Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture – 36
Problem Solving Session – 8.

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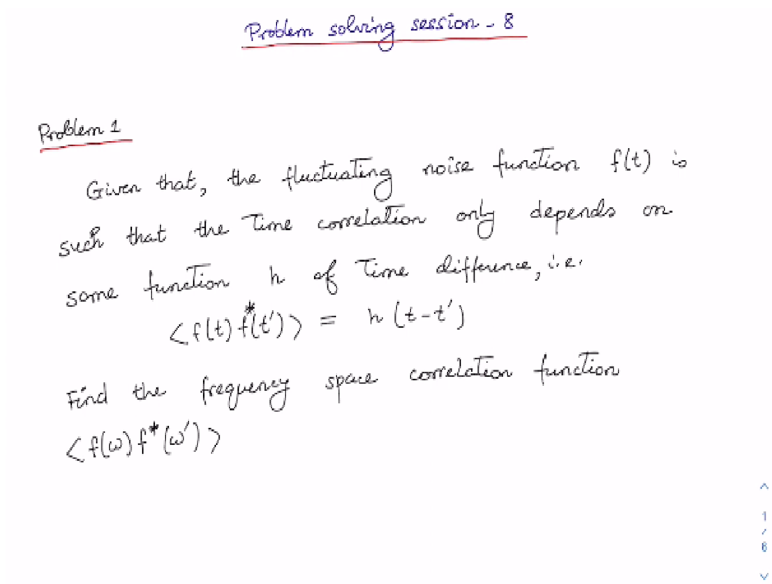

Welcome to the problem solving session number 8. In this problem solving session, we are going to solve some problems related to classical Langevin equation particularly the correlation functions and so on and also some basic cavity optomechanics. Now, the first problem, given that the fluctuating noise function f of t is such that the time correlation only depends on some function h of time difference, you are asked to find the corresponding frequency space correlation function.

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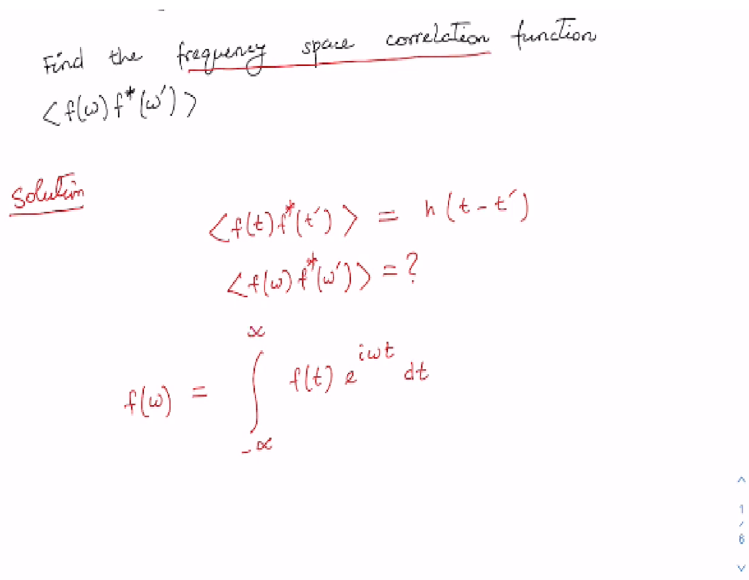

Let us do it. You are given the time correlation function for f, f is the fluctuating noise function and f star of t is the complex conjugate of the function f of t and this time correlation depends on only on a time difference, the function such that it is basically a function of the time difference, we are asked to find out what is f of omega and f star of omega dash. Now, we know that f of omega is the Fourier transformation of this function f of t.

So, the Fourier transformation we can write it in this form, integration from minus infinity to plus infinity. So, the Fourier transform of the function f start of t, that is, the complex conjugate of f of t that would be equal to again in the similar way it would be f star of omega.

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Thus:

$$\langle f(\omega) f^*(\omega') \rangle = \left\langle \left(\int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \right) \left(\int_{-\infty}^{\infty} f^*(t') e^{i\omega' t'} dt' \right) \right\rangle$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt\, dt' \, \langle f(t) f^*(t') \rangle e^{i(\omega t + \omega' t')}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt\, dt' \, h(t - t') \, e^{i(\omega t + \omega' t')}$$

Therefore, because the definition of Fourier because it is just a function and this is also function and kernel is e to the power i omega t as you know. So, therefore, we can now write this frequency correlation f of omega f star of omega dash, I can write it as f of omega we know that would be minus infinity to plus infinity f of t e to the power i omega t dt. And the other one is minus infinity to plus infinity f star of t dash e to the power i omega dash t dash dt dash.

We can now write it in this way. We have this integration minus infinity to plus infinity minus infinity to double integration is there. We have these dt dt dash average of f of t f star of t dash e to the power i omega t + omega dash t dash, this is I think very straightforward. Now, this is already given that it is a function of h, it is minus infinity to plus infinity here minus infinity to plus infinity dt dt dash and this guy here is h t - t dash e to the i omega t + omega dash t dash.

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Take:

$$t = t' + \tau \qquad t' = t' \quad \longrightarrow \quad J = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(t, t') \rightarrow (\tau, t') \qquad |\det(J)| = 1$$

So, $dt\, dt' \longrightarrow dt'\, d\tau$

$$\langle f(\omega) f^*(\omega') \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt'\, d\tau\, h(\tau)\, e^{i\omega(t'+\tau)}\, e^{i\omega' t'}$$

I can make my life simple if I go to some different variables because I know that I have here time differences is there. So, let us say I have t = t dash + tau and let me keep t dash = t. So, I mean to say that we are now going from the variable t, t dash to tau, t dash. So, now, if you can see that from here the Jacobian of transformation here I can write it as 1 1 1 0. So, if I take the magnitude of the determinant of this Jacobian there is you can see that will be simply equal to 1.

So, therefore, I can have this dt dt dash I can immediately write it as dt dash d tau. So, this will lead me to this integration. I am going to, now the variables, let me actually write here again, let me do it this way, let me write f of omega f star of omega dash, that would be equal to minus infinity to plus infinity, minus infinity to plus infinity. Now, I have dt dash d tau and here this function I can write it as h of tau e to the power i omega t I am now going to replace it by t dash + tau and I have e to the power i omega dash t dash.

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$$\langle f(\omega) f^{*}(\omega') \rangle = \int_{-\infty} \int_{-\infty} dt' d\tau \, h(\tau) \, e \qquad e$$

$$= \int_{-\infty}^{\infty} dt' \, e^{i(\omega+\omega')t'} \int_{-\infty}^{\infty} d\tau \, h(\tau) \, e^{i\omega\tau}$$

$$\Rightarrow \boxed{\langle f(\omega) f^{*}(\omega') \rangle = 2\pi \, \delta(\omega+\omega') \, S_{ff^{*}}(\omega)}$$

where, $$S_{ff^{*}}(\omega) = \int_{-\infty}^{\infty} d\tau \, h(\tau) \, e^{i\omega\tau}$$

Now, I can write minus infinity to plus infinity dt dash e to the power i omega + omega dash t dash and minus infinity to plus infinity d tau h of tau e to the power i omega tau, I think it is very simple and as you know this guy is nothing it is related to the theta delta function and that would be 2 pi delta omega + omega dash and let me define this quantity as this function S ff star of omega where I am writing S ff star of omega is the spectral noise density and this is the Fourier transformation of this time correlation function h of tau.

And you know that this is nothing but the Wiener Khinchin theorem. And in fact, I can now write using this expression let me write here f of omega f star of omega dash that we utilize this.

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where, $S_{ff^*}(\omega) = \int_{-\infty} d\tau \, h(\tau)\, e$

$$\int \langle f(\omega) f^*(\omega') \rangle \, d\omega' = 2\pi \int \delta(\omega + \omega') \underbrace{S_{ff^*}(\omega)} \, d\omega'$$

$$\Rightarrow \frac{1}{2\pi} \int \langle f(\omega) f^*(\omega') \rangle \, d\omega' = S_{ff^*}(\omega) \underbrace{\int_{-\infty}^{\infty} \delta(\omega + \omega') \, d\omega'}_{=1}$$

$$\Rightarrow S_{ff^*}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle f(\omega) f^*(\omega') \rangle \, d\omega'$$

And using this what I can do if I integrate both sides by over omega dash f of omega f star of omega dash d omega dash and then I will have here 2 pi delta omega + omega dash S ff star omega d omega dash. Now this is a function of only omega. So, I can therefore write the whole thing if I take 2 pi to the other side then I have 1 / 2 pi integration, f of omega f star of omega dash d omega dash that would be equal to S ff star omega integration delta omega + omega dash d omega dash.

And you know that integration on over all these things it is simply going to give you 1. So, this spectral noise density I can write it as equal to 1 / 2 pi integration minus infinity to plus infinity f of omega f star of omega dash t omega dash. So, actually what we were asked? We are asked to find out this quantity and so, our answer actually we have worked it out. So, this is what we have.

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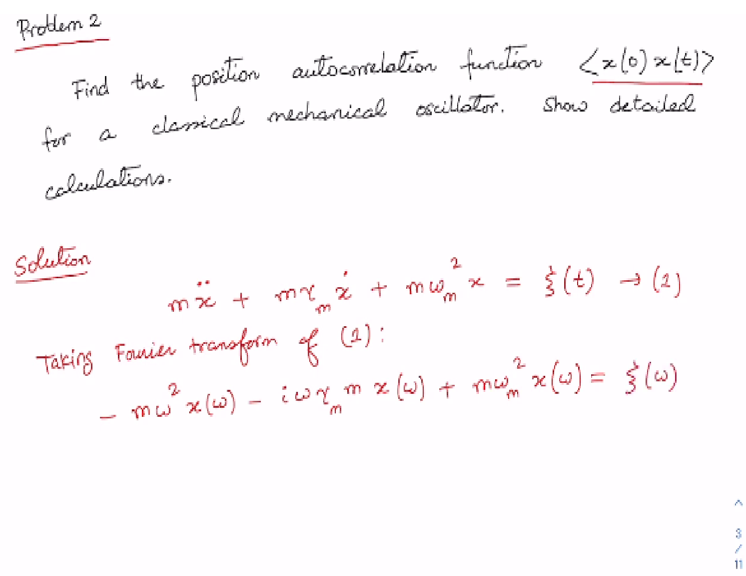

Let us now work out this problem. Find the position autocorrelation function for a classical mechanical oscillator, you have to show detailed calculations. So, let us do it. To do this problem, we need to start with the classical Langevin equation for the harmonic oscillator which has been taught to you in the lecture class and classical Langevin equation for the harmonic oscillator is m x double dot + m gamma m x dot, gamma m is the decay rate of the harmonic oscillator + m omega m square x.

Omega m is the resonance frequency of the harmonic oscillator and this is equal to the so-called Langevin force. Rather than dealing with in the time domain we can go into the frequency domain that will make our calculations easier. And to do that, we can take the Fourier transformation of this equation 1 and taking Fourier transformation we can write equation one in this form that will be minus taking Fourier transform of 1 we can write minus m omega square x of omega, x of omega is the Fourier transform of the position variable.

I have - i omega gamma m, m is also there, x of omega + m omega m square x of omega, that is equal to Xi of omega. So, this is in the frequency domain.

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$$m\ddot{x} + m\gamma_m \dot{x} + m\omega_m^2 x = \xi(t) \rightarrow (1)$$

Taking Fourier transform of (1):

$$\boxed{-m\omega^2 x(\omega) - i\omega\gamma_m m\, x(\omega) + m\omega_m^2 x(\omega) = \xi(\omega)} \rightarrow (2)$$

$$x(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{i\omega t}\, dt$$

$$m\left[\omega_m^2 - \omega^2 - i\omega\gamma_m\right] x(\omega) = \xi(\omega)$$

And x of omega is the Fourier transform of the position variable and that is integration minus infinity to plus infinity, x of t is the position in the time domain e to the power i omega t dt. Now, we are going to analyze this equation. Let us say this is equation number 2, this equation I can write in this form. I can write it as m omega m square - omega square - i omega gamma m x of omega that is equal to Xi of omega.

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$$\Rightarrow \quad x(\omega) = \left\{\frac{1}{m\left[\omega_m^2 - \omega^2 - i\gamma_m\omega\right]}\right\}\xi(\omega)$$

$$\boxed{x(\omega) = \chi(\omega)\,\xi(\omega)} \rightarrow (3)$$

$$\chi(\omega) = \left[m\left(\omega_m^2 - \omega^2 - i\gamma_m\omega\right)\right]^{-1} \rightarrow (4)$$

$$x(\omega)\, x(\omega') =$$

Or I can write it as x of omega = 1 divided by m into omega m square - omega square - i gamma m omega into xi of omega, this quantity we can name it as the mechanical susceptibility and this is denoted as a chi of omega. So, x of omega = chi of omega into Xi of omega, Xi of omega is the Fourier transform of the Langevin noise or Langevin force and where chi of omega is the mechanical susceptibility and this is m into omega m square - omega square - i gamma m omega to the power -1.

Now, let us find out what is, say, x of omega x of omega dash the product of these 2 functions that would be as you can see from here let me say this is my equation number 3, this is equation number 4.

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$$\chi(\omega) = \left[m\left(\omega_m^2 - \omega^2 - i\gamma_m\omega\right)\right]^{-1} \rightarrow (4)$$

$$\boxed{\langle x(\omega)\, x(\omega')\rangle = \chi(\omega)\, \chi(\omega') \langle \xi(\omega)\, \xi(\omega')\rangle}$$

$$\langle f(\omega)\, f(\omega')\rangle = 2\pi\, \delta(\omega+\omega')\, S_{ff}(\omega)$$

From equation 3 you can see that I can have chi of omega chi of omega dash xi of omega xi of omega dash. Now if I take the average, then the average would be on this variable only. So, this is an important equation we get. Now from the previous problem what we have got there we have f omega f of omega dash the average of this quantity that is a frequency correlation is equal to 2 pi delta omega + omega dash S ff this is a spectral density S ff omega.

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$$\boxed{\langle x(\omega)\, x(\omega') \rangle = \chi(\omega)\, \chi(\omega') \langle \xi(\omega)\, \xi(\omega') \rangle} \rightarrow (5)$$

$$\langle f(\omega)\, f(\omega') \rangle = 2\pi\, \delta(\omega+\omega')\, S_{ff}(\omega) \rightarrow (6)$$

Using (6) in (5):

$$2\pi\, \delta(\omega+\omega')\, S_{xx}(\omega) = \chi(\omega)\, \chi(\omega')\, 2\pi\, \delta(\omega+\omega')\, S_{\xi\xi}(\omega)$$

So, we can utilize it and using this we can now write let me say from equation 5 I can using this equation say 6, using 6 in 5 we can write it is 2 pi delta omega + omega dash S xx of this is the position spectral density function this is equal to chi of omega chi of omega dash 2 pi delta omega + omega dash S Xi Xi of omega that is the spectral noise density for Langevin function.

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Integrating both sides

$$\int \delta(\omega+\omega')\, S_{xx}(\omega)\, d\omega' = \int \chi(\omega)\, \chi(\omega')\, \delta(\omega+\omega')\, S_{\xi\xi}(\omega)\, d\omega'$$

$$\Rightarrow S_{xx}(\omega) \int \delta(\omega+\omega')\, d\omega'$$

So, if I integrating both sides over frequency omega dash I can write delta omega 2 pi 2 pi will get cancelled out because this is on the both sides delta omega omega dash S xx omega d omega dash that would be equal to chi of omega as you can see, this is very straightforward and you have to do the calculations methodically you have your delta omega + omega dash S Xi Xi of omega d omega dash. So, integration is over the frequency variable omega this so, therefore, I can write S xx omega integration delta omega + omega dash d omega dash.

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$$\Rightarrow S_{xx}(\omega) \underline{\underline{\int \delta(\omega+\omega')\, d\omega'}}$$

$$= S_{\xi\xi}(\omega) \int \underline{\chi(\omega)\, \chi(\omega')\, \delta(\omega+\omega')\, d\omega'}$$

$$\Rightarrow S_{xx}(\omega) = S_{\xi\xi}(\omega)\, \underline{\underline{\chi(\omega)}}\, \underline{\underline{\chi(-\omega)}}$$

$$\Rightarrow \boxed{S_{xx}(\omega) = |\chi(\omega)|^2\, S_{\xi\xi}(\omega)}$$

That is equal to S xi xi omega I can take it out because this is only dependent on the frequency variable omega not omega dash, but here I have chi omega chi omega dash delta omega + omega dash d omega dash. So, this one is obviously equal to 1. So, therefore, I have S xx of omega that is equal to, now, I can use the property of the diract delta function here and then I will have it will be S xi xi of omega chi of omega into chi of - omega as you can see that these 2 function this chi of - omega is the complex conjugate of chi of omega.

So, therefore, I can write S xx of omega = modulus of chi omega square S xi xi of omega. So, this shows how the spectral density for the position is related to the spectral density of the Langevin noise or Langevin force. So, now, you see since, the oscillator position is drive by Langevin noise it is also a stationary variable.

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$$S_{xx}(\omega) = \int \langle x(t)\, x(0) \rangle\, e^{i\omega t}\, dt$$

$$\Rightarrow \quad \langle x(t)\, x(0) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{\underline{S_{xx}(\omega)}}\, e^{-i\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, |\chi(\omega)|^2$$

Now, according to Wiener Khinchin theory which we discussed in the class, we know that this spectral density for the position S xx omega is nothing but the Fourier transform of the correlation x of t x of 0 e to the power i omega t dt. So, this is the S xx. So, S xx this particular function is the Fourier transform the correlation function so, from here we can write x of t x of 0 = 1 / 2 pi that is the inverse Fourier transform I am writing that will be S xx of omega e to the power - i omega t d omega integration limit is from minus infinity to plus infinity.

Now, then I can write one by 2 pi integration minus infinity to plus infinity d omega i know that this is related to the Langevin noise that is by this relation chi of omega square yes this is what we have written.

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$$\Rightarrow \quad \langle x(t)\,x(0)\rangle = \frac{1}{2\pi}\int_{-\infty} \ldots$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, |\chi(\omega)|^2\, S_{\xi\xi}(\omega)\, e^{-i\omega t}$$

$$S_{\xi\xi}(\omega) = 2m\gamma_m k_B T$$

This would be S Xi Xi of omega e to the power -i omega t. Now, you recall from our lectures 27 that S Xi Xi of omega, the spectral noise density is equal to twice m gamma m K B T.

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$$\boxed{\langle x(t)\,x(0)\rangle = \frac{\gamma_m k_B T}{\pi m}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}\, d\omega}{(\omega_m^2-\omega^2)^2+(\gamma_m\omega)^2}}$$

$$\oint \frac{e^{-i\omega t}}{(\omega_m^2-\omega^2)^2+(\gamma_m\omega)^2}$$

So, using this I can write the correlation function x of t x of 0 = gamma m K B T divided by pi m integration minus infinity to plus infinity and I have here e to the power - i omega t and I just have to put chi of modulus of chi omega square. So, if I put it then I will get omega m square - omega square whole square + gamma m omega whole square. So, this is what ultimately I have. Now, as you can see the whole problem now boils down to solving this particular integral.

So, now, let us do that well there should be a d omega term should also be there. I assume that all of you know complex analysis and know how to solve contour integration. Now, let us consider these contour integration e to the power - i omega t omega m square - omega square whole square + gamma m omega whole square.

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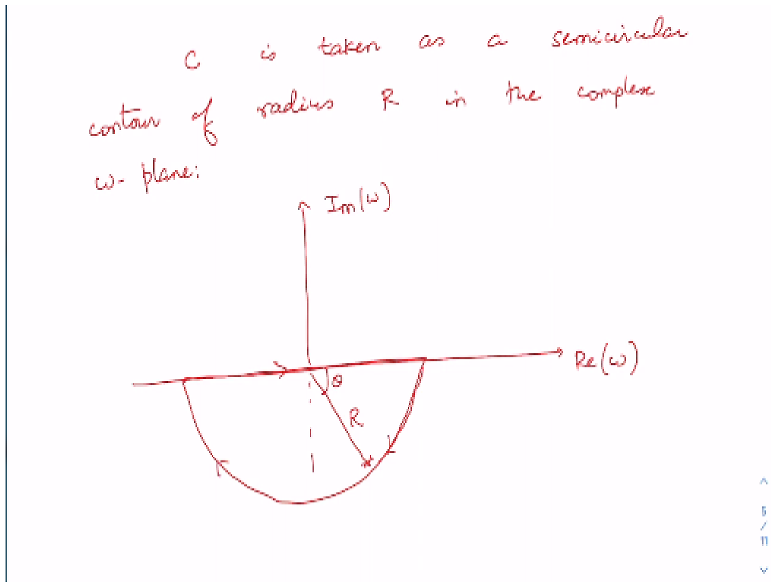

So, this contour C we will take it as a, let me right here, contour C is assumed or taken as a semi circular. I will give you the diagram, semi circular contour of radius R in the complex omega w plane, so, the contour I am taking of this form, so, I have this is a my real axis real of omega then this is imaginary omega I take a contour of this type. So, from here I go this side and then go this way and so, this has a radius the semicircle has a radius R this is theta. So, this is what I have. This is my contour. Now, the poles you have to work out, poles of this function I think all of you know how to work out what is this call contour integration.

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$$F(\omega) = \frac{e^{-i\omega t}}{(\omega_m^2-\omega^2)^2+(\gamma_m\omega)^2}$$

Poles of $F(\omega)$ are at ω- planes for which $\boxed{\omega_m^2-\omega^2 = \pm i\omega\gamma_m}$

So, first of all we have this function f of omega that is e to the power this is my function. So, let me write it as f of omega = e to the power - i omega t omega m square - omega square whole square + gamma m omega whole square this is my function and we have to find out the poles of this function. So, poles of f of omega are at w values or omega values or w values for which you will see that omega m square - omega square = + - i omega gamma m because for this when this equation is satisfied, then this particular function blows up. So, these are the poles.

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We have poles at:

$\omega = -\frac{i\gamma_m}{2} + \omega_m'$

$\omega = -\frac{i\gamma_m}{2} - \omega_m'$

$\omega = \frac{i\gamma_m}{2} + \omega_m'$

$\omega = \frac{i\gamma_m}{2} - \omega_m'$

- $\omega^2 + i\omega\gamma_m - \omega_m^2 = 0$

$$\Rightarrow \omega = \frac{-i\gamma_m \pm \sqrt{-\gamma_m^2+4\omega_m^2}}{2}$$

$$= \frac{-i\gamma_m}{2} \pm \omega_m'$$

$$\omega_m' = \sqrt{\omega_m^2 - \frac{\gamma_m^2}{4}}$$

So, if we solve this equation that will give us the poles so, we have poles at omega = - I, this is very easy to solve you can do it, i gamma m / 2 + omega m dash. I will tell you what is omega m dash or better why not let us quickly solve it I have omega squared I have to just solve this equation omega square if I let me just solve one out of this there are 2 equation out of these 2 because let me just solve quickly 1 equation say omega squared + i omega gamma m - omega m squared = 0 this quadratic equation would have the solution.

Omega = -i gamma m + - - gamma m square + 4 omega m square divided by 2 and these I can write as -i gamma m by 2 + - omega m dash where omega m dash is square root of omega m square - gamma m square / 4. So, as you will see that from 1 equation, I have 2 poles and from the another equation I will have another pole so, there will be in total 4 poles. So, this is one pole another pole would be at omega = - i gamma m / 2 - omega m dash. Then you will have omega = i gamma m / 2 + omega m dash and finally, you will have omega = i gamma m / 2 - omega m dash.

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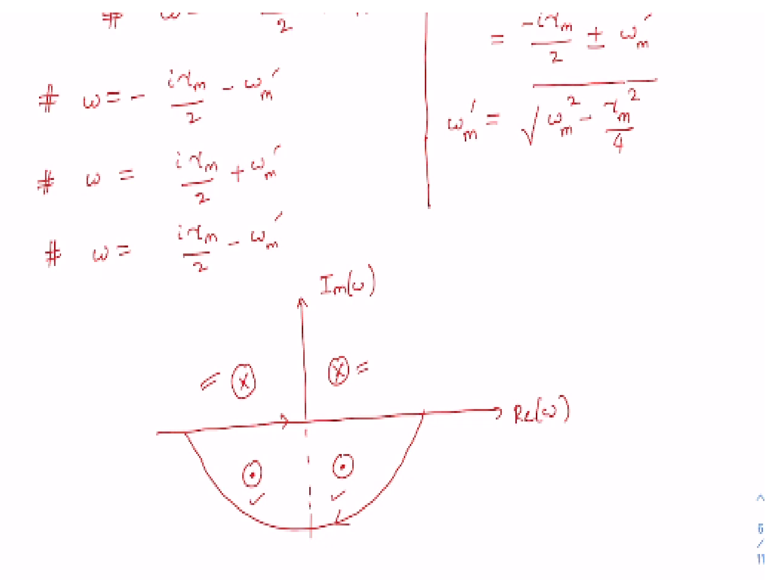

So, if you try to locate the poles in this contour plane you will have here this is the imaginary omega, this is real omega and this is my contour. So, as you will see, inside this contour, I have 2 poles one pole is located here the other one is symmetrically located to the other side and the other 2 poles one is located here and one is located here as these 2 poles are outside this contour so, we will not bothered about it, will bother only about this these 2 poles and because we are going to apply the so called residue theory.

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$$\oint_C F(\omega)\,d\omega = \lim_{R\to\infty}\int_{-R}^{R} d\omega\, F(\omega) \qquad \Big|\ \omega = Re^{i\theta}$$

$$+\underbrace{\lim_{R\to\infty}\int_{0}^{-\pi} d\theta\, F\left(Re^{i\theta}\right) iRe^{i\theta}}_{=\,0}$$

Now, we have the integration is this, we have this contour integration F of omega d omega is equal to in the limit say R tends to infinity we have from you see you can go from -R to +R. So, this is your R say -R to +R d omega F of omega this is not a complex analysis class so I am being very brief here and then we are having limit R tends to infinity integration. Now if you go by this semicircle thing integration around the semicircle you go from 0 to - pi d theta F of R e to the power i theta.

Because omega inside this semicircle I am taking it as R e to the power i theta and then I will have i R e to the power i theta. So, omega I am taking as inside this semi circle I am taking omega as R e to the power i theta and this integration is in the clockwise direction. So, in fact, you will find that using the so called Jordan's lemma, this contour contribution from this integral will go to 0.

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Thus, $\boxed{\oint_C F(\omega)\,d\omega} = \int_{-\infty}^{\infty} d\omega\, F(\omega)$

$$\oint_C F(\omega)\,d\omega = 2\pi i \left(\text{Sum of residues}\right)$$

So, therefore, we will have to bother about or we left out with integration F of omega d omega, this would be equal to when I take the limit R tends to infinity + - infinity I will have minus infinity to plus infinity d omega F omega. So, solving the integral ultimately boils down to solving this contour integration as you can see, and now, we can apply the so called residue theorem as you know that this complex integration F of omega d omega = 2 pi i into the sum of residues and we have 2 poles.

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$$\oint_C F(\omega)\,d\omega = -2\pi i \left(\text{Sum of residues}\right)$$

Residue at $\omega = -\frac{i\gamma_m}{2} + \omega_m' = \underline{\underline{R_1}}$

Residue at $\omega = -\frac{i\gamma_m}{2} - \omega_m' = \underline{\underline{R_2}}$

So, we have to calculate the residues, residue at omega = one pole the pole that lie inside the contour one is – i gamma m / 2 + omega m dash and let us say this is a residue R 1. And another one we have to work it out at residue at omega = - i gamma m / 2 - omega m dash. So, this is residue 2. So, these 2 residues we have to work out. Now, only thing you have to keep in mind is that when I have taken this contour integration, I am going in a clockwise direction and because of that, I have to take a minus sign here.

So, we will do that. So, first let us work out what is R 1 and what is R 2. I will just show you the calculation for R 1 R 2 you can do it in the similar.

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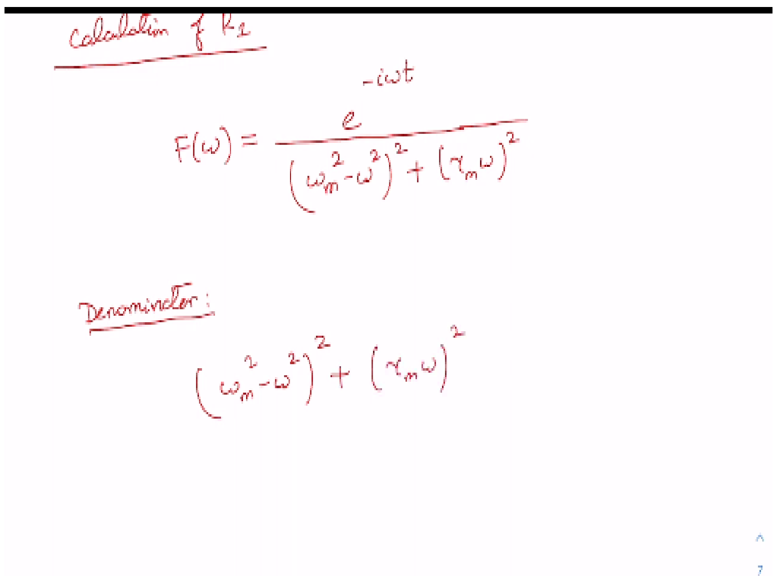

Before I do the calculation let me first simplify this function F of omega, F of omega = e to the power - i omega t omega m square - omega square whole square + gamma m omega whole square. So, this denominator let me simplify first denominator that is omega m square - omega square whole square + gamma m omega whole square.

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$$= \left(\omega^2 - \omega_m^2 - i\gamma_m \omega\right)\left(\omega^2 - \omega_m^2 + i\gamma_m \omega\right)$$

$$\boxed{\omega_m'^2 = \omega_m^2 - \frac{\gamma_m^2}{4}}$$

$$= \left[\left(\omega - \frac{i\gamma_m}{2}\right)^2 - \omega_m'^2\right]\left[\left(\omega + \frac{i\gamma_m}{2}\right)^2 - \omega_m'^2\right]$$

$$= \left(\omega + \omega_m' - \frac{i\gamma_m}{2}\right)\left(\omega - \omega_m' - \frac{i\gamma_m}{2}\right)\left(\omega + \omega_m' + \frac{i\gamma_m}{2}\right)\left(\omega - \omega_m' + \frac{i\gamma_m}{2}\right)$$

These I can write as omega square - omega m square - i gamma m omega into omega square - omega m square + i gamma m omega. Now, we know that omega m dash square = omega m square - gamma m square / 4. So, if I utilize it in this expression, so, let me just simple manipulation you have to do very straightforward we will get omega - i gamma m / 2 whole square - omega m dash square into you will have let me take it then you will have omega + i gamma m / 2 whole square - omega m dash square.

These you can further write as omega + omega m dash - i gamma m / 2 into omega - omega m dash - i gamma m / 2 into omega + omega m dash + i gamma m / 2 into there will be 4 terms in total product of 4 terms omega - omega m dash + i gamma m / 2.

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$$\left(\omega - \omega_m' + \frac{i\gamma_m}{2}\right)$$

$$R_1 = \lim_{\omega \to -\frac{i\gamma_m}{2} + \omega_m'} \left(\omega + \frac{i\gamma_m}{2} - \omega_m'\right) \frac{e^{-i\omega t}}{\left[\left(\omega+\omega_m' - \frac{i\gamma_m}{2}\right)\left(\omega-\omega_m' - \frac{i\gamma_m}{2}\right)\left(\omega+\omega_m' + \frac{i\gamma_m}{2}\right)\left(\omega-\omega_m' + \frac{i\gamma_m}{2}\right)\right]}$$

$$\Rightarrow R_1 = \frac{e^{-i\left(\omega_m' - \frac{i\gamma_m}{2}\right)t}}{\left(2\omega_m' - i\gamma_m\right)\left(-i\gamma_m\right)\left(2\omega_m'\right)}$$

So, residue at R 1 that is the residue at omega is equal to let me write now workout first one that will be limit omega tends to you are going to calculate the residue at - i gamma m / 2 + omega m dash and this would be omega + i gamma m / 2 - omega m dash and then this function F of omega is there so e to the power - i omega t and these whole all these terms you have to put and I think let me write here anyway.

You will have omega + omega m dash - i gamma m / 2 into omega - omega m dash - i gamma m / 2 into omega + omega m dash + i gamma m / 2 into omega - omega m dash + i gamma m / 2 as you can see, this way, you can very easily calculate it you just put the numbers there then he will get R 1 let me write the final expression you are going to get R 1 as e to the power - i omega m dash - i gamma m / 2 into t divided by twice omega m dash - i gamma m into - i gamma m twice omega m dash this is what we will have as R1.

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Sly, $\quad R_2 = \dfrac{e^{-i\left(-\frac{i\gamma_m}{2} - \omega_m'\right)t}}{\left(-2\omega_m' - i\gamma_m\right)\left(-2\omega_m'\right)\left(-i\gamma_m\right)}$

$$\int_{-\infty}^{\infty} F(\omega)\, d\omega = -2\pi i \left(R_1 + R_1\right)$$

Similarly, please verify it yourself you will get R 2 the residue at the other pole R 2 would be equal to e to the power - i - i gamma m / 2 - omega m dash into t divided by - twice omega m dash - i gamma m into - twice omega m dash - i gamma m. So, this is what you will get. So, now, we can work out this integration. So, integration minus infinity to plus infinity f of omega d omega is 2 pi i into some of the residues I think this is a little bit algebra, but straightforward algebra.

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$$\int_{-\infty} F(\omega)\, d\omega = -2\pi i \left(R_1 + R_2\right)$$

$$= \frac{\pi e^{-\gamma_m t/2}}{\gamma_m \omega_m^2}\left[\cos\omega_m' t + \frac{\gamma_m}{2\omega_m'} \sin\omega_m' t\right]$$

Finally,

$$\langle x(t)\, x(0)\rangle = \frac{\gamma_m k_B T}{\pi m} \, \frac{\pi e^{-\gamma_m t/2}}{\gamma_m \omega_m^2}\left(\cos\omega_m' t + \frac{\gamma_m}{2\omega_m'} \sin\omega_m' t\right)$$

Please do that users have to add these 2 terms and do the manipulation then finally, you should get pi e to the power - gamma m t / 2 divided by gamma m omega m square then here you will have cos omega m dash t + gamma m divided by twice omega m dash sin omega m dash t so, this is what we will get. So, therefore, finally, we can now obtain, because the integration we have walked out. So, we have this autocorrelation for the position x of t x of 0 is equal to we had this term gamma m K B T / pi m and that integral was there.

So, integral now, we have worked out that is pi e to the power - gamma m t / 2 gamma m omega m square. So, let me once again right here that is cos omega m dash t + gamma m / 2 omega m dash sin omega m dash t.

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$$\langle x(t)\,x(0)\rangle = \frac{\gamma_m K_B T}{\pi m}\;\frac{\quad}{\gamma_m \omega_m^2}\left(\cos\omega_m' t + \frac{\quad}{2\omega_m'}\ \cdots\right)$$

$$\Rightarrow \boxed{\langle x(t)x(0)\rangle = \frac{K_B T}{m\omega_m^2}\, e^{-\gamma_m t/2}\left(\cos\omega_m' t + \frac{\gamma_m}{2\omega_m'}\sin\omega_m' t\right)}$$

Note that $\langle x(0)x(0)\rangle = \frac{K_B T}{m\omega_m^2}$

$$\Rightarrow \frac{1}{2} m\omega_m^2 \langle x(0)^2\rangle = \frac{1}{2} K_B T \quad //$$

So, if I simplified further I have x of t x of 0 = K B T divided by m omega m this is square omega m square e to the power - gamma m t / 2 and we have cos omega m dash t + gamma m / twice omega m sine omega m dash t. So, this is the required answer. So, if you are not convinced or finding it difficult you can quickly verify whether this makes sense for example, if you note that x of 0 x of 0 that you will see that you will get it as K B T / m omega m square. And from here you can write say half m omega m square x of 0 square is equal to as you can see, this will be simply half K B T and you know that this is the so called equipartition theory.

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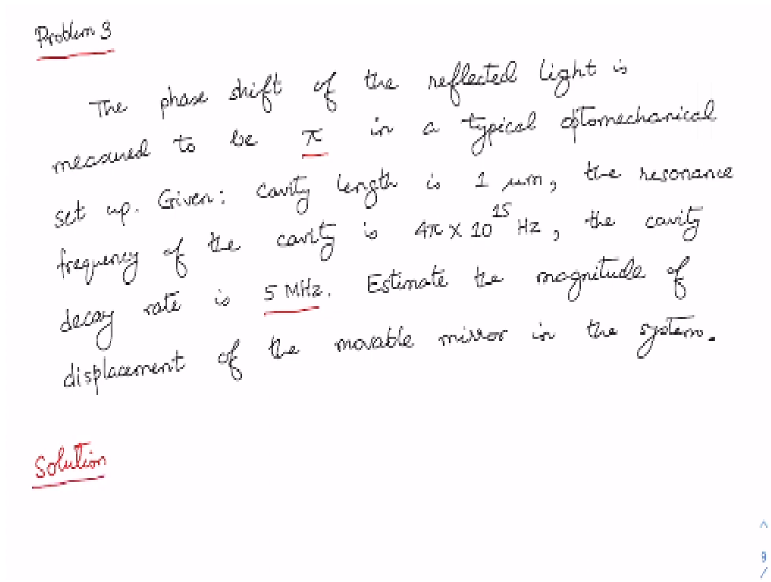

Now, let us work out this problem the phase shift of the reflected light is measured to be pi in a typical optomechanical setup. Given cavity length is 1 micrometer, the resonance frequency of the cavity is 4 pi into 10 to the power of 15 Hertz, the cavity decay rate is 5 mega Hertz. Estimate the magnitude of displacement of the movable mirror in the system. Let us solve it as you know that light can enter into the cavity when the resonance condition is made and are reflected light undergoes a phase shift.

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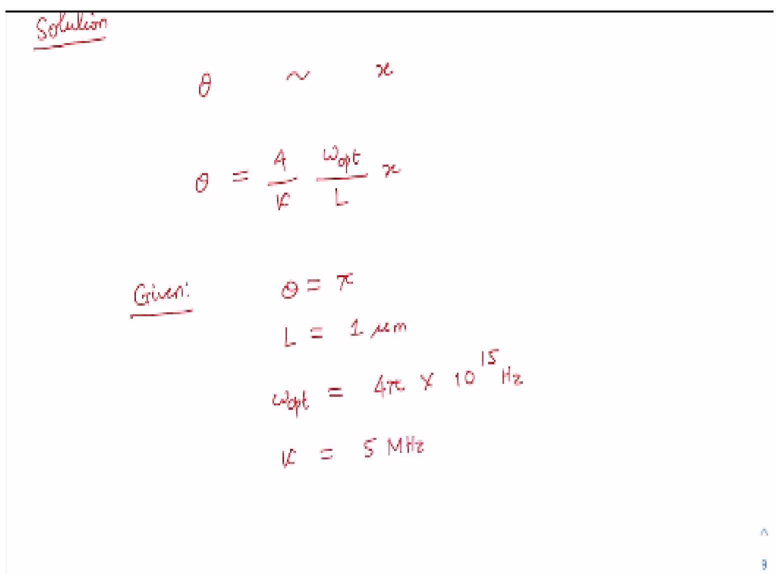

For small displacement, the phase shift theta linearly depends on the mirror displacement x and this is given by this formula which we talked about in the lecture class theta = 4 / kappa, kappa is the cavity decay rate omega optical that is the resonance frequency of the cavity divided by L, L is the length of the cavity and x is the displacement. So, in this particular problem, we just need to apply this formula and we are given theta = pi, L = 1 micrometer resonance frequency is given to be 4 pi into 10 to the power 15 Hertz and the cavity decay rate is 5 megaHertz. What is not given is the displacement and we have to find it out.

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$$\omega_{opt} = 4\pi \times 10 \; Hz$$

$$\kappa = 5 \; MHz$$

$$x = ?$$

$$x = \frac{\kappa L}{4\,\omega_{opt}}\,\theta = \frac{5\times 10^{6} \times 10^{-6} \times \pi}{4 \times 4\pi \times 10^{15}}$$

$$= \frac{5}{16} \times 10^{-15} \; m$$

$$= 3.125 \times 10^{-16} \; m$$

So, that is from the formula we have x = kappa L divided by 4 omega optical into theta. So, if I put the parameters here, kappa is 5 megahertz, so, that is 5 into 10 to the power 6 hertz and an L = 1 micrometer that is 10 to the power - 6 meter and theta = pi and we have 4 into omega optic 4 pi into 10 to the power 15. So, if you put all this then we will obtain 5 divided by 16 into 10 to the power -15 meter or I can write it as 3.125 into 10 to the power -16 meter. So, this is the answer..