Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture-42
Linearized Cavity Quantum Optomechanics.

Hello welcome to lecture 31 of the course. This is lecture number 10 of module 3. In this lecture we are going to investigate the linear response of the cavity optomechanical system around the steady state, then we will study the quantum limit for ground state cooling of the mechanical oscillator, so let us begin.

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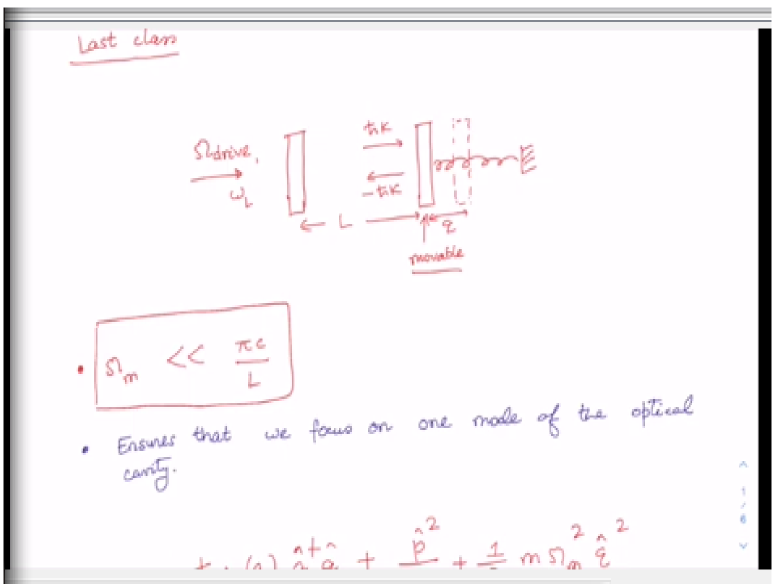

In the last lecture we started discussing the quantum regime of cavity optomechanical system. We assumed that the resonance frequency of the mechanical mode is much smaller than the so-called free spectral range of the cavity which ensures that we focus on only one mode of the optical cavity.

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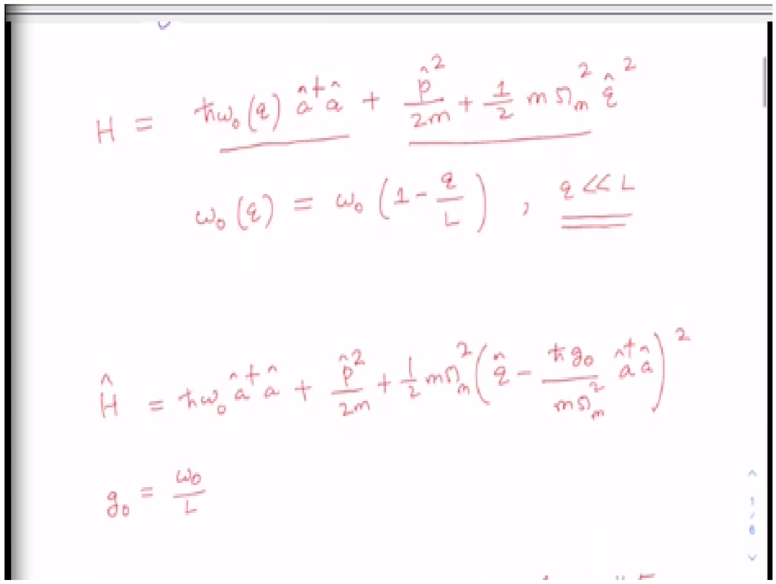

And under that assumption we wrote down the Hamiltonian quantum optomechanical Hamiltonian for the system. And while we have written it we just considered the optical mode and the mechanical oscillator only. And the interaction between the optical mode and the mechanical oscillator comes due to the very tiny shift of the mechanical oscillator due to optical force and this tiny shift is assumed to be much smaller than the cavity length.

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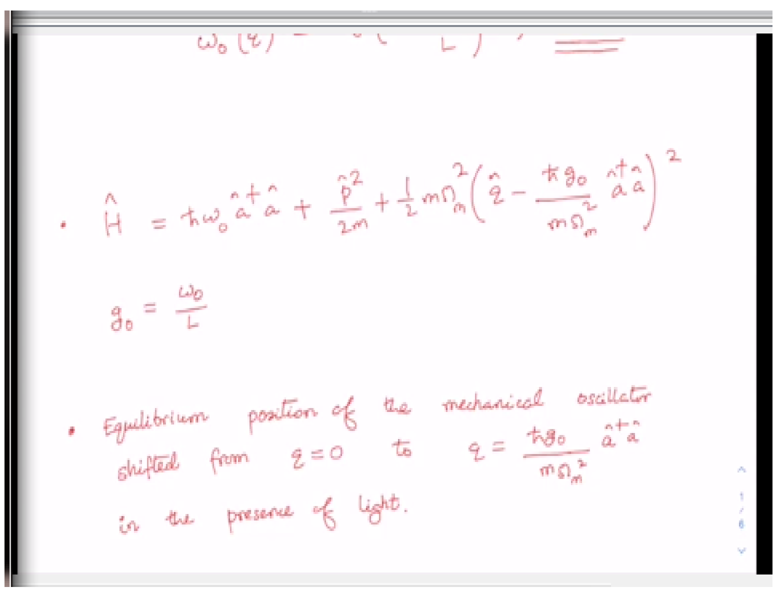

And this Hamiltonian is written in a different form also which tells us that the equilibrium position of the mechanical oscillator gets shifted from its equilibrium position 0 to a non-zero q value when light or the light mode is present there.

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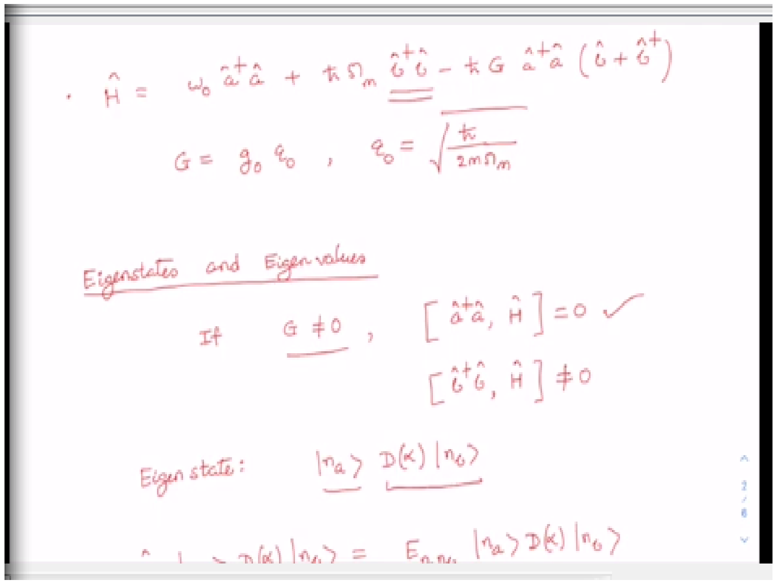

Then we can write this Hamiltonian in terms of the creation and annihilation operator of the mechanical oscillator as well and we have written it in that form. After that we work out the Eigen state and the Eigen values of this Hamiltonian. It turns out that and it is very easy to see that when there is coupling between the light and the mechanics the photon number gets conserved but the phonon number is no longer conserved.

In fact, we can assume the Eigen state for this system in the presence of light when G is non-zero to be a direct product of the number state of the photon and the displaced number state of the phonons.

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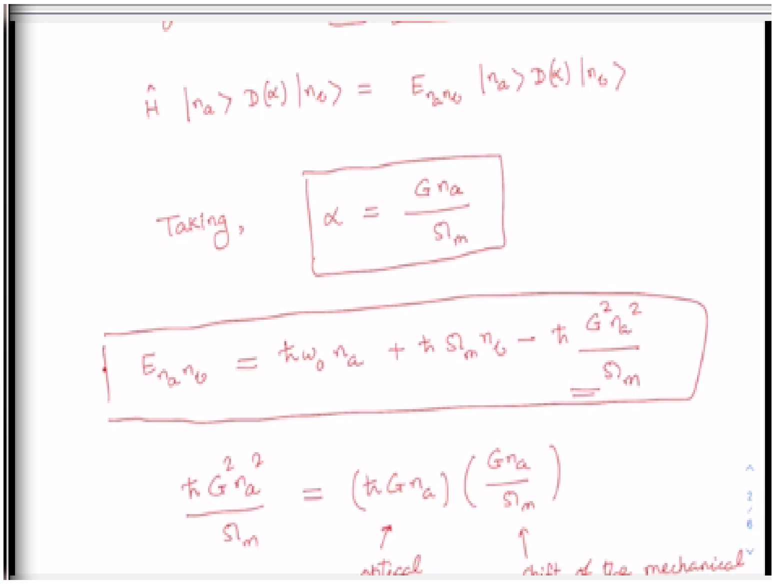

And by solving the Eigen value equation and taking this parameter alpha to be this, we find that the energy Eigen value for the system can be worked out to be this one. And it tells something interesting, it tells that this extra term that is coming when there is coupling between the mechanics and the light is the energy loss by the optical oscillator due to it is interaction with the mechanical oscillator.

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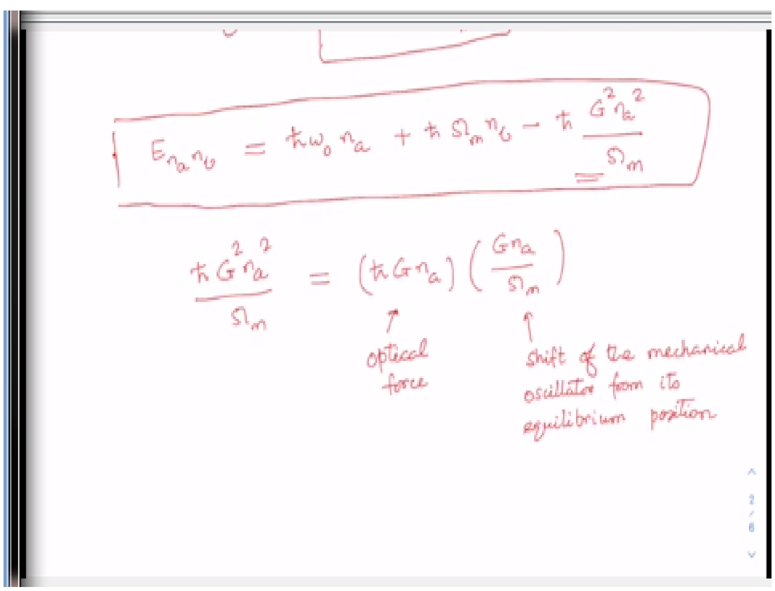

And this is basically a product of the optical force into the displacement of the mechanical oscillator from its equilibrium position.

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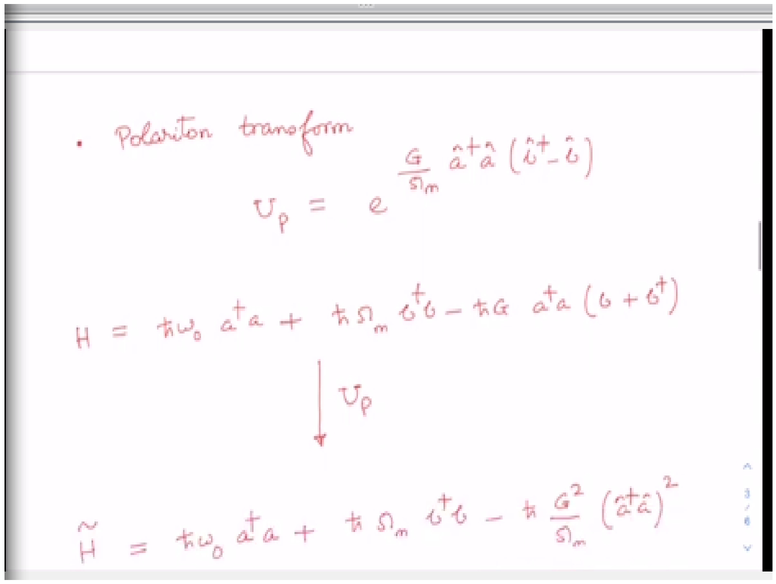

We also learned how to apply this so-called polariton transform.

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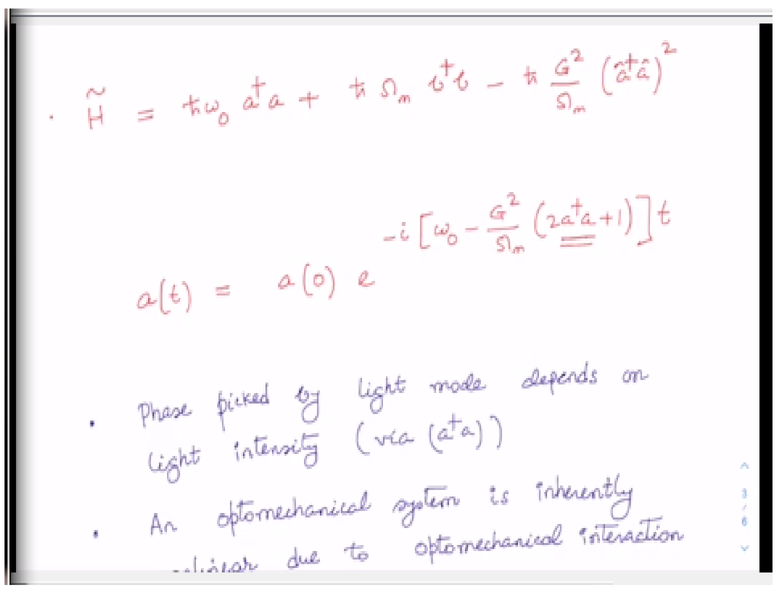

And by using the polariton transform the Hamiltonian can be converted into a different form and which tells us that the phase picked by the light mode depends on the light intensity because of the presence of this term a dagger a which is the photon number. So, an optomechanical system is inherently non-linear due to optomechanical interaction.

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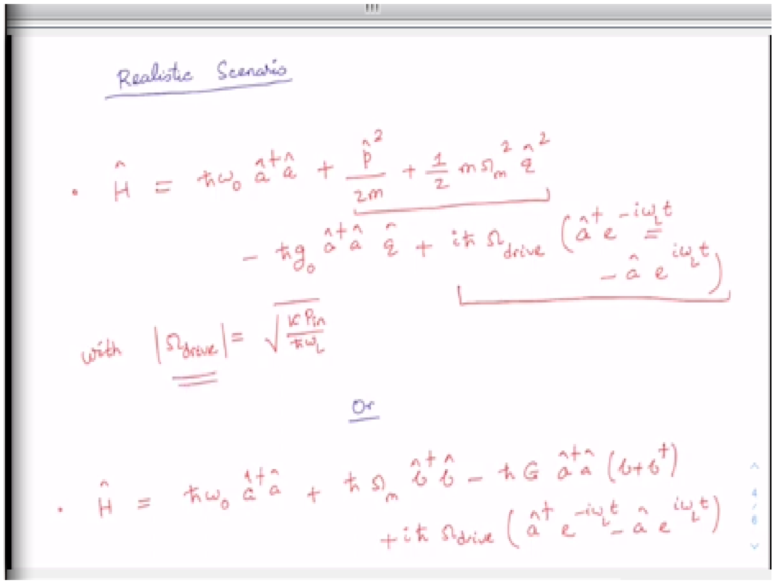

Till now we considered the optical mode and the mechanical mode only but to get into a realistic scenario we have to consider the laser drive also. Because the Fabry-Perot cavity is now externally driven by a laser with frequency omega L and laser amplitude omega drive. And this Hamiltonian can also be written in terms of the annihilation and creation operator of the mechanical oscillator.

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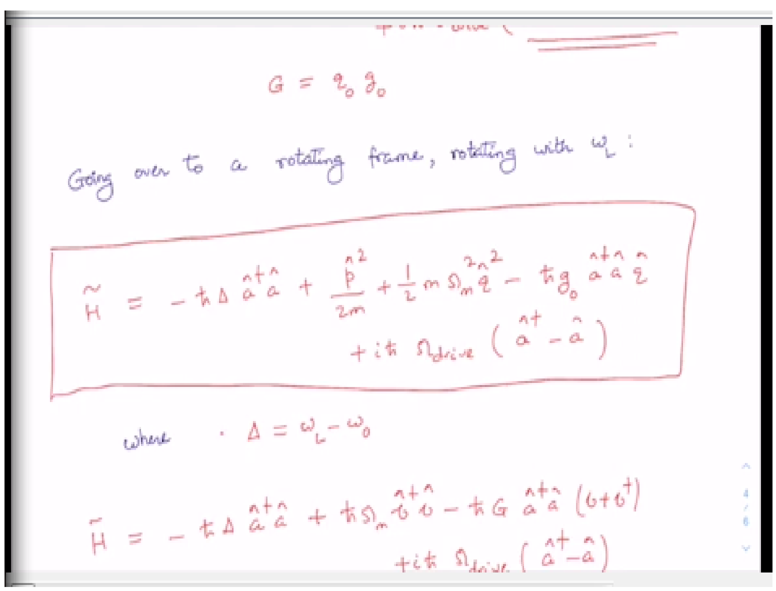

And as you can see that in this Hamiltonian there is explicit time dependence is there and to get rid of this time dependence we can go to a rotating frame of reference. And we can rewrite our

Hamiltonian in this rotating frame of reference in this particular form where this delta is the detuning parameter in terms of creation and annihilation operator.

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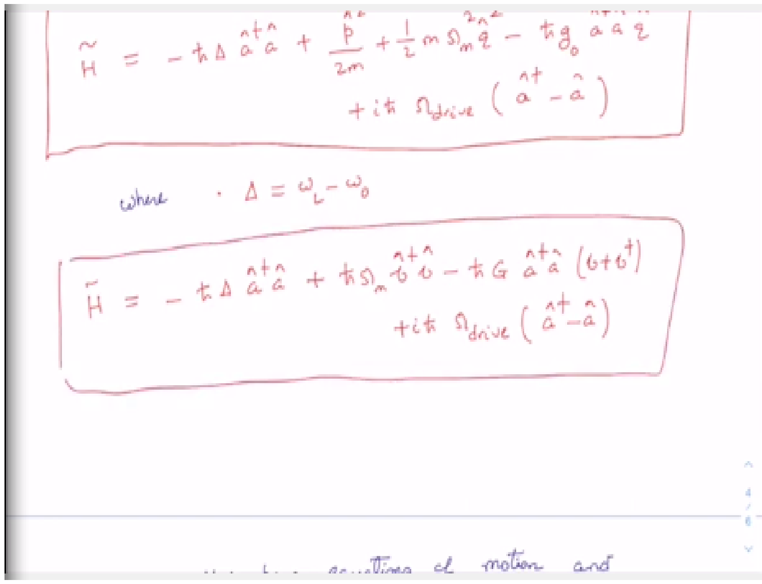

Of course, you can write it in this form as well.

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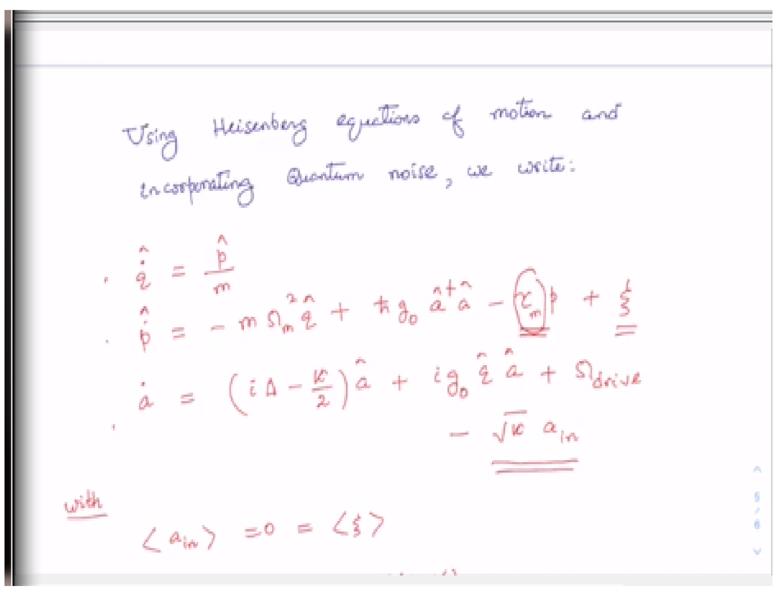

Now using the Heisenberg equation of motion and incorporating quantum noise we can get the equation of motion time evolution equations for the various operators, position, momentum and the optical mode. And here as you can see we have incorporated the quantum noise, this is the Langevin noise that we discussed earlier in previous class. And gamma m p this particular term refers to the mechanical damping, this gamma m is the mechanical damping rate. And this particular term is the noise that is entering into the optical cavity.

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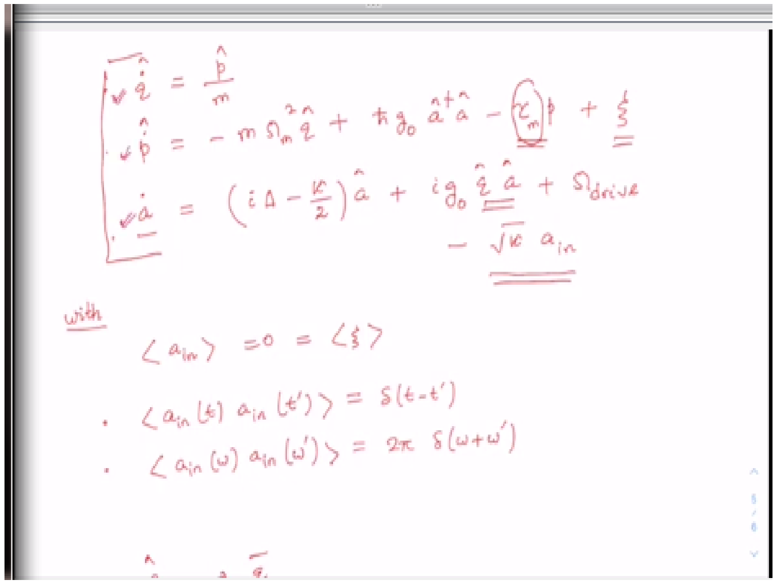

And this noise has 0 mean as we know because these are in nature these are Langevin noise. And also we are aware of the time correlation or the autocorrelation function in a time domain as well as in the frequency domain. Now you please note that these quantum Langevin equations here, these are the quantum Langevin equation these equations are non-linear.

For example as you can see that the time evolution of the cavity mode operator a depends on the product q a here. And this is the product of 2 operators. So no exact analytical solutions to these quantum Langevin equations are at the moment available. So, however we can find the steady state solution in exact algebraic form and let us do that and to find the steady state solution for this position momentum

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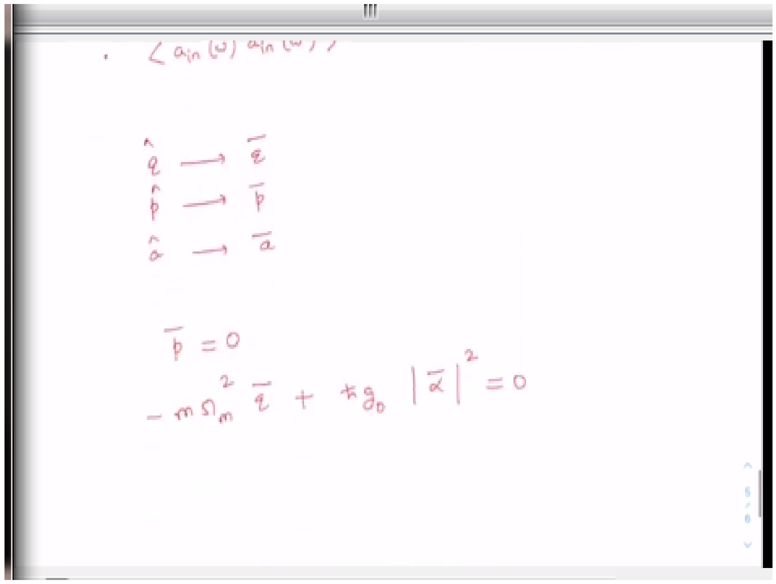

and the optical mode let us denote the steady state solution corresponding to q by q bar and momentum variable p by p, these are q bar p bar as a steady state solution and corresponding to the operator a the steady state solution is say a bar. What we are going to do? We are going to just here make the left-hand side of these Langevin equations to be 0 because in steady state there has to be 0.

And if we do that, then as you can see for example from this equation if q dot = 0 immediately we can write that p bar = 0 and also from the equation of motion time evolution equation for the momentum operator. From there we can write - m omega m square q bar + h cross g 0 mod alpha bar square = 0. You see here it would be q bar in the steady state and a dagger a will become modal for square and p the steady state it is 0. And because we have to take the average of that, so this fluctuation if we take the average it is anyway going to be 0.

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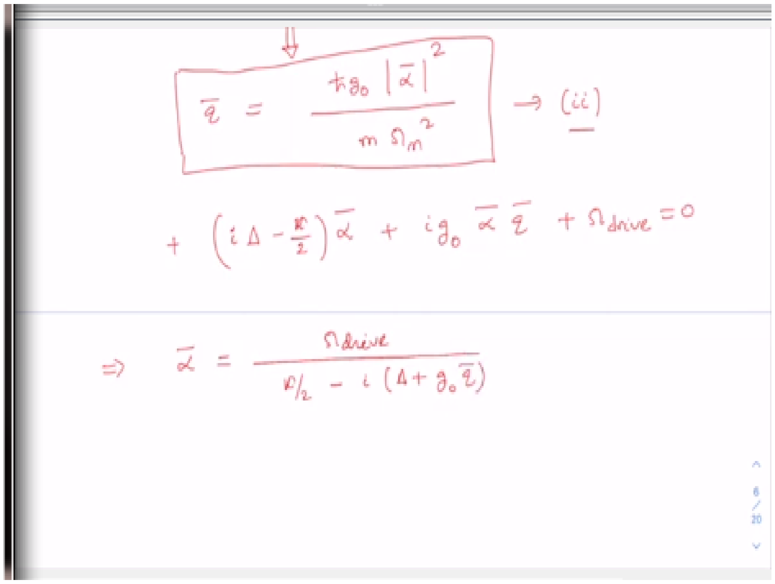

So, from this equation we can immediately write that q bar we can write it as h cross g 0 mod alpha bar square divided by m into omega m square, so this one expression we get. And from the equation of motion for the optical mode in the steady state in the similar way we can write it let us look at here.

From here we will see that we can write it in the following way. So, what we can do is this say minus actually it is + i delta - kappa by 2 alpha bar. Then we have the term i g 0 alpha bar q bar + omega drive = 0. And from here we can write alpha bar = omega drive divided by kappa by 2 - i into delta + g 0 q bar.

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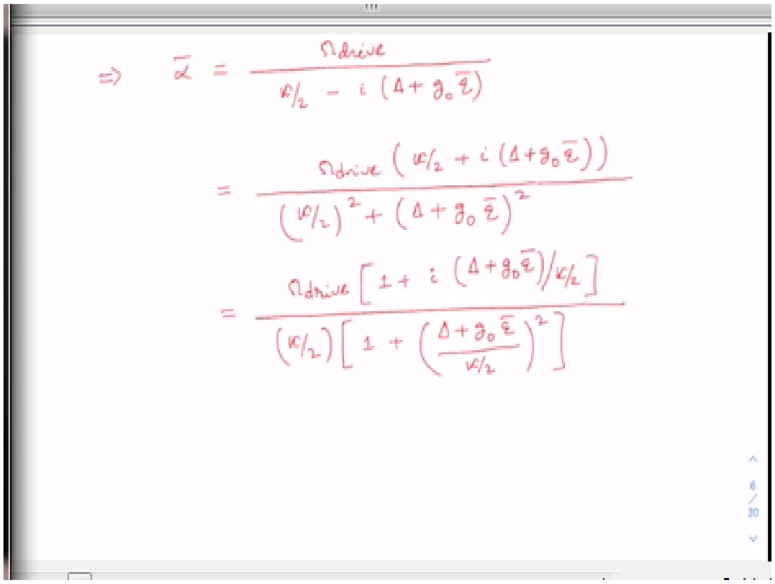

This I can further write as very simply I will have kappa by 2 whole square + delta + g 0 q bar whole square and in the numerator, I will have omega drive into kappa by 2 + i into delta + g 0 q bar. Further I can write this in the following form. If I take kappa by 2 out in the both numerator and the denominator I will have here kappa by 2 and here I will have omega drive, let me write the denominator first here I have 1 + delta + g 0 q bar divided by kappa by 2 whole square. And here in the numerator I have 1 + i into delta + g 0 q bar divided by kappa by 2.

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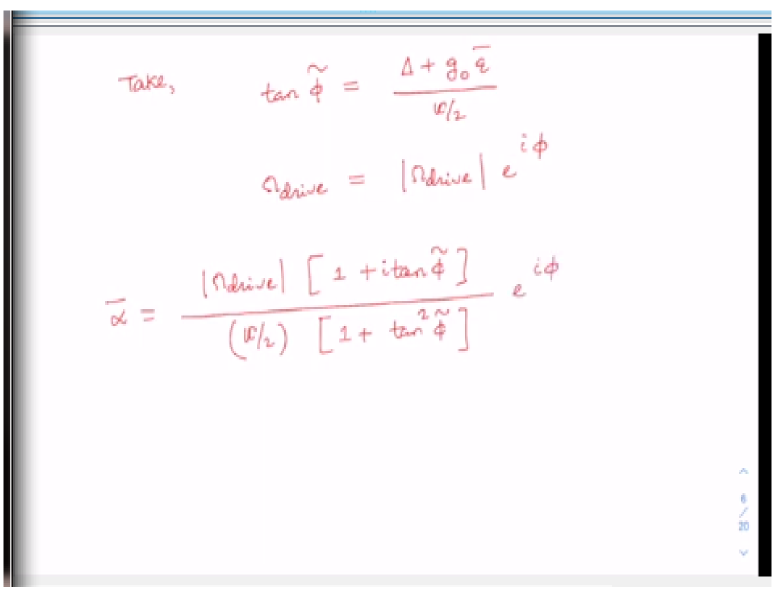

If I now take say tan phi tilde = delta + g 0 q bar divided by kappa by 2. And omega drive if I now write it as its amplitude and its phase in this form, so its phase is phi. Using this I can write alpha bar as it would be equal to omega drive it is actually mathematical trick I am applying here, you can do the calculation yourself, you will have 1 + i tan phi tilde. And here you have kappa by 2, 1 + tan square phi tilde e to the power i phi.

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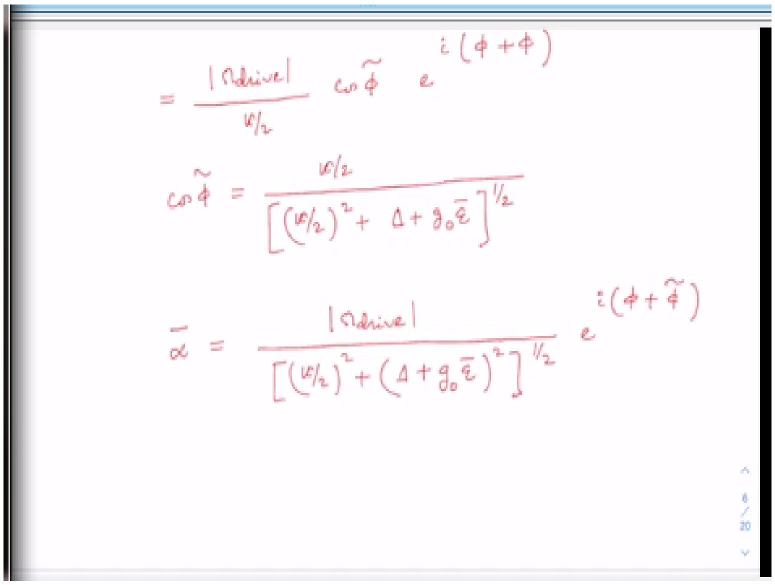

And it is very simple to show that this is actually lead you to the term modulus of omega drive divided by kappa by 2 and here you will have cos phi tilde. And this you can write it as e to the power i phi + phi tilde. Now you can easily read out that cos phi tilde is nothing but because you know tan phi tilde, so you can make out what is cos phi tilde would be. That would be equal to kappa by 2 divided by kappa by 2 whole square + delta + g 0 q bar whole to the power half.

Now using this we can write alpha bar as equal to modulus of omega drive divided by kappa by 2 whole square + delta + g 0 q bar whole square to the power half. And here you will have e to the power i phi + phi tilde.

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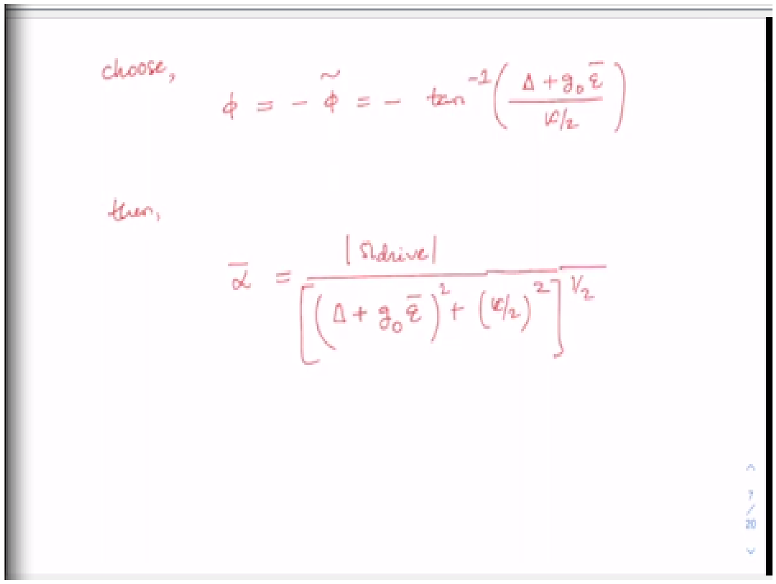

Now if we choose the phase of the drive phi such that it is equal to - phi tilde which is actually equal to - tan inverse of delta + g 0 q bar divided by kappa by 2. Then, alpha would be a real quantity and alpha bar we can write as omega drive divided by delta + g 0 q bar whole square + kappa by 2 whole square to the power half. So, in this case alpha bar is now a real quantity.

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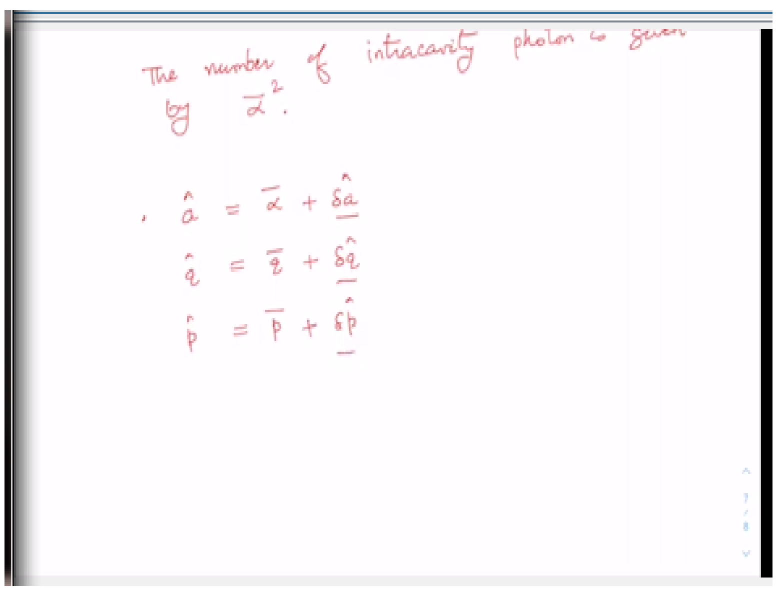

And also, you see the number of intracavity photon is given by alpha bar square. Now we will investigate the linear response of the optomechanical Fabry-Perot around the steady state values which already we have found out. And this is going to be a semi classical approximation where we will write each dynamical variable.

For example, this optical mode which is a dynamical variable and the mechanical oscillator variables, the position of the mechanical oscillator and this momentum these are the dynamical variable. And we are going to write it as a sum of 2 parts, one is the classical part that is the steady state value, for example for the optical mode we can write a = alpha bar + its deviation, the time dependent quantum fluctuation part.

And the position operator of the mechanical oscillator its steady state value is q bar and its corresponding quantum fluctuation is delta q. And for the momentum of the mechanical oscillator its steady state value is p bar and the corresponding quantum fluctuation is delta p. And this delta a, delta q and delta p they have 0 mean quantum fluctuation.

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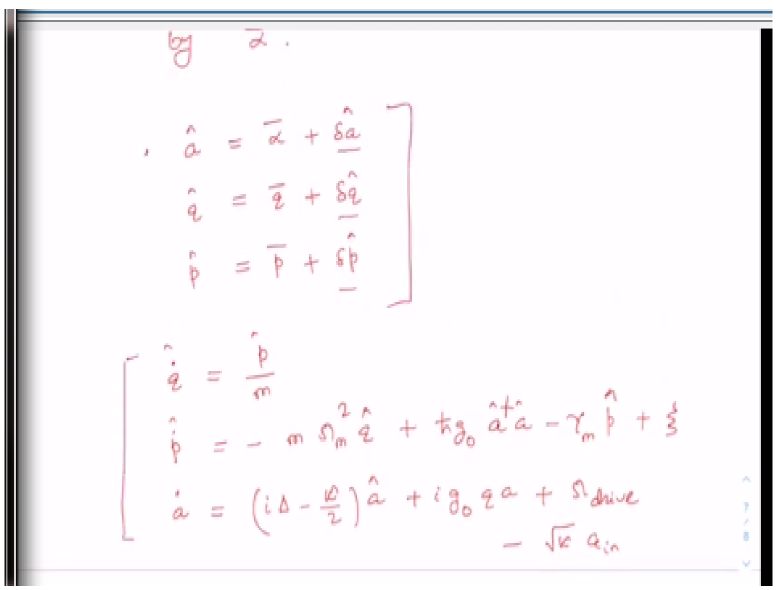

And then if we insert these quantities in our quantum Langevin equation, let me write it again q dot = p by m, all these are operators. So, sometime I may not write it but you please understand that I am now talking about quantum operators here. We have m omega m square q + h cross g 0 a dagger a - gamma, it is gamma m p and the Langevin noise Xi and we have q dot is equal to, in fact q dot I have written already.

We have a dot = i delta - kappa by 2 a, i g 0 q into a + omega drive and the quantum noise that is square root of kappa a in. Now if we put this answers in this quantum Langevin equation

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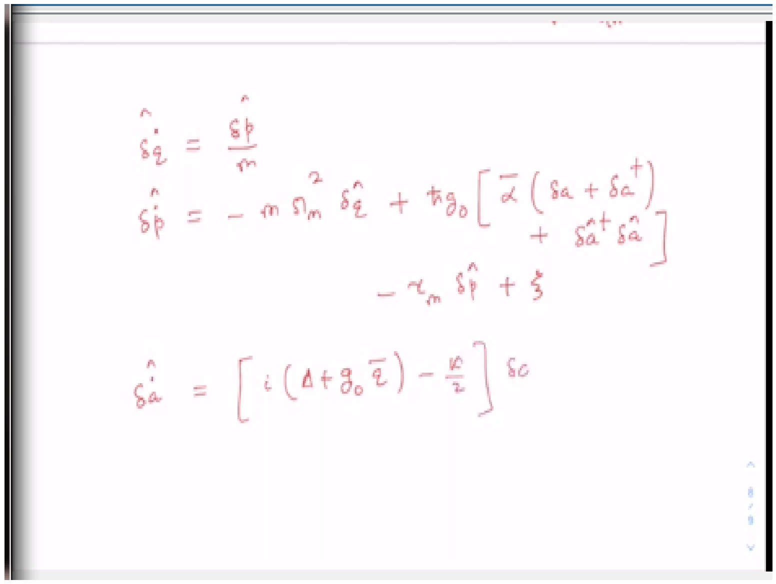

then we can write, you can easily show it you will get delta q dot = delta p by m, then all these are operators again. And we have delta p dot = - m omega m square delta q + h cross g 0 and I will urge you to verify it, it is very simple or otherwise maybe we can straight away show it in our problem solving session as well. But it is very straightforward it is delta a dagger delta a and we have - gamma m delta p + Xi. And for the optical mode the fluctuation is delta a dot = i delta + the detuning parameter + g 0 q bar - kappa by 2 delta a.

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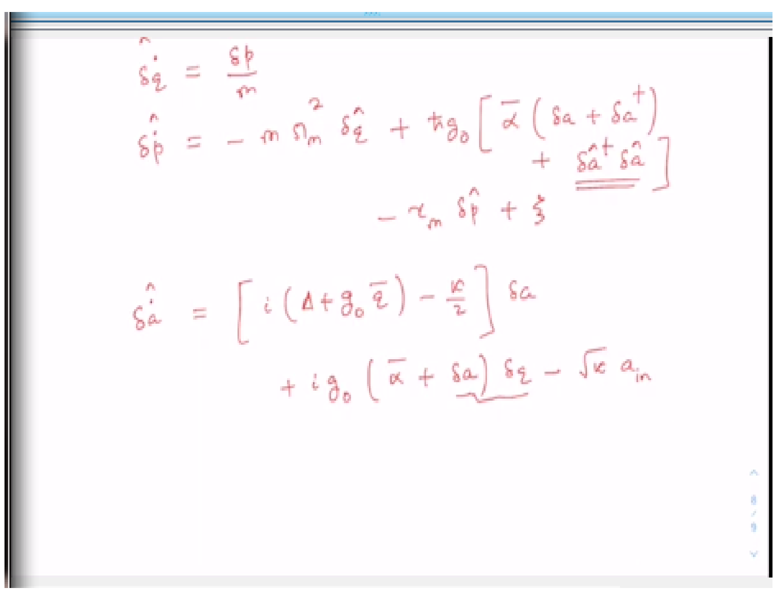

And I have i g 0 alpha bar + delta a delta q - square root of kappa will have a in here. Now you see since these fluctuations are assumed to be small, we are going to retain only those terms which are linear in the fluctuation. So, if we go to retain only the linear terms and then the terms which are bilinear. For example delta a dagger delta a this is a bilinear term and the product of delta a delta q which you are going to encounter here for example, delta a delta q this we have to remove.

Because these would be very further small and we will just concentrate into the linear terms only. Then this is going to simplify our these Langevin equations corresponding to this fluctuation.

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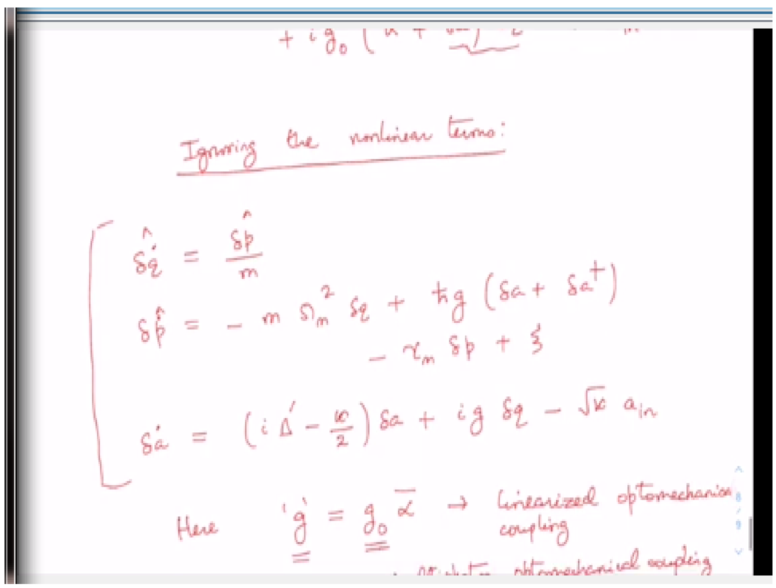

Then we can write delta q dot, so let me first write that what we are ignoring. We are ignoring the nonlinear terms and that is why we are doing the linearization, ignoring the nonlinear terms we get delta q dot = delta p by m. Then delta all these are again operators delta p dot = - m omega m square delta q + h cross g delta a + delta a dagger - gamma m delta p + Xi and we have here delta a dot = i delta dash.

This is the modified detuning parameter effective detuning parameter that I am going to define soon. - kappa by 2 delta a + i g delta q - square root of kappa a in.

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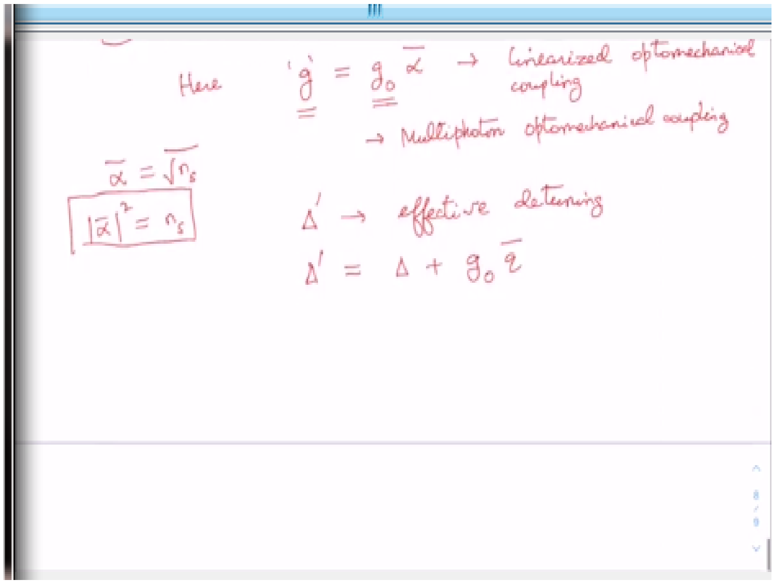

Now here this parameter g = g 0 into alpha bar and this particular parameter g is termed as the linearized optomechanical coupling, it is called linearized optomechanical coupling for obvious reason. It is also sometimes called multiphoton optomechanical coupling. Because in the steady state coupling, for example g 0 this is enhanced by the steady state photon number ns because alpha bar you see this is actually square root of the steady state photon number.

Because mod alpha bar square gives the intensity and that is equal to the number of photons in the cavity in the steady state. So, therefore you see that your g 0 is now getting enhanced by the amount by alpha bar and that is the reason this parameter g is called also as the multiphoton optomechanical coupling. And this quantity delta dash which this is called effective detuning and this is delta dash = delta + g 0 into q bar.

So, these set of equations actually constitute the linearized quantum Langevin equations for the optomechanical Fabry-Perot cavity and it contains the full linear response of the system. In fact, this set of equations could be derived entirely in terms of creation and annihilation operators as well.

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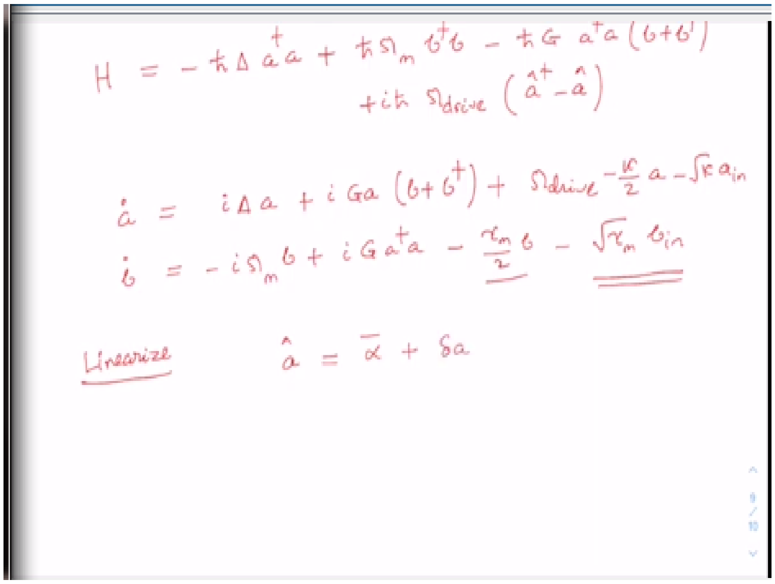

And to do that we are going to start with the Hamiltonian where we write the whole Hamiltonian in terms of only creation and annihilation operator in this form - h cross delta a dagger a + h cross omega m b dagger b - h cross g a dagger a into b + b dagger + i h cross omega. This actually we have written earlier also, so hope you are getting it, it is a dagger - a, this is written exclusively in terms of creation and annihilation operators for the optical mode as well as the mechanical mode.

And from here we can as usual get the Heisenberg equation of motion for the optical mode and a mechanical mode a dot and b dot. And you will get it as say i for optical mode you will have i delta a + i Ga b + b dagger and + omega drive - actually we have written it earlier - kappa by 2 - square root of kappa a in. And for the mechanical mode you will have - i omega m b + i G a dagger a - gamma m by 2 b - square root of gamma m b in.

Now here this is the quantum noise that is entering into the mechanical mode, a mechanical substrate or the mechanical oscillator this is the damping of the mechanical damping. Now if we want to linearize it, so we can do that exactly by the same procedure that we have already adopted here a would we write this an annihilation operator for the optical mode we will write it as 2 parts. That is the steady state value and its deviation from the steady state value that is the quantum fluctuation part.

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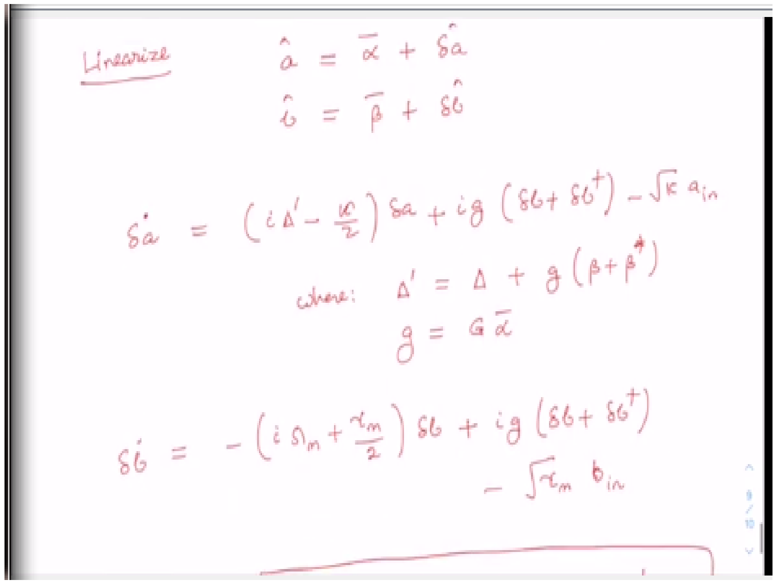

And for the mechanical mode also annihilation operator would be steady state value + delta b, ok. So, similar calculations we can actually carryout here also and this will lead us to these equations for delta a and delta b. For delta a it will be delta a dot = i delta dash - kappa by 2 delta a + ig delta b + delta b dagger - square root of kappa a in. And here by the way delta dash is equal to the effective detuning here would be delta + g into beta + beta star. And g is equal to that is the linearized optomechanical coupling here would be G + alpha bar and delta b dot = - i omega m.

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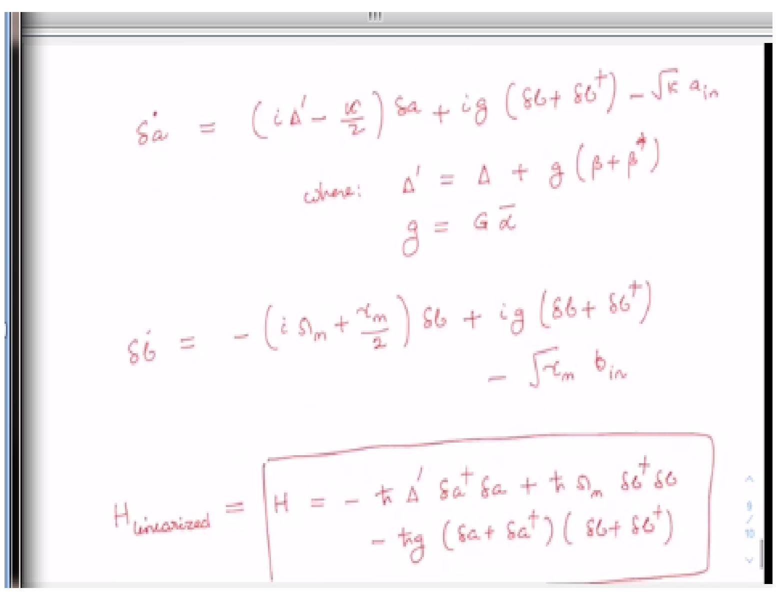

I request you to please do this yourself and please verify it whether I am writing it correctly here. You have delta b + ig delta b + delta b dagger - square root of gamma m b in. The Hamiltonian corresponding to this linearized regime can be written in this form that is H is equal to, now we are going to term it as the linearized Hamiltonian. Or rather we will simply write it as a simple H but you understand that now I am talking about linearized Hamiltonian.

That would be - h cross delta dash delta a dagger delta a + h cross omega m delta b dagger delta b - h cross g delta a + delta a dagger into delta b + delta b dagger. So, this is the linearized Hamiltonian in the absence of damping and other quantum noise. And you can verify whether this Hamiltonian that I have written is correct or not just by applying the Heisenberg equation of motion to get back these equations, time evolution equation for the operator delta a and delta b in the absence of the corresponding damping and the quantum noise.

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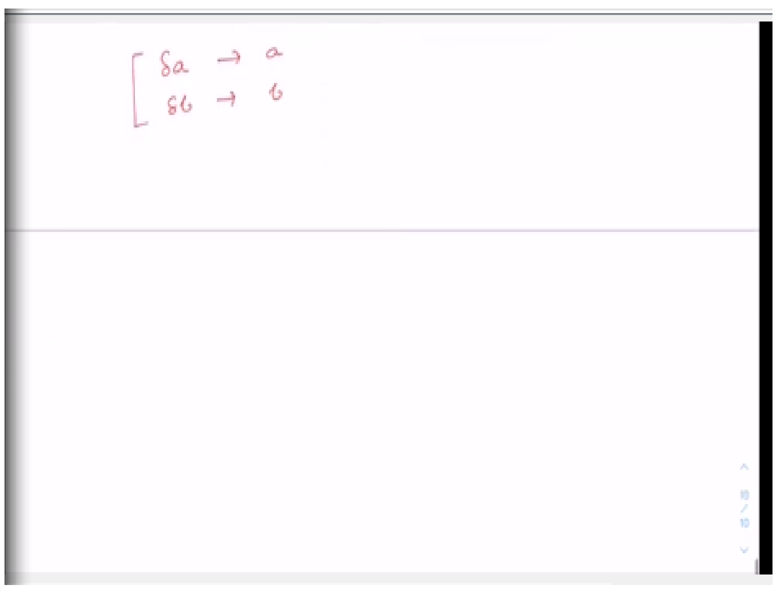

In fact, in many cases or many times people prefer to write delta a as simply a and delta b as b, with the understanding that they represent the quantum fluctuations.

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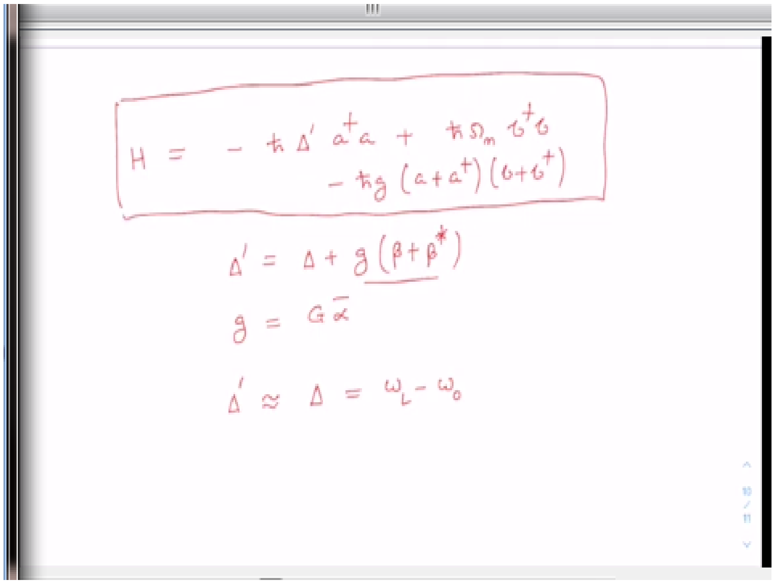

So, if we do that then we can rewrite this linearized Hamiltonian as H = - h cross delta dash a dagger a + h cross omega m b dagger b - h cross g a + a dagger into b + b dagger. And here delta dash = delta + the linearized optomechanical coupling constant g into beta + beta star. And here let me remind you that this g is related to the G and this is the alpha bar. What we were going to do?

This particular part is generally not that very great. So, now onwards in the rest of our treatment we are going to take delta dash to be nearly equal to delta. And where you know that delta this detuning parameter is omega L - omega 0, where omega 0 is the resonance frequency of the optical cavity. The physics described by this linearized Hamiltonian can be depicted schematically and let me show you. Because in essence what we are having is, we are having 2 harmonic oscillator.

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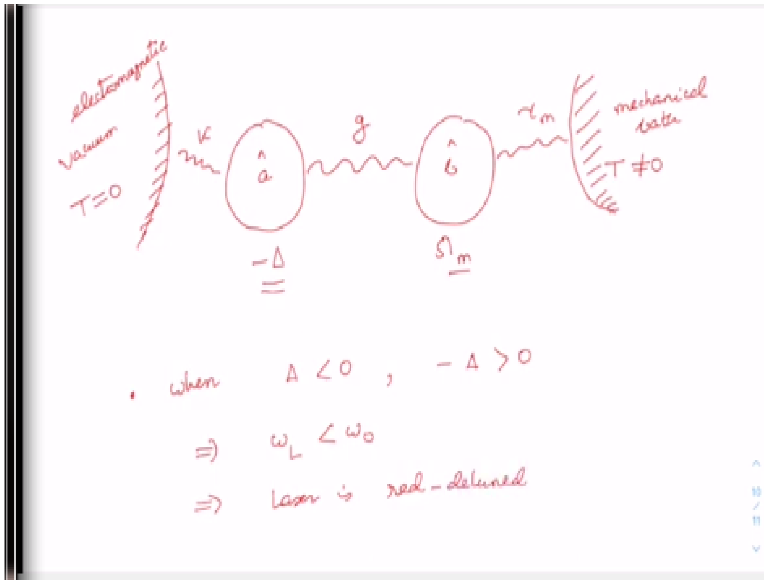

One oscillator is due to the optical cavity and this is represented by the operator a and this oscillator has frequency oscillation frequency - delta as you can see from here. And other oscillator is the mechanical oscillator and this is represented by the operator b and it has oscillation frequency omega m. And these 2 oscillators are coupled via this optomechanical coupling and this described by this coupling constant.

Their coupling is described by this linearized optomechanical coupling constant g. Also, you see that this mechanical oscillator it has a damping and it damps at the rate of gamma m and actually what happens is that? You know the quanta of these mechanical oscillators are called phonons and these phonons get decayed to some kind of a substrate or some bath**.** So, we can term them as mechanical bath and they are at some finite temperature.

So, they get decayed to this mechanical bath also the photons also has a dissipation constant as you know that this decay at the rate kappa and they decay into the environment which is described by the so-called electromagnetic vacuum. So, this is electromagnetic vacuum and you know this electromagnetic vacuum is modeled as harmonic oscillator and they oscillate at optical frequency and because of that electromagnetic vacuum is effectively at a temperature T = 0.

So, we have a situation that when this delta is less than 0 that means in that case - delta would be greater than 0 and we are going to have a positive optical oscillator. And the mechanical oscillator is oscillating at frequency omega m and as you know it is already connected to a mechanical bath at finite temperature. So, effectively speaking the mechanical oscillator is a hot oscillator and the optical oscillator is a cold oscillator.

So, this would effectively bring down the temperature of the mechanical oscillator due to its coupling to the optical oscillator. Now this is basically in a very simple way the principle behind from this pictorial diagram it is clear that this is how the mechanical oscillator gets cooled due to it is coupling to the optical oscillator. When delta is less than 0 it implies that omega L is less than omega 0. And it physically means that the laser is red detuned and we are going to discuss considering the laser to be red-detuned.

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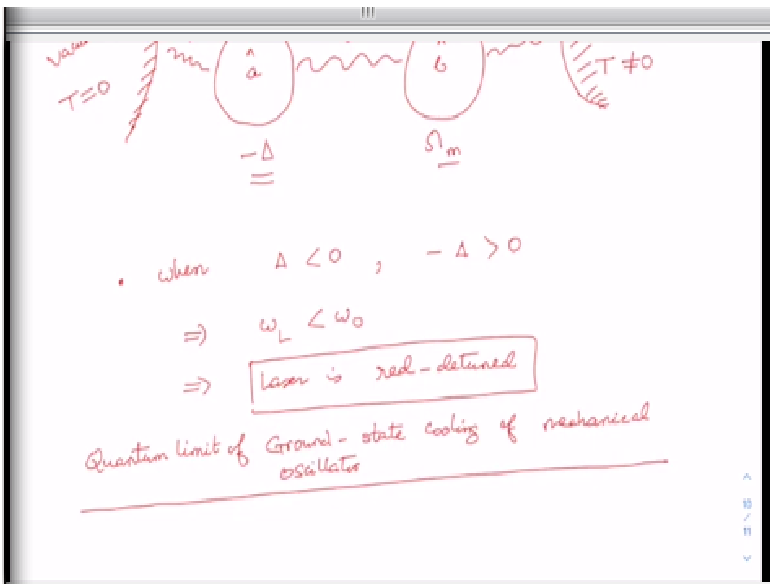

And because now we can achieve what is called ground-state cooling of mechanical oscillator. But because of quantum physics we will see that there is a limit to this cooling, so we are now going to discuss quantum limit of ground state cooling of mechanical oscillator. Before I go on to discuss the quantum limit for ground state cooling of mechanical oscillator let me remind you why receiving ground state is of paramount importance.

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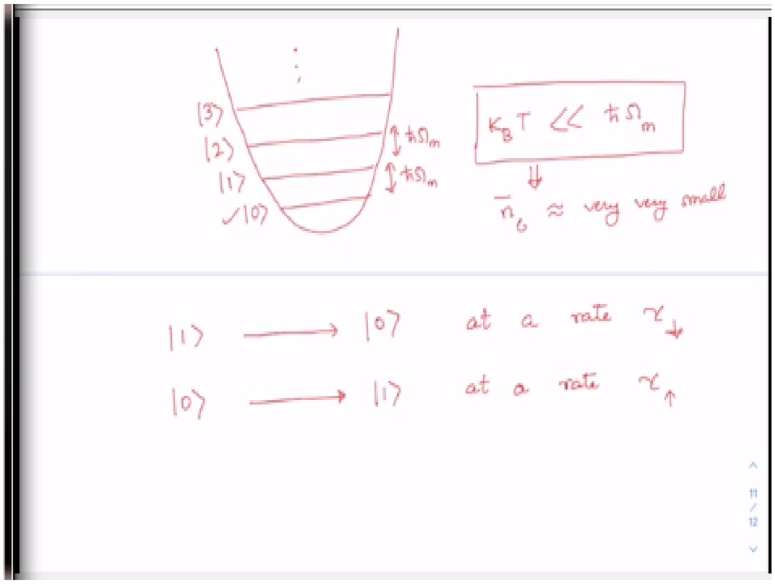

It is important if we want to study quantum phenomena in macroscopic objects. As you know the mechanical systems can be created as harmonic oscillator and the adjacent energy levels of these harmonic oscillators are equally spaced and if we want to study the quantum feature of this massive mechanical system or mechanical oscillators then the discreteness of the energy levels has to be there.

And it will be there provided the energy due to the thermal fluctuation which is K B T it is much, much smaller than the energy spacing that is h cross omega m that already we know. So, this is extremely important and this is why we need to cool the mechanical oscillator to a very low temperature. In terms of phonon picture this actually says that it amounts to saying that the number of phonons or the average number of phonons has to be very, very small.

We can get the ground state provided the average number of phonons is too small. However, as we will see that because of quantum mechanics it is not possible to lower the average number of photons arbitrarily. And now the question is what happens if the mechanical oscillator gets coupled to an optical cavity? Say the mechanical oscillator is in 1 photon state, let us say this is our 1 proton state, this is the ground state, this is the 2 photon state, this is the 3 photon states and so on.

Then if it is now coupled to an optical oscillator or an optical cavity then the mechanical oscillator which is we are assuming it to be in the 1 phonon state. It may go to the ground state at a transition rate say gamma down due to the coupling with the optical oscillator. On the other hand, if the oscillator is already in the ground state that is represented by this ket 0 here. If it is already in the ground state then there is a very small probability that it derives some energy from the optical cavity and transit to the 1 phonon state with at a rate let us say gamma up.

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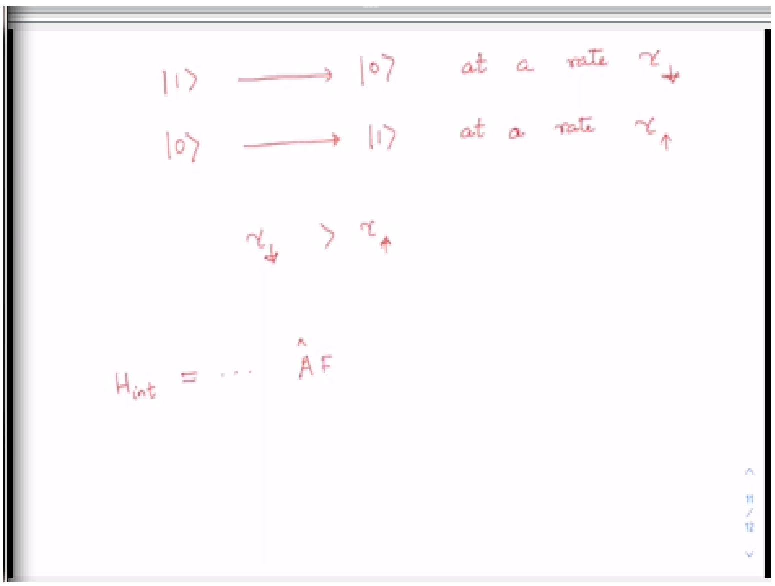

For cooling it is obvious that the downward transition that is gamma down has to be greater than the upward transition rate. Now in general if a quantum system is coupled to some environment the interaction Hamiltonian is given by this form. So, say the system operator is represented by A and the fluctuating environment is represented by the operator F.

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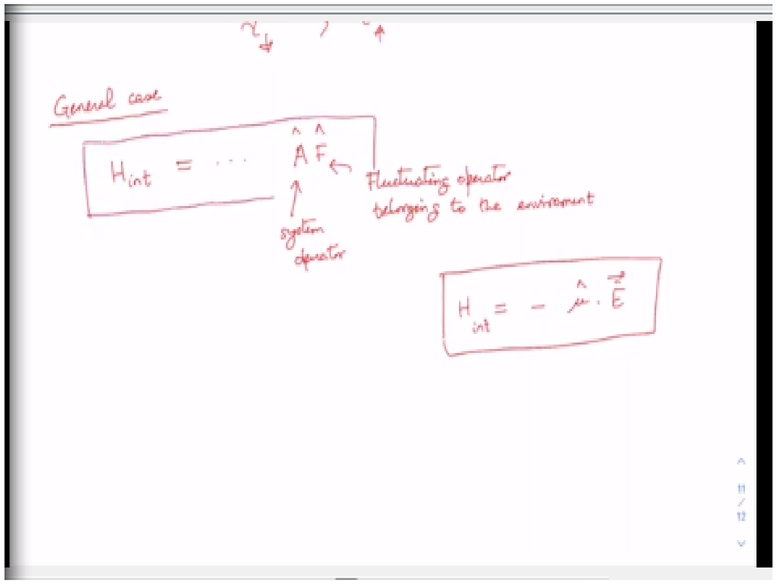

Here A is the system operator and F is the fluctuating operator belonging to the environment. Here I am considering a general case and then we will apply this case after we discuss it we will apply to our specific case. To give you an idea you know that when an atom and an electric field interact, the interaction Hamiltonian is given by - mu dot E, mu dot is the dipole moment operator.

Suppose we are considering a 2 level atom then mu dot is the dipole moment operator and E is the electric field operator, so this is the interaction Hamiltonian. So, similar way we can talk about the general case in this form. Now the transition rate within the system.

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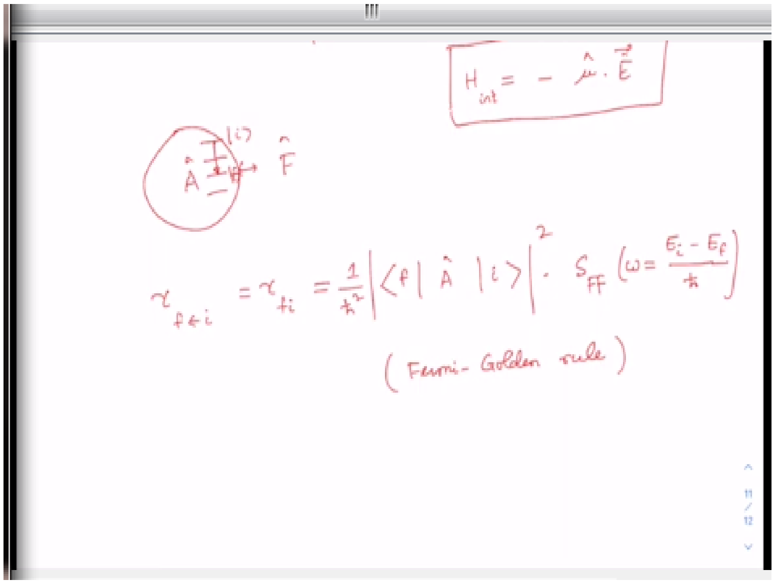

So, we have this system A and the fluctuating environment is represented by this F, so they are getting coupled to each other and that is why we are having this interaction Hamiltonian here. And within the system suppose we want to know say how the system is transiting from some initial state i to some say final state f, then this transition rate is given by the so-called Fermi golden rule and this rate is represented by say gamma i to f.

Or simply I can write it as gamma f i is equal to as per the Fermi golden rule it is we are going from the initial state in the system to the final state f. So, this is the matrix element, this is the probability of going from the initial state i to the final state f within the system. So, mod square will give you the probability then we have 1 by h cross square and we have to calculate this spectrum of the fluctuating operator at the frequency

that I am going to write it as omega which represents the energy difference from the initial energy state of the system to the final energy state of the system divided by h cross. So, this is the formula, this is known as the Fermi golden rule and this formula is applicable when the interaction between the system and the environment is weak.

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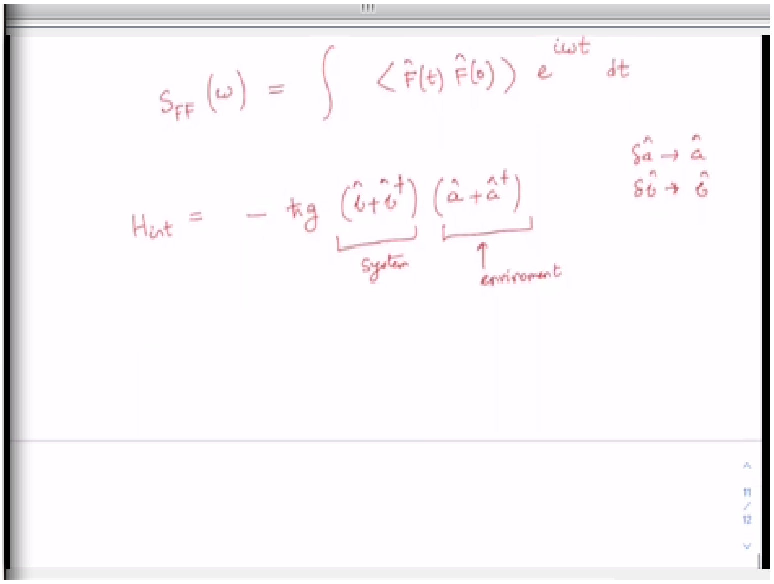

So, by the way you recall that this spectrum omega, S FF omega is if you remember that this is nothing but the Fourier transform of the correlator of this operator F. So, this is the Fourier transformation you have to calculate and we will see how to do that for our specific case. Now in our case we have our interaction Hamiltonian to be like this, this is equal to - h cross g into b + b dagger into a + a dagger all of them these are quantum operators.

Because you remember we have replaced delta a that is the quantum fluctuation belonging to the optical cavity we replaced it by a and similarly the quantum fluctuation belonging to the mechanical system is represented by this simply replaced by b. And here we have this our system, this part is a system and we can take it as our environment, we can say that our mechanical oscillator is surrounding by optical cavity which is here acting as the environment.

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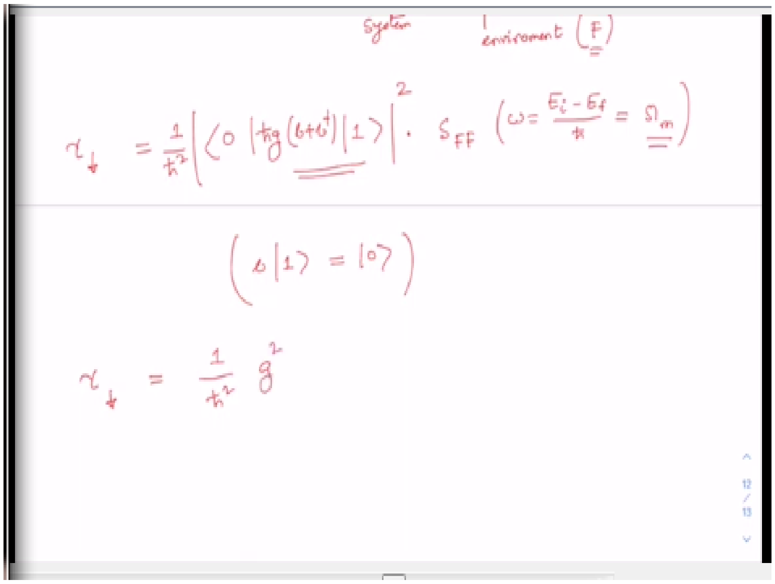

Now using the Fermi golden rule, we can write the rate, for example if we are going from the 1 phonon state in the system to the 0 phonon state and the system operator let me write it as h cross g b + b dagger. And this is the matrix element we have to take the mod square and 1 by h cross square and here we are going from the initial state to the final state. So, upper state to the lower state, so this would be the downward transition, so let us say it is gamma -.

And we have this fluctuating environment represented by a + a dagger, so for the moment let me just write F here and we have to calculate S FF. And at frequency initial we are going from the higher energy state 1 phonon state to the 0 phonon state. So, therefore this would be simply the mechanical frequency of our system or the oscillator. We have to evaluate omega at omega m.

So, you see this part is easy to calculate because we know that when this annihilation operator operates on the 1 phonon state, we will get simply 0. And therefore because of this you can immediately see that this downward transition rate from here I will get the term g square.

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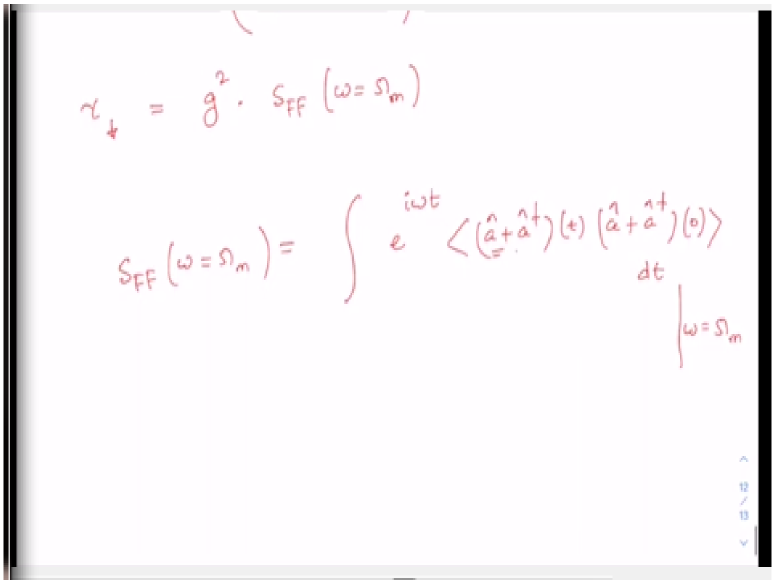

And we have of course only g square because h cross square will get cancelled out, so we will have g square and we have to calculate the spectrum at the oscillation frequency of the mechanical oscillator. Let us calculate it, so here S FF at omega = OMEGA m is equal to the Fourier transformation of the term a + a dagger, it is at time t and a + a dagger at time t = 0, we have to calculate it.

And finally, all the calculations have to be carried out at omega = OMEGA m. Now you note that this already as I said that this operator a is the quantum fluctuation corresponding to the optical mode and it is basically in the ground state.

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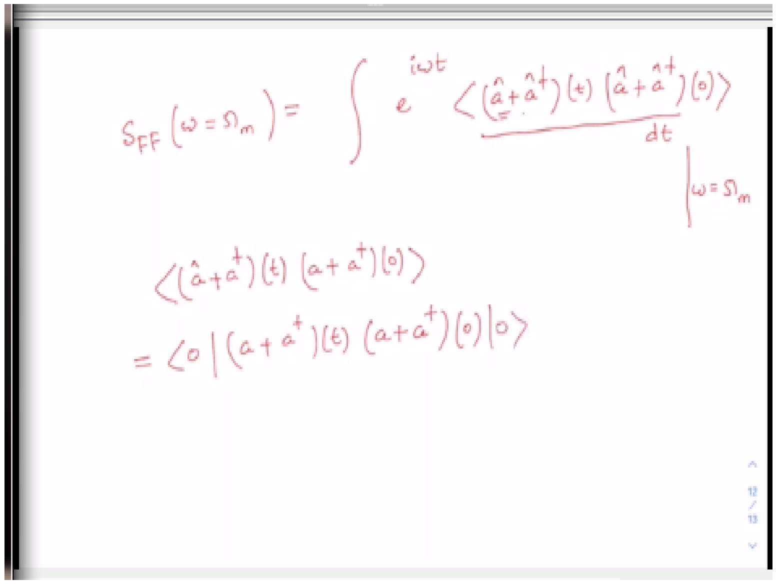

So, we have to calculate the expectation value of this particular term here a + a dagger at time t and a + a dagger at time t = 0, this we have to calculate with respect to the ground state. Because it is already in the optical oscillator is in the ground state because effectively it is at temperature 0 as you know because the optical frequency is in the around 10 to the power 15 hertz or so. So, that is the reason we can calculate it in the ground state. Now if we break it, ok, we will do this first.

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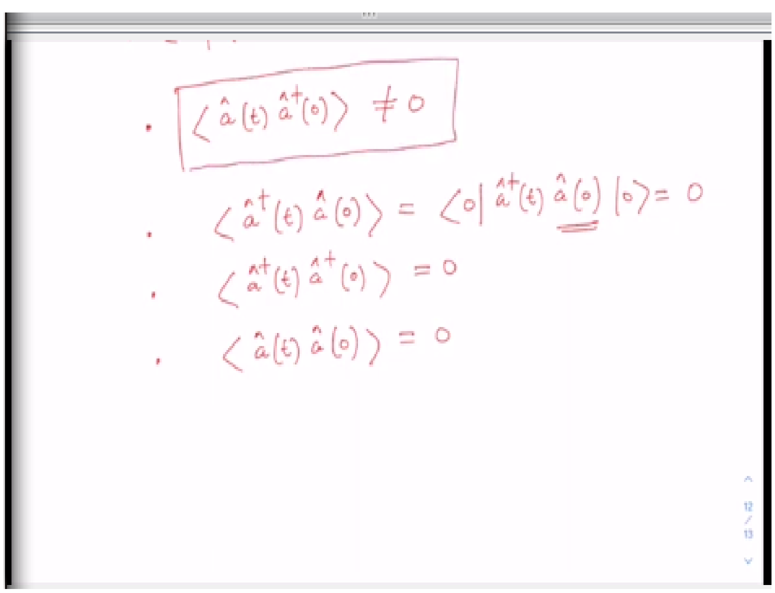

If we break it, we will get 4 terms and most of the terms will vanish except as you will see that only term that will remain non-zero would be a of t and a dagger of 0, this will remain non-zero but rest of the terms will 0. Let me show how? Actually, it is very trivial to see. For example, you have a dagger t and a of 0 and because you are calculating it in the ground state.

So, therefore you can immediately see that when this annihilation operator operates on the ground state you are going to get 0. Similar way you will have a dagger t, a dagger of 0 that is also going to give you 0. And then also you can see that you will have also a of t, a of 0 would be equal to 0. So, only term that would remain non-zero would be this one, so we have to calculate this as.

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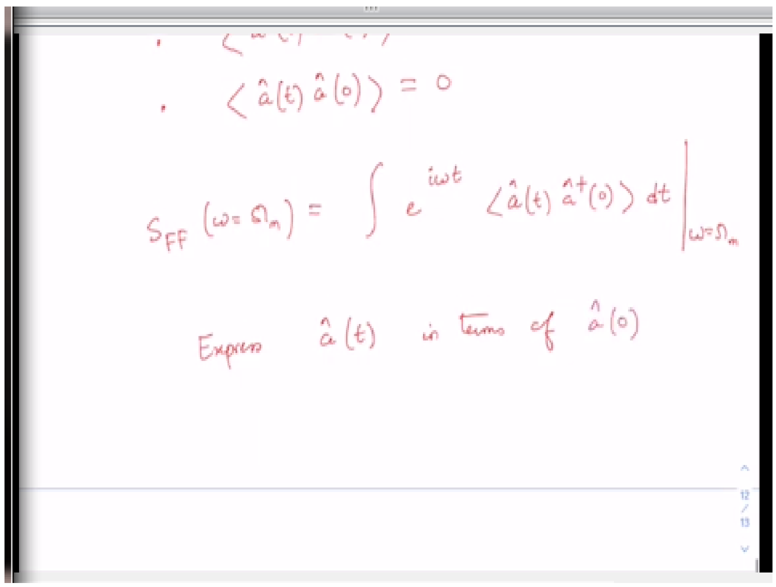

So, we will be left out with only one term and we have to calculate the Fourier transformation of this would be a of t, it would be a of t a dagger of 0 and dt evaluating here omega m, let us do it. But before we do that first we have to express this quantity express a of t in terms of a of 0 and that we can do it because we know the quantum Langevin equation for the optical oscillator.

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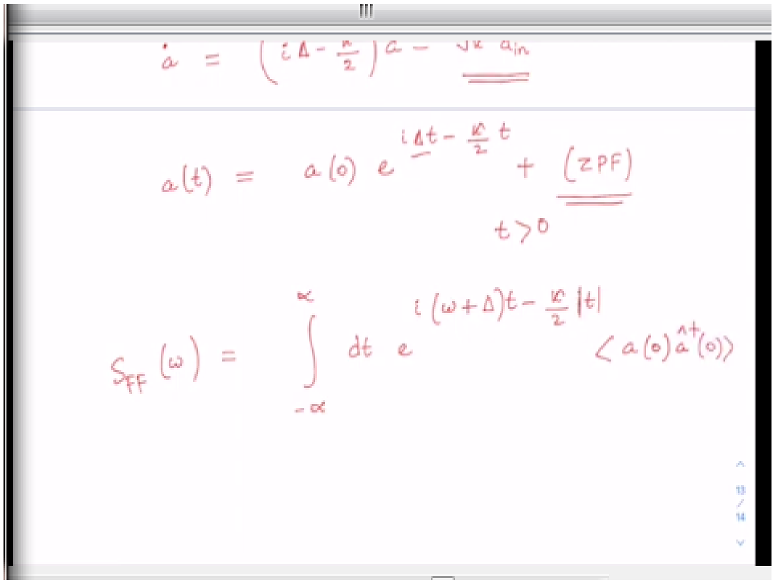

So, that is a dot is equal to in the absence of mechanics I can write it as - i delta kappa by 2 a - square root of kappa a in. And the solution is trivial and that is a of t would be equal to a of 0 e to the power i delta t - kappa by 2 t. And there will be terms belonging to the zero point fluctuation because of this. But as you will see that if we take the average then this fluctuation part is going to vanish and this we have written for t greater than 0.

And so therefore what we will have is this spectrum would be minus infinity to plus infinity dt e to the power i from the Fourier part we have i omega. And now here delta is there omega + delta t - kappa by 2 mod t a of 0 a dagger of 0. Now here I am taken more t because I have taken from the t less than 0 also is taken into account because the integration is from minus infinity to plus infinity. So, that is the reason we have put a mod sign here to take into both the cases, so let us now work it out.

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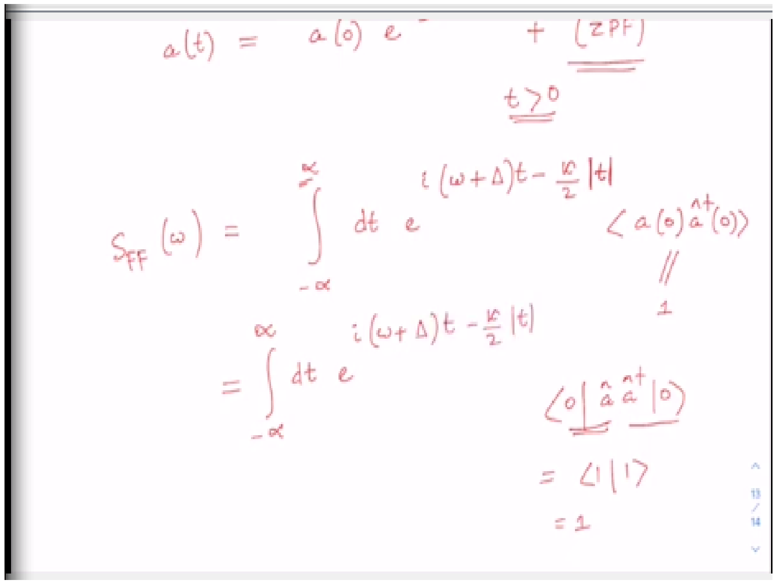

One thing you can immediately see that this quantity is simply equal to 1, because it is trivial to see. Let me just show you we have a, a dagger 0, so because of this you will get 1 and because of this when it operates on the bra part you will get 1 and that is equal to simply 1. So, we are left only to evaluate this very simple integration, it is dt e to the power i omega + delta t - kappa by 2 mod t.

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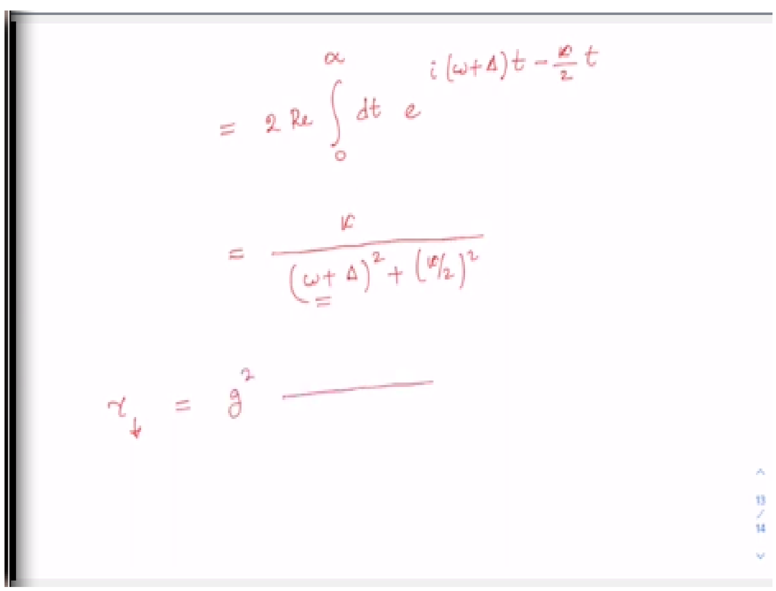

In fact, you see that this I can evaluate it very simply, let me do it here, I can take the integration from 0 to infinity then I have to border about the real part. So, you can see you have 2 into real part dt e to the power i omega + delta t - kappa by 2 t, that I can write. And if I evaluate this integral simple algebra actually and you will get it as kappa divided by omega + delta square + kappa by 2 whole square. Therefore, the downward transition rate, remember we have to evaluate it at OMEGA m.

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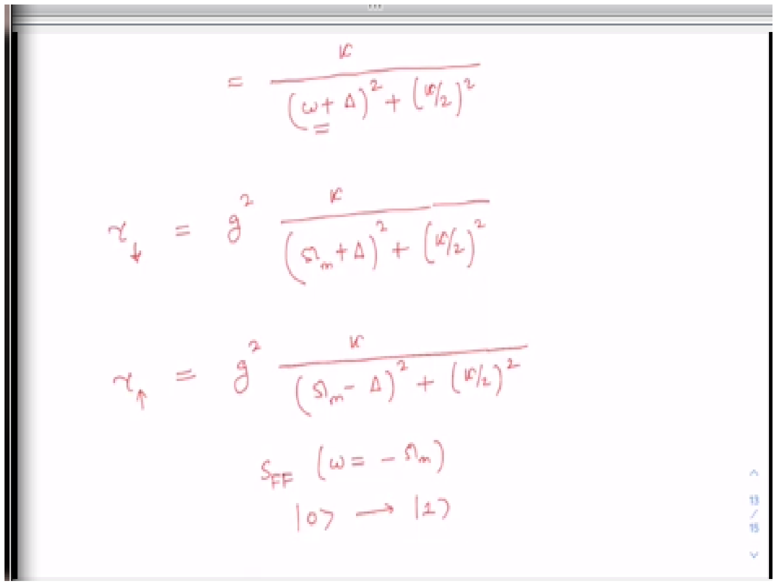

So, therefore I will have g square into kappa divided by omega m + delta whole square + kappa by 2 whole square. Exactly in the similar way we can calculate the upward transition rate gamma up and that would be equal to g square into kappa divided by omega m - delta whole square + kappa by 2 whole square. Now in this case we have evaluated the spectrum at the frequency - omega m because here we are going from the 0 phonon state to the 1 phonon state.

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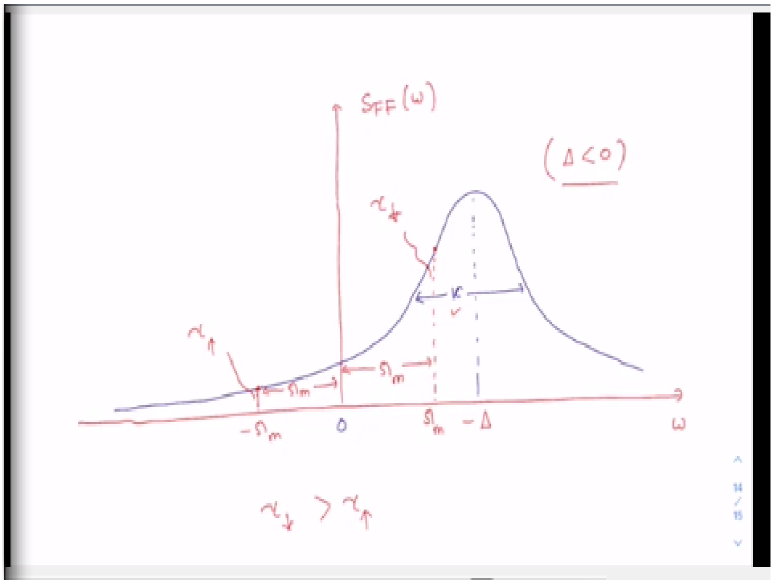

We can now plot the fluctuation spectrum as a function of frequency for delta less than 0 because we are interested in cooling. As you can see that this spectrum is Lorentzian and it has a maximum at - delta and the width is given by the cavity decay rate kappa. And if we want to obtain the downward transition rate as you have already seen that we have to evaluate the spectrum at frequency omega = + omega m.

Let us say we are interested in calculating the downward transition rate then we have to evaluate it omega m. So, from here this distance in frequency is omega m and this will give us the downward transition rate. On the other hand, the upward transition rate can be obtained if the spectrum is evaluated at the other end that is at frequency - omega m. So, this would be somewhere lying here in distance in frequency unit, so this would be - omega m, so this is our gamma up.

It is clear that from this plot as well that downward transition rate is higher than the upward transition rate. Let us now calculate the phonon numbers to which our mechanical oscillator settles down to. The idea is to calculate the average phonon number and see if we can somehow suppress the phonon numbers to 0 by optimizing the parameters of the optomechanical system achieving a pure ground state. But firstly, let us find out what is the phonon number in the steady state.

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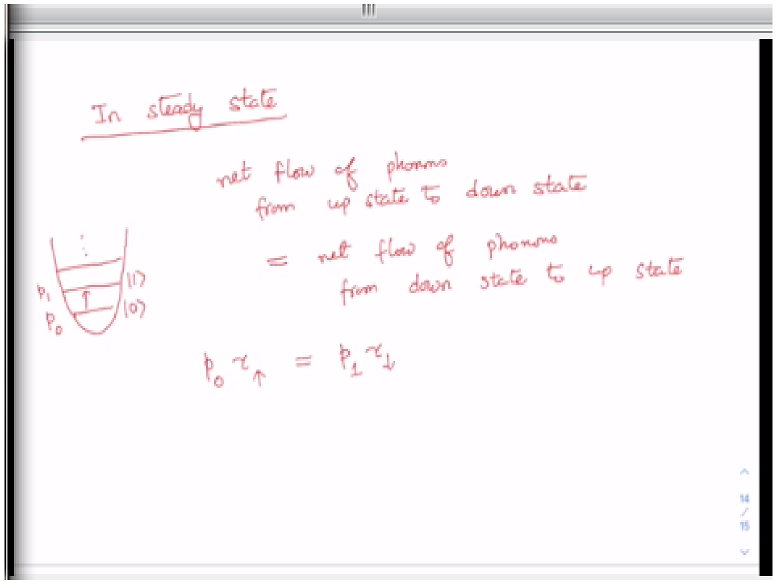

In the steady state the net flow of phonons from the upward state or upward state to the downward state. Let me write here net flow of phonons from upstate to downstate must be balanced by net flow of phonons from down state to upstate then we will accept the steady state. And in terms of probabilities we can write it as say we are in the ground state that is the probability of being in the ground state is say p 0.

We are having in the ground state and corresponding probability is p 0 and we go from the ground state to the upstate and upstate occupation probability is p 1. And this is the one problem state and so on, we have other state but we are confining our discussion to the 1 phonon state and the ground state only. So, it would be the net flow phonons from the upstate to the downstate would be p 0 gamma up and it has to be balanced by, now we are in the upstate that is p 1 and the rate of transition in the downward is given by gamma down.

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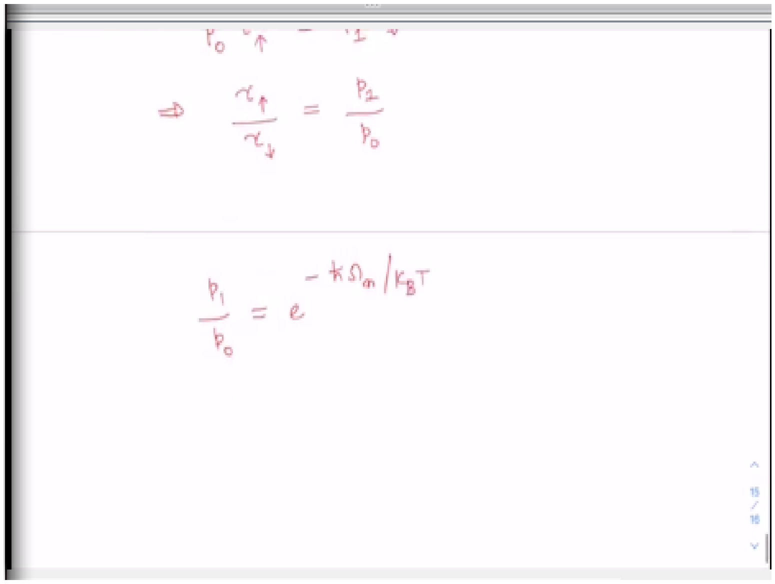

You can write down the ratio of this upward transition and the downward transition as p 1 divided by p 0 as is evident from this expression. Now invoking the Boltzmann distribution for probabilities p 1 by p 0 = e to the power - the energy difference between these 2 energy levels and that would be h cross omega m divided by K B T.

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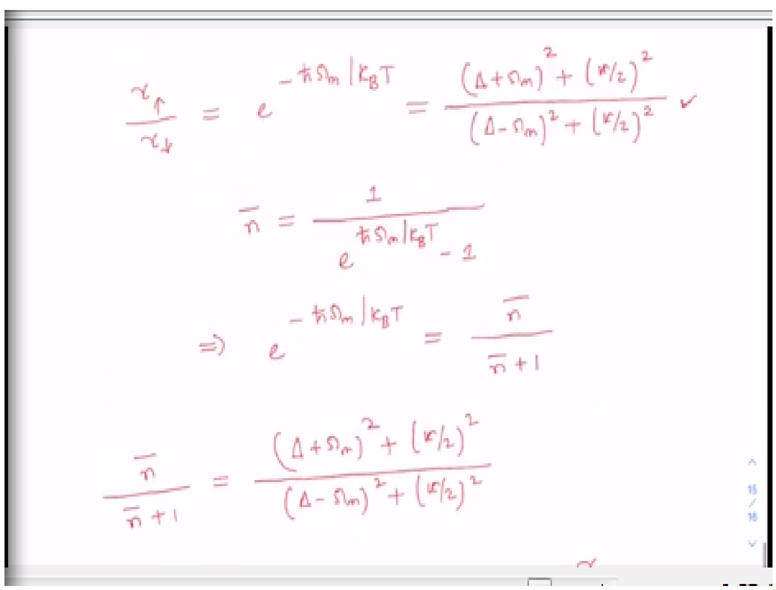

And from here we can immediately write that the ratio between the transition rate for the upward and the downward is equal to e to the power - h cross omega m by K B T. Again, we know what is the upward transition rate and the downward transition rate, already we have derived the corresponding expressions for them. So, if we put those terms here and we will get this ratio as delta + omega m whole square + kappa by 2 whole square divided by delta - omega m whole square + kappa by 2 whole square.

Also, we know that the average number of phonons is given by this expression, that is 1 divided by e to the power h cross omega m by K B T - 1. And from here we can quickly write that e to the power - h cross omega m by K B T = n bar divided by n bar + 1. So, we then can write that this ratio + n bar divided by n bar + 1 = delta + omega m whole square + kappa by 2 whole square, I am just repeating here.

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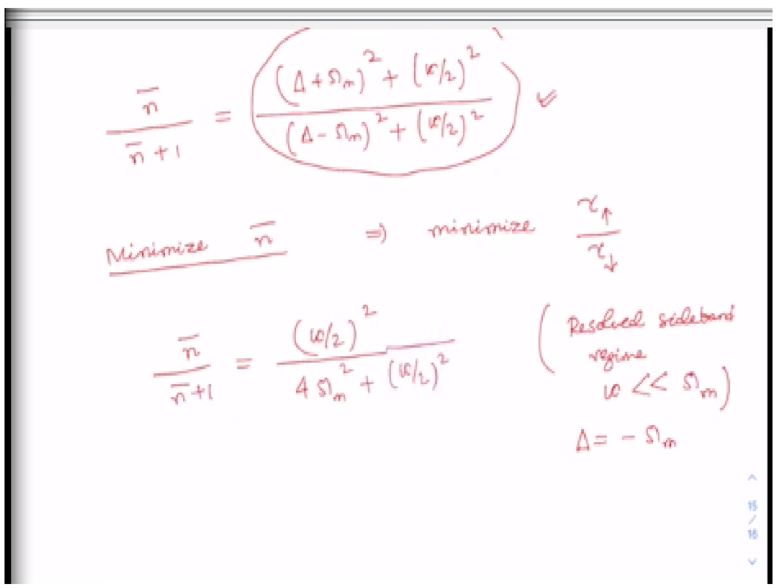

And then we have delta - omega m whole square + kappa by 2 whole square. So, now let us optimize the parameters because our goal is to minimize the average number of phonons, so we have to minimize n bar. So, to do that we have to minimize the ratio, so this implies that we have to minimize this ratio of gamma up and gamma down, this ratio of gamma up by gamma down.

That means we have to concentrate on this expression or then you will see that I can then write if I invoke the resolved sideband regime. If we work in the resolved sideband regime then I can write n bar by n bar + 1. By the way resolve sideband regime we have in that regime our kappa has to be much, much smaller than the resonance frequency of the mechanical oscillator.

And also, we are going to take delta = - omega m and if I take that I can write this expression as kappa by 2 whole square divided by 4 omega m square. As you can see I have just put delta = - omega m here and then here I have kappa by 2 whole square.

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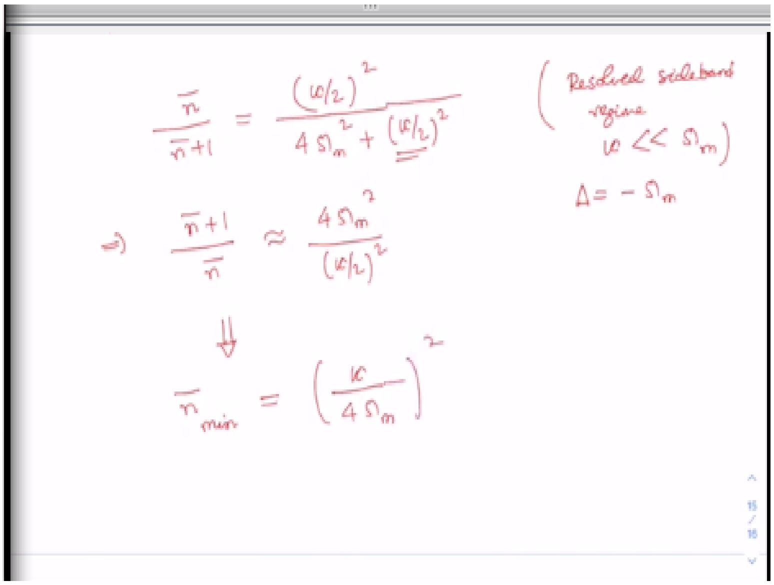

And we can write this as n bar + 1 divided by n bar is equal to invoking the resolved sideband regime. That means I can ignore this term and then I have here it will be 4 omega m square divided by kappa by 2 whole square. And therefore, from here you see that the optimized that means the minimum average number of phonons would turn out to be the ratio of kappa by omega m, so this is the expression we are going to have.

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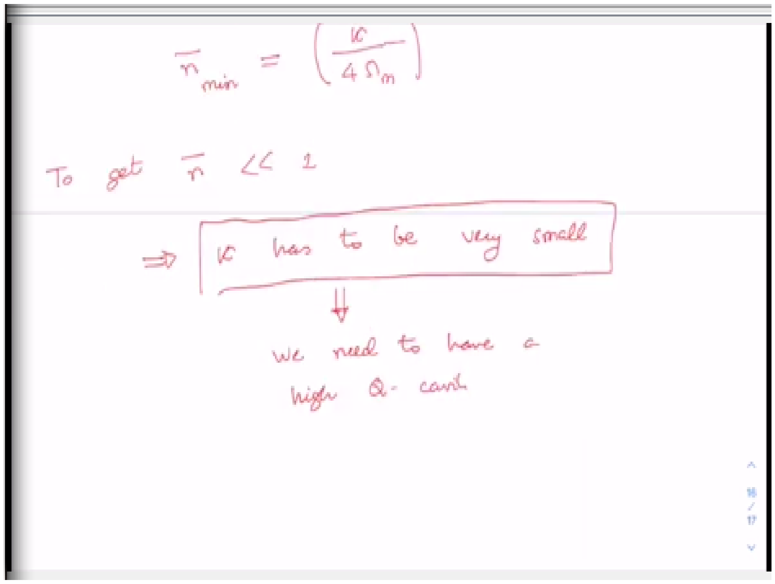

Now clearly if we want to have the phonon number to be much, much less than 1 to get n bar to be much, much smaller than 1, so that we can get the ground state of the mechanical oscillator. We need to make kappa has to be very small, lower the kappa or the cavity rate is the probability that we will reach into the ground state of the mechanical oscillator.

And lower kappa cavity decade means that we need to have a very high Q cavity. Let me stop here for today. In this lecture we have studied the linear response of the cavity optomechanical system around it is steady state and which led us to the regime of linearized cavity quantum optomechanics. Then we also studied the quantum limit for the ground state cooling of the mechanical oscillator.

In the next lecture we are going to study various other phenomena related to cavity quantum optomechanics such as squeezing and normal mode splitting and so on. So, see you in the next lecture, thank you.