow, slow enough to neglect the second order time derivative of A and B. So, we can neglect A double dot as well as we can neglect B double dot.

Now if we put this solution in equation 1 and also take these conditions into account, that means A and B variables are slowly varying in time. And after some straightforward algebra we can get these equations for the time evolution of A and B. A dot would be equal to +epsilon omega 0 by 2 A and B dot would be equal to -epsilon omega 0 by 2 B.

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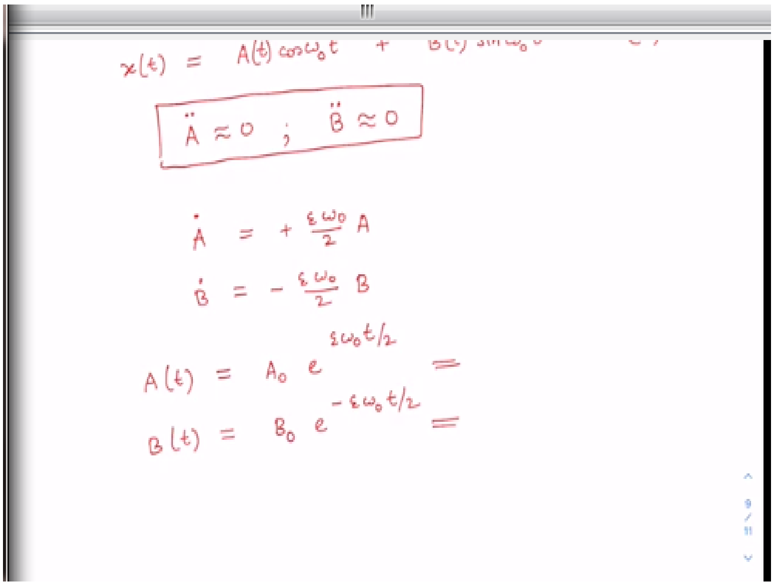

And these 2 equations has this obvious solution, so A of t would be equal to A 0 e to the power epsilon omega 0 t by 2 and B of t would be equal to B 0 e to the power -epsilon omega 0 t by 2. This implies that one quadrature component is increasing exponentially with time at the cost of the other one because the other one is here B is decreasing exponentially in time.

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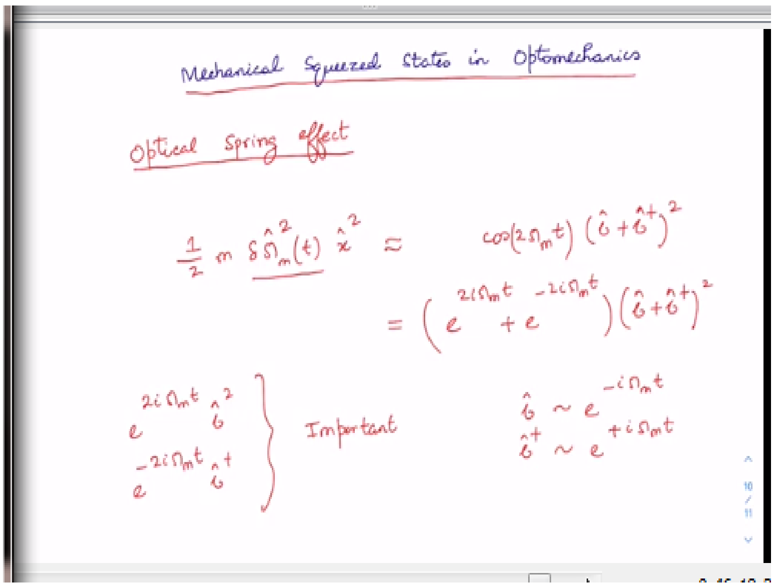

Now, hopefully you can appreciate if I tell you that in optomechanics we can get mechanical squeezed states by exploiting the so-called optical spring effect. An optical spring effect we have already studied in great details in one of the lecture classes. As you know the optical spring effect simply means that the light field changes the spring constant of the mechanical oscillator and we can change the spring constant as a function of time by changing the laser intensity.

Now you can easily guess that if we can modulate the spring constant of the mechanical oscillator at twice it is frequency the particular mechanical mode could be squeezed. Now let me give you a quick quantum treatment of the whole idea. Because now in addition to the usual Hamiltonian we have a part involving the modulated spring constant. Say that part is half m delta omega m square, this particular piece is coming because of the modulation and we have x square also.

Now we can write this term would be proportional to the main terms that are, let me just write them. We are modulating it at twice the frequency of the mechanical mode say that is cos twice omega m t. You can take sine also does not matter, I am here taking cos and then from x square part we have b + b dagger whole square. These I can write as e to the power cos 2 omega m t I can write as e to the power twice i omega m t + e to the power - twice i omega m t and we have term b + b dagger square.

Now we claim that only 2 terms are of significance and these 2 terms are e to the power 2 i omega m t b square and e to the power -twice i omega m t b dagger square. These 2 terms are important and you can immediately see why it is so. Because b varies as e to the power -i omega m t, on the other hand b dagger varies as e to the power +i omega m t.

And other 2 terms are going to oscillates highly and therefore we can neglect those 2 terms because of the so-called rotating wave approximation. Using the rotating wave approximation and going over to a rotating frame which is rotating with the frequency omega m.

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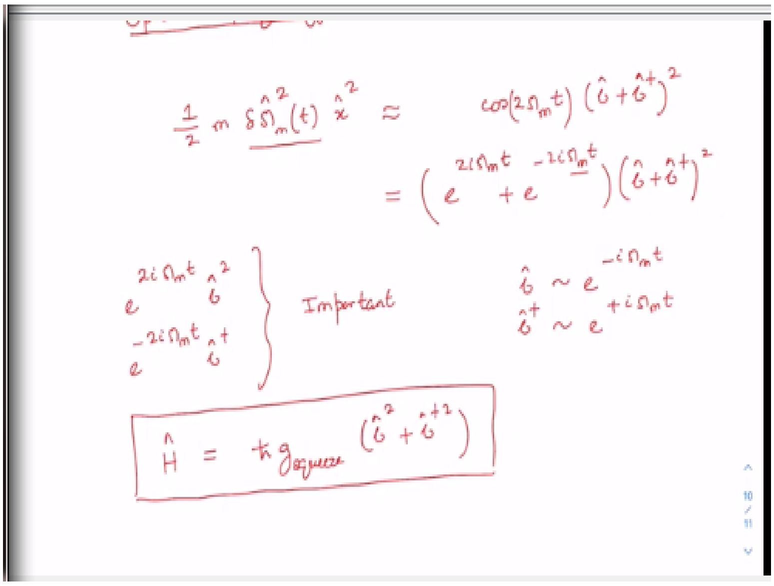

We can write our Hamiltonian by gathering all the other constant putting it under this term h cross g, let us say this is squeezing parameter and we will have b square + b dagger square. This is the Hamiltonian which must be familiar to you and this Hamiltonian is going to give us squeezing; I hope you get the main idea. Let me stop here for today. In this course we have covered 2 primary platforms of quantum technology, namely the circuit quantum electrodynamics and cavity quantum optomechanics.

In this context we have learned all the essential fundamentals and also covered some mathematical techniques. I am sure this will help you to understand scientific literatures in the area and some of you may even take up advanced studies or research, thank you so much.