Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture – 34
Cavity Optomechanics: Classical Regime I

Hello ,welcome to lecture 25 of the course, this is lecture number 4 of module 3; in this lecture we will discuss the classical regime of cavity quantum optomechanics. This will enable us and it will help us to appreciate the quantum regime later on. **(Video Starts: 00: 50)** So, let us begin in the last class, we discussed how one can obtain information regarding what is going on inside the cavity by the so called displacement readout as the optomechanical system is an interferometer.

It is easy to measure the phase shift suffered by the circulating light inside the cavity. As the phase shift is dependent on the displacement as you can see from the expression that we discussed in the last class, the phase shift is dependent on displacement and it is inversely proportional to the cavity decay rate kappa. So, one can essentially work out all the essential details we found that measurement of phase shift leads to the displacement versus time plot.

Which can give us an idea about the temperature of the harmonic oscillator just from the amplitude of fluctuation it turns out that it is wise to look at the noise spectrum of the harmonic oscillator. And also we found out that this noise spectrum which is here given by S xx and it is a function and this noise spectrum is nothing but it is the Fourier transform of the correlator we discussed in detail in the last class.

And this fact that the noise spectrum is a Fourier transformation of the correlator is also known as the Wiener Khinchin theorem. Then, we went out to discuss the so called fluctuation dissipation theorem. And it is basically a relationship between the noise spectrum and the linear response of the system and linear response is characterized by a quantity called mechanical susceptibility as we know that the displacement of the movable mirror in an optomechanical system is directly proportional to the radiation pressure force.

And this proportionality factor is the so called linear susceptibility and fluctuation dissipation theorem connects this linear susceptibility with simply the noise spectrum. And this noise spectrum as you can see from this expression here is related to this noise spectrum S xx is related to the imaginary part of the susceptibility. And imaginary part of the susceptibility is related to dissipation of the mechanical system.

And this expression is actually in the classical domain then we have a expression for the quantum regime also and we discussed this in the problem solving session in a little bit in detail at very high temperature limit this quantum expression is actually essentially boils down to the classical expression and of course, that would be the case. Then we applied the fluctuation dissipation theorem to a classical damped harmonic oscillator.

In very straight calculations, we can do in the frequency domain and from there we can find out the expression for the susceptibility and we get the expression for the susceptibility from the expression we get the imaginary part of the susceptibility. And as we know the imaginary part of the susceptibility is related to the noise spectrum via this entity and we are discussing in the classical domain only and we get the expression for the noise spectrum and under the situation where the dissipation is small enough.

And if we expand or workout or simplify this expression basically around the resonance frequency, then we get this very simple expression and this expression if we plot then we get this typical plot which already we got earlier in our qualitative discussions. And from this plot noise spectrum plot we can derive immediately lot of information such as the resonance frequency of the harmonic oscillator, the damping rate gamma and also the temperature of the harmonic oscillator just by working out the area under the curve. **(Video Ends: 05:44)**
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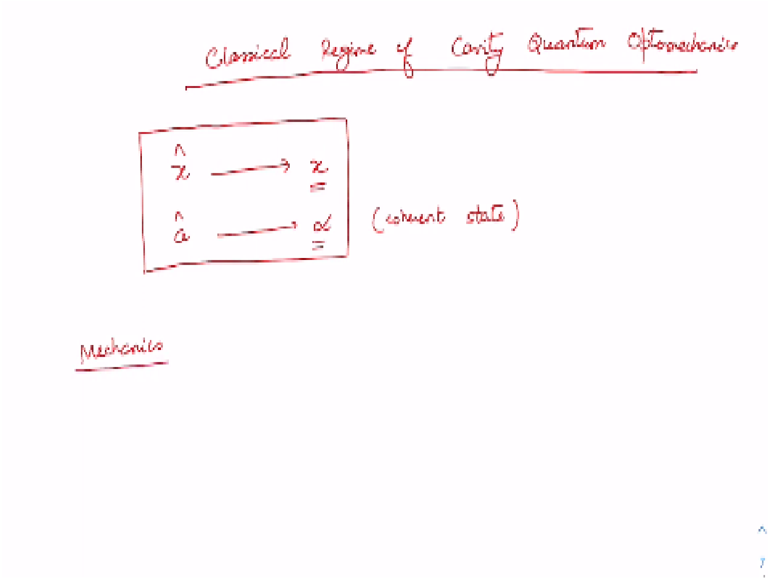

Now, we embark on our study on the classical regime of cavity quantum optomechanics because this is going to help us in understanding the quantum regime better. So, now we are going to discuss classical regime of cavity quantum optomechanics. In the classical limit we'll replace the position operator x cap by the corresponding classical variable x. On the other hand, the light field which is generally denoted by this annihilation operator in quantum mechanics.

This is going to be replaced by the parameter alpha which refers to the so called coherent state and you know that the coherent state is the most classical state of a harmonic oscillator so alpha basically refers to coherent state. And we already know that in the quantum regime light which is an electromagnetic field behaves like a harmonic oscillator. So, this is essentially the regime that we are going to discuss the key thing to remember is that x cap the position operator is going to be replaced by the variable x.

And the annihilation operator who is refers to the quantize electromagnetic field or here in this case light this is electromagnetic field is replaced by the parameter alpha which refers to the coherent state. Now, the equation of motion will be in terms of this parameter x and this alpha and there are 2 parts one is the mechanics part and another one is the light part and we are going to consider only one single mode of light.

Now, coming to the talking about the mechanical mirror this is attached to a spring that is the movable mirror of the Fabry perot cavity, it could be modelled as a damped harmonic oscillator and acted upon by a radiation pressure force and we says due to the light field and we have already discussed about the damped harmonic oscillator that equation of motion in the last class so here let me write it again the mechanics part is going to be described by the this classical damped harmonic oscillator model.

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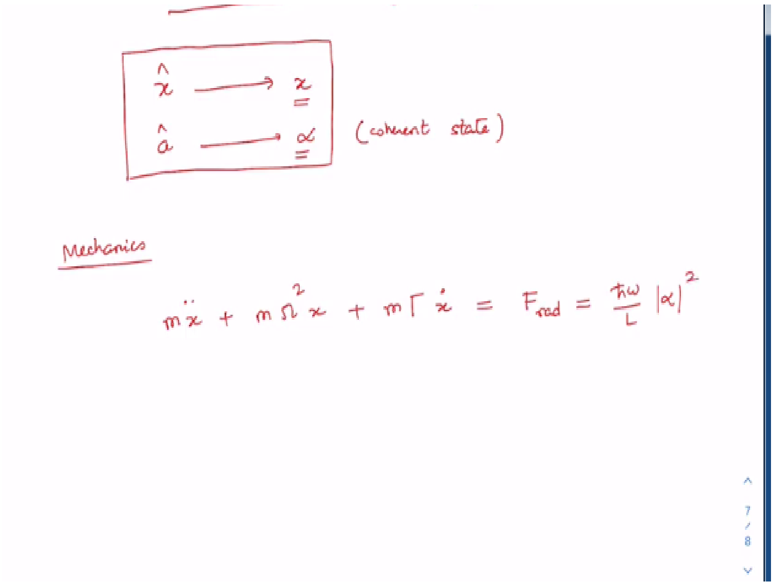

So, in that case we have this equation of motion so, m x double dot + m capital omega square x then we have m gamma x dot and now we are having only the radiation pressure force who is actually you know that this is nothing but h cross omega / L mod alpha square.

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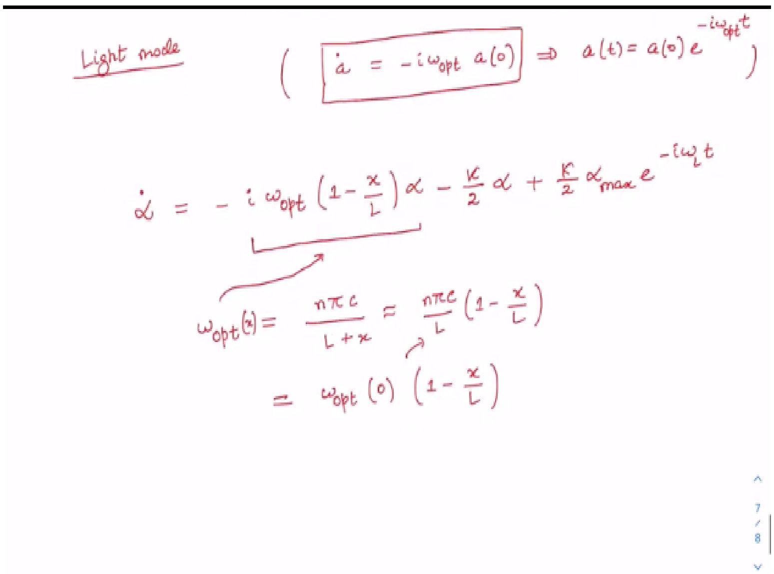

On the other hand, what about the light mode? As I saying that I am just going to consider only one mode of light. So, light mode would be described by just you know that we earlier discussed that how this annihilation operator evolves in time. So, the equation of motion for the annihilation operator is this, a dot = - i omega this is the optical frequency within the cavity and then you have this initial this thing .

Now just maybe better I say that we know that a of t = a of 0 e to the power - i omega optical, this is the optical resonance frequency of the cavity this is what we know. So, based on this we have this equation now, because this annihilation operator is replaced by the parameter alpha, so we can write alpha dot = - i omega optical, but this is now modified due to the movable mirror so this would be 1 - x / L alpha.

And apart from that we will have other 2 terms that I am now going to discuss one is this decay term of the optical field decay is happening at the rate kappa, amplitude is decaying at the rate kappa / 2 and the density will decay at the rate kappa and apart from that is another term which I am going to explain soon that would be kappa / 2 alpha max e to the power - i omega L t, omega L is the laser frequency.

Now before I go further let me again clarify these how this particular term is coming, because we have already we know that omega optical that is the resonance frequency of the cavity is given by this expression n pi c divided by L because of the fact that the mirror is displaced by a small amount x this one we can write using the Taylor series expansion n pi c / L into 1 - x / L. And this particular term we can write it as omega optical, when there is both the mirror are fixed or not moving so we have this particular term. And that is how we are getting this one so, this is basically a function of x so I hope you are getting the idea.

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$$\dot{\alpha} = \underbrace{-i\,\omega_{opt}\left(1-\frac{x}{L}\right)\alpha}\ \underbrace{-\ \frac{\kappa}{2}\alpha}\ + \underbrace{\frac{\kappa}{2}\alpha_{max}\,e^{-i\omega_L t}}_{\text{Laser drive}}$$

$$\omega_{opt}(x) = \frac{n\pi c}{L+x} = \frac{n\pi c}{L}\left(1-\frac{x}{L}\right)$$

$$= \omega_{opt}(0)\left(1-\frac{x}{L}\right)$$

Now, the second term here, this refers to the decay of the amplitude of the photon and this particular term this term takes the laser drive into account because the laser drive is a classical field that is where in say cosinusoidally or sinusoidally so, this takes the laser field into account and this particular parameter alpha max here indicates the fact that the value of this alpha will settle down to the value of alpha max at resonance frequency, I think this would be clear soon as you will see.

So, the amplitude of the laser drive is characterized by giving the amplitude of the light field inside the cavity at resonance by this alpha max. Now, we can get rid of this time dependence by going over to a rotating frame of reference.

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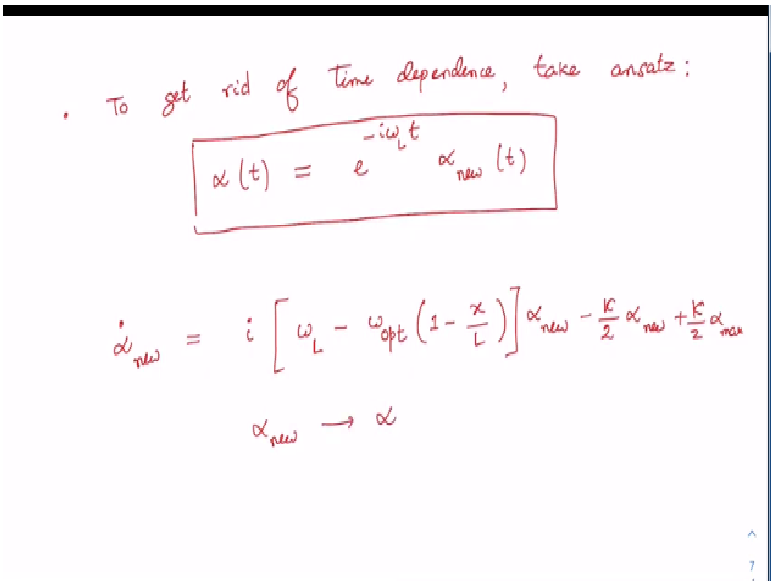

So, take the answers to get rid of time dependence in the equation for the light mode take the answers that means, actually we are going over a rotating frame of reference, we have changing the frame of reference, and that is what that's the reason we are taking these answers. So, we will take alpha of t is equal to say it is rotating in the laser with frequency say omega L t and there is a new variable let me define it is alpha new of t.

Now, if we put these answers in the equation here in this equation you can do it and if you do that, you will see that you will get alpha dot new = i omega L - omega optical that is resonance frequency omega L is the laser frequency and this is 1 - x / L. These are very straightforward calculation if you take your pen and paper you will be able to see very easily it is very trivial and you will have kappa / 2 here it is the new variable alpha new + kappa / 2 alpha max.

In the process as you see in this equation, we are getting rid of the explicit time dependence. Now, other than keeping alpha new are going to rename it again alpha new to alpha. So, then we can rewrite this equation by this one.

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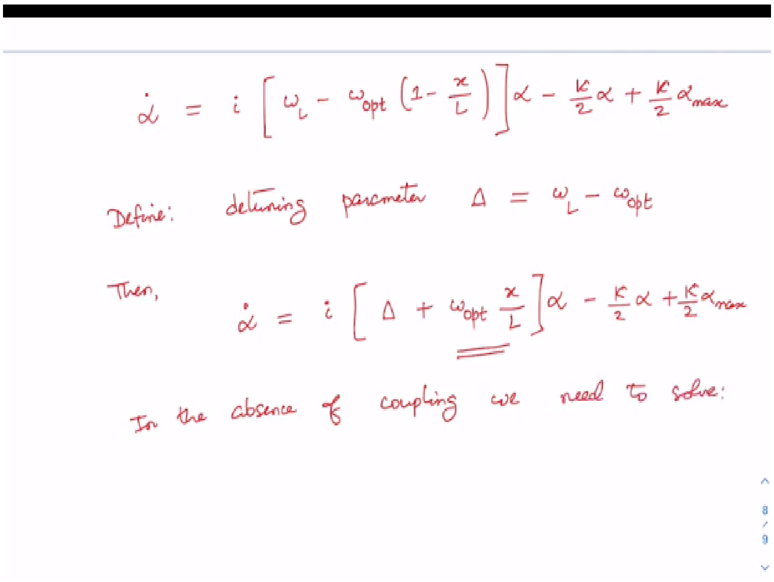

So, we will have alpha dot in the rotating frame we are now having i into omega L - omega optical 1 - x / L, then this is alpha - kappa / 2 alpha + kappa / 2 alpha max. So, this is what we are having now we can define a parameter called the detuning parameter. So, we will define detuning parameter say delta denoted by delta = omega L the laser frequency minus the resonance frequency of the cavity.

Then we will have we will be able to write this equation is alpha dot = i into we will have delta + omega optical into x / L as you can see very easily from the equation into alpha - kappa / 2 alpha + kappa / 2 alpha max. So, this is what we will have? Now, you see in the absence of coupling when to the mechanics we need to solve in the absence of coupling this term would not be there let us write in the absence of coupling we need to solve.

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$$\dot{\alpha} = i\Delta\alpha - \frac{\kappa}{2}\alpha + \frac{\kappa}{2}\alpha_{max}$$

In the steady state: $\dot{\alpha} = 0$, for $\alpha = \bar{\alpha}$

$$i\Delta\bar{\alpha} - \frac{\kappa}{2}\bar{\alpha} + \frac{\kappa}{2}\alpha_{max} = 0$$

$$\Rightarrow \bar{\alpha} = \frac{\kappa/2\,\alpha_{max}}{\frac{\kappa}{2} - i\Delta} = \frac{\alpha_{max}}{1 - \frac{i\Delta}{\kappa/2}}$$

$\Rightarrow$ Right at resonance $\Delta = 0$, $\bar{\alpha} = \alpha_{max}$

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So, let me just actually why I am doing it I just want to show you the significance of this term alpha max. So, in the absence of coupling I just have to solve alpha dot = i delta alpha - kappa / 2 alpha + kappa / 2 alpha max and in the steady state because you know decay time will ultimately lead to the steady state. In the steady state you will have alpha dot = 0 for say alpha is equal to some value alpha bar which is the steady state value.

So, if you put it in this equation here then you will get i delta alpha bar - kappa / 2 alpha bar + kappa / 2 alpha max = 0 and from here you can immediately find out the steady state value of alpha this is very trivially you can get it would be kappa / 2 alpha max divided by kappa / 2 - i delta or I can simply write it as alpha max divided by 1 - i delta divided by kappa / 2. Now, as you see what I said earlier that right at resonance.

So, this implies that right at resonance that is when the detuning parameter is exactly equal to 0 or the laser frequency matches the resonance frequency we have alpha bar = alpha max. So, that is the significance of the term alpha max.

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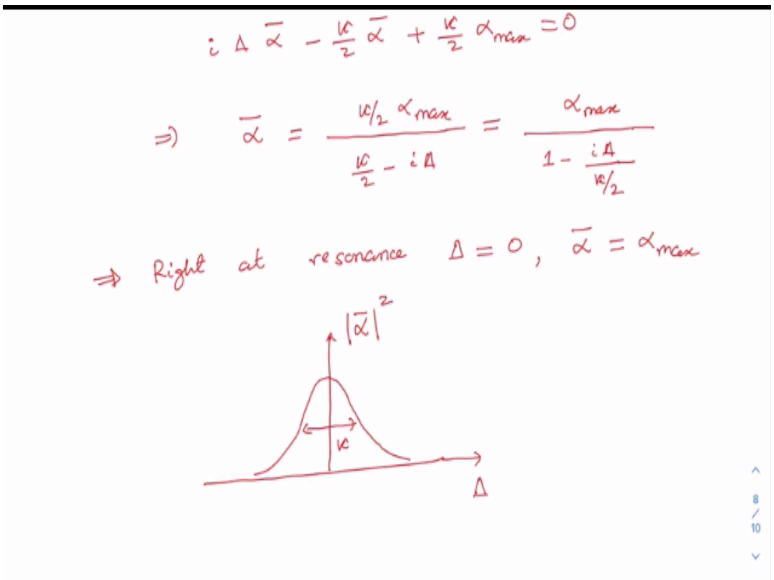

Also you can see that if we plot the steady state intensity of the light field that means, as a function of the detuning parameter that means, if I take mod alpha bar square and here if I plot delta then I will get a Lorentzian and so, you will get a plot like this kind of a plot you will get a Lorentzian and width of this full width at half maximum would be given by the cavity decay rate kappa.

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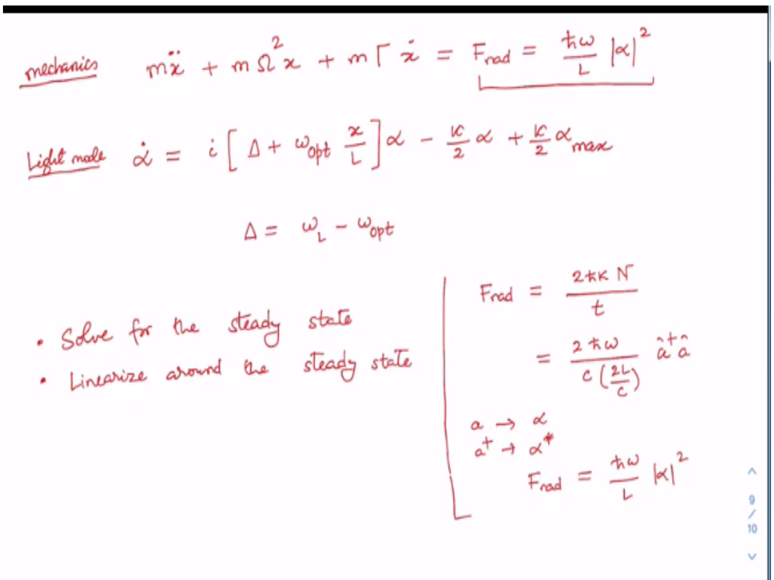

Now, in order to analyze this equation of motion due to mechanics and light field or light mode we will assume that the coupling between the light field and mechanics will lead to only a small deviation around the equilibrium. By the way some of you may be puzzled that how these particular term damped radiation pressure forces coming, let me just quickly remind you again let me take you back to the basic as you remember that when we talked about radiation pressure force.

The momentum exerted by a single photon on the movable mirror the change in momentum due to a single photon was twice h cross k and if there are N number of photons then the total momentum change would be twice h cross k into N divided by the time so, this is the rate of change of momentum that is the force radiation pressure force this I can further write as you know k = omega / c.

So, omega / c and time is equal to one round trip time would be 2L divided by c and N is quantum mechanically speaking it is your photon numbers so it is a dagger a. Now in the classical limit as you know already that a is replaced by alpha and a dagger would be replaced by alpha star. So, immediately you see that I can write radiation this radiation pressure force would be h cross omega / L mod alpha square I hope it is clear to you let me proceed further.

So, the idea is now to solve for the steady state and linearized around the steady state and asks for the solutions basically. So, what we are going to find is that the light field modifies the mechanical behaviour resulting in effects like optomechanical damping, which we discuss qualitatively earlier. Now, we will see that quantitatively and also the so called optical spring effect that is the light induced frequency shift of the mechanical oscillator. So, what we are going to do now?

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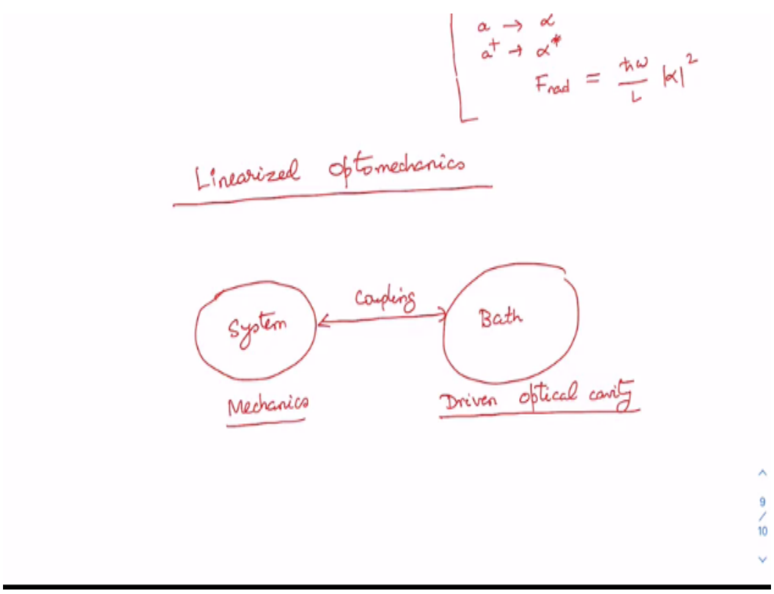

We are going to enter into the domain of what is called or what is known as the linearized optomechanics. So, we are going to discuss let me explain what I mean by that more clearly, you see say we have this system suppose we have a system and in our case our system is mechanical oscillator or the mechanical system. So, in our case the mechanics because we are focused to what this mechanical system is doing or the mechanical oscillator and generally the system is coupling to a bath or the environment.

Let us say bath and in our case our bath that is the driven optical cavity you know it is driven by laser field from outside. So, our system is the mechanics and bath is that driven optical cavity and they are getting coupled to each other by the so called radiation pressure force. So, we are going to look for same kind of effect for a mechanical system the kind of we get when a particle moves to a crystalites you know when a particle moves through a crystalites it distorts the crystalites and acquires in different masses the so called effective mass.

So, similar kind of analogous effect we are going to expect only thing here is that so that we can go into the domain of linearized optomechanics will assume that coupling is weak.

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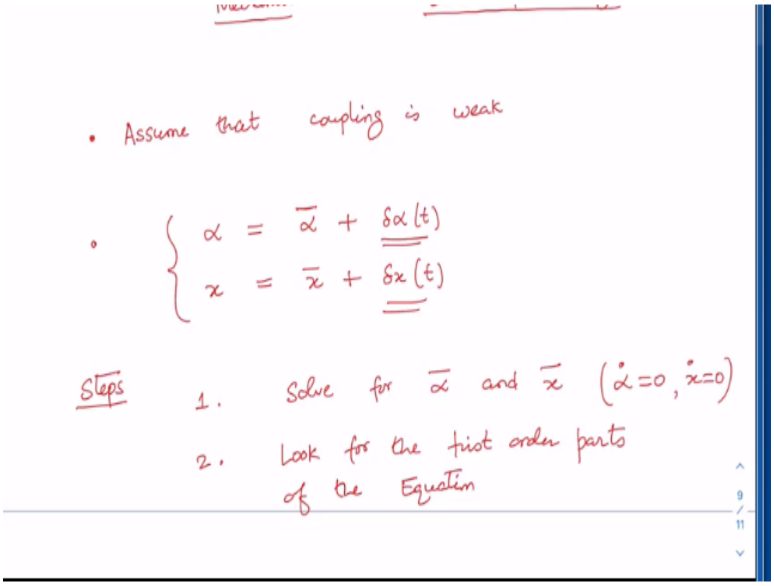

So, we are going to assume that this coupling between the bath and the system or the mechanics and the driven optical cavity we are going to assume that coupling is weak. And also what we are going to do? We are going to write this parameter now alpha, this alpha is getting deviated from the steady state by among say delta alpha t. And these mechanics this displacement is deviated from its steady state we will do x bar by an amount delta x t so this is very important.

So, here alpha bar repeat again that alpha bar and x bar are the steady state solutions, while these delta alpha t and delta x of t are the corresponding deviations from the steady state. Now, the dynamics of the system is going to be modified due to the coupling to the bath and one way is to do it basically to linearize it what we are going to do are the following steps that we are going to adopt first of all, let me just write here the steps that we are going to do.

First step is solve for steady state solve for alpha bar and x bar in that case users have to put alpha dot = 0 in the steady state alpha dot would be equal to 0 and x dot would be equal to 0. And then in the second step, we are going to look for the first order parts of the I think it will be more clearer as we will do the calculation look for the first order parts of the equation of motion.

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$$m\ddot{x} + m\Omega^2 x + m\Gamma\dot{x} = F_{rad} = \frac{\hbar\omega}{L}|\alpha|^2$$

$$x \to \bar{x} + \underline{\underline{\delta x}}, \qquad \alpha \to \bar{\alpha} + \delta\alpha$$

$$|\alpha|^2 = \alpha\alpha^*$$

$$\boxed{m\delta\ddot{x} + m\Gamma\delta\dot{x} + m\Omega^2\delta x = \frac{\hbar\omega}{L}\left(\bar{\alpha}^*\delta\alpha + \bar{\alpha}\delta\alpha^*\right)}$$

$$\boxed{\delta\dot{\alpha} = i\Delta\delta\alpha + i\frac{\omega_{opt}}{L}\left(\bar{x}\delta\alpha + \bar{\alpha}\delta x\right) - \frac{\kappa}{2}\delta\alpha}$$

Let us assume that we already know now we will do the calculations but let us assume that we already know alpha bar and x bar let us assume now, first look at the equation of motion for the mechanics I have already written down the equation of motion for the mechanics so, let me just copy it from here. And then let me then x I am going to replace this x by x bar + delta x and I will be able to get an equation for delta x because steady state you know x it is a steady state it is a constant value so, if I take the double derivative it will go to 0.

And you can anyway put and then you will be able to get this equation of motion for this deviation part. So, that will be m delta x double dot + m gamma delta x dot + m capital omega square delta x dot I think only you will just have only m omega square delta x here and this would be equal to h cross omega / L and you will have similarly you are replacing alpha by alpha bar + delta alpha so mod alpha square is actually alpha into alpha star.

So, if you put that you are going to keep the term up to first order only. So, you will have alpha star delta alpha + alpha bar delta alpha star complex conjugate basically. So, this is what you are going to get this is for the mechanics part. And similarly, if you do that for the light part you will get delta alpha dot = i delta delta alpha + i actually will get it more if you take your pen and paper and do the calculations along with me so it is very straightforward.

So, we will get here x bar delta alpha + alpha bar delta x - kappa / 2 delta alpha. So, this is the equation of motion for the deviated light mode. So, this is what you are going to get? So, as you can see from these equations that the driving term vanishes as it is already taken into account while finding the steady state you remember earlier we have found out the steady state below alpha bar here you see this we worked out earlier and here this driving term was actually included there through this alpha max term. So, therefore, this term is not appearing explicitly in this equation of motion.

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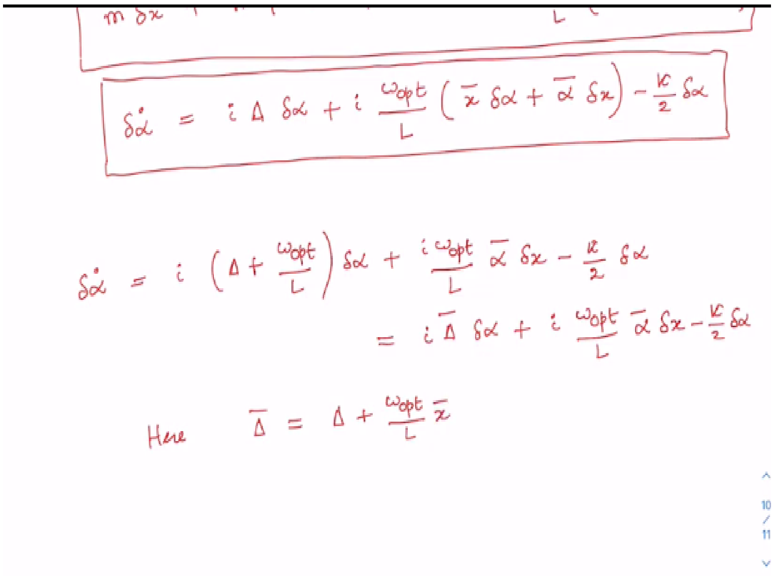

So, now what we can do? We can further write this equation of motion a little bit let me write down the equation of motion put a light mode delta alpha dot = i into delta + omega optical / L delta alpha + i omega optical / L I will have alpha bar delta x - kappa / 2 delta alpha = i delta bar I will explain what is delta bar here is that is then I have delta alpha + i omega optical / L alpha bar that is the steady state value of the light mode delta x here - kappa / 2 delta alpha. And here this delta bar is delta this detuning parameter is now getting modified due to the coupling to the mechanics that is omega optical / L into x bar.

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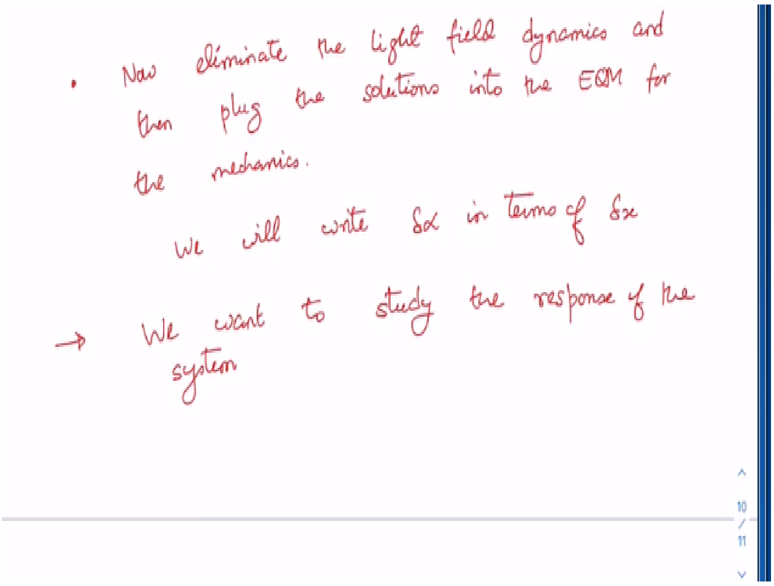

So, now the idea is to eliminate now, what we are going to do? Now, eliminate the light field dynamics and I will explain what I mean by this light field dynamics and then plug the solutions and you will see how to do that plug the solutions into the equation of motion for the mechanics. So, this is what now we are going to do that means, we will write delta alpha this light field in terms delta x by the way finally what do we want?

We want to study this is our core goal we want to study the response of the system in this case our mechanical system mechanical oscillator, we want to study the response of the system to an external force say F denoted by F. So, we will now we are going to add it to the equation of motion for the mechanics. So, equation of motion for the mechanics already I have written here and this part was coming due to the radiation pressure force which is intrinsic to the system into bath. Now, apart from that, now we are going to add an external force to it.

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$$\boxed{m\,\delta\ddot{x} + m\Gamma\,\delta\dot{x} + m\Omega^2\,\delta x = \frac{\hbar\omega}{L}\left(\bar{\alpha}^*\delta\alpha + \bar{\alpha}\,\delta\alpha^*\right) + F}$$

Go over to frequency domain:

$$\frac{d}{dt} \to i\omega \qquad \frac{d^2}{dt^2} \to -\omega^2$$

$$\delta\dot{\alpha} = i\bar{\Delta}\,\delta\alpha + i\frac{\omega_{opt}}{L}\bar{\alpha}\,\delta x - \frac{\kappa}{2}\delta\alpha$$

$$\downarrow$$

$$\left(-i\omega - i\bar{\Delta} + \frac{\kappa}{2}\right)\delta\alpha(\omega) = \frac{i\omega_{opt}}{L}\bar{\alpha}\,\delta x(\omega)$$

So, the equation of motion that I am going to write here would be m delta x double dot + m gamma delta x dot + m capital omega square delta x = h cross omega / L alpha star delta alpha + alpha bar delta alpha star and we are now going to add this external force I hope we are getting the idea because we want to study the response of the system to this external force that studies and we are adding this particular term there. Now, what we are going to do? We can solve these equations very easily if we go to the Fourier space.

So, let us go over to the frequency domain now. If we go over to the frequency domain already you know that in the frequency domain say if you have say delta t is there let me remind you what you can easily say if you say delta t or dt. So, if you say d of dt is there, then you replace it by i omega if d 2 dt double time derivative is there then you replace it by - omega square and so on.

So, we will go over the frequency domain .Take the equation of motion for the light mode that is delta alpha dot = i delta bar delta alpha + i into omega optical / L alpha bar delta x - kappa / 2 delta alpha in the frequency domain we can write it as - i omega let me take this into the left hand side - i delta bar and also this one let me take into the left hand side and I will have + kappa / 2. Now, we are writing it in the frequency domain. So, I have delta alpha of omega and on the right hand side we will have a i omega optical / L alpha delta x of omega.

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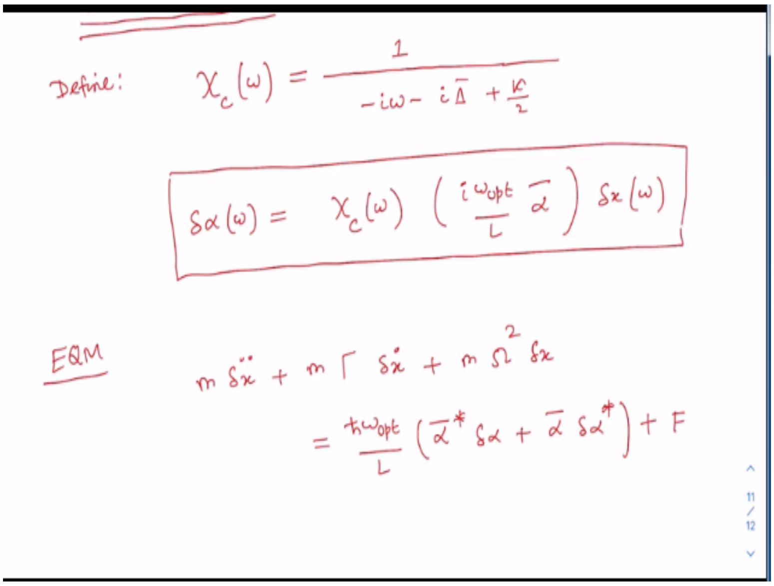

Now, let us define the cavity response to a susceptibility parameters say chi c, if I define a parameter K B T susceptibility if I say I am just defining it, say that we define this particular parameter actually if I define chi c is 1 divided by - i omega - i delta bar + kappa / 2 then we can then write this expression delta alpha of omega is equal to I will have it a chi c of omega that is the susceptibility corresponding to the cavity into i omega optical / L alpha bar and I have delta x omega.

So, basically this susceptibility parameter gives you the idea that how the light field is getting modified due to the mechanics as a response to the mechanics how the light field is getting deviated from its steady state so that is the whole idea. Now, consider the equation of motion for the mechanics let me rewrite the equation of motion for the mechanics we had m delta x double dot + m capital gamma delta x dot + m omega square delta x = h cross omega optical / L I have here alpha bar star delta alpha + alpha bar delta alpha star and now we are having also this external force.

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$$m\,\delta\ddot{x} + m\Gamma\,\delta\dot{x} + m\Omega^2\,\delta x = \frac{\hbar\omega_{opt}}{L}\left(\bar{\alpha}^*\delta\alpha + \bar{\alpha}\,\delta\alpha^*\right) + F$$

$$\downarrow$$

$$\left(-m\omega^2 + m\Omega^2 - im\Gamma\omega\right)\delta x(\omega) = \frac{\hbar\omega_{opt}}{L}\left(\bar{\alpha}^*\delta\alpha(\omega) + \bar{\alpha}\,\delta\alpha^*(\omega)\right) + F(\omega)$$

Now, if we write this thing in the frequency domain will have - m omega square + m capital this omega square - i m gamma omega. Then this is in the frequency domain, I have delta x of omega. And on the right hand side, I have all these terms, h cross omega optical / L alpha bar star delta alpha now it is in the frequency domain, and then I have alpha bar delta alpha star again in the frequency domain and this force also in the frequency domain.

Now, if I substitute the expression for delta alpha here, you have this expression here. Now, let me substitute delta alpha omega and its complex conjugate using this expression here and if we do that.

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$$\delta\alpha(\omega) = \chi_c(\omega)\left(\frac{i\omega_{opt}}{L}\bar{\alpha}\right)\delta x(\omega)$$

$$\begin{cases} \delta\alpha^*(\omega) = \left[\delta\alpha(-\omega)\right]^* \\ \delta x^*(\omega) = \delta x(\omega) \end{cases}$$

$$\left(-m\omega^2 + m\Omega^2 - im\omega\Gamma\right)\delta x(\omega)$$

So, what I am doing now in the next step? I am going to substitute delta alpha omega is equal to I write say chi c of omega i omega optical / L alpha bar delta x of omega then what we will have I am going to write but before that, while I do that, we have to use this facts that delta alpha star of omega if I take the complex conjugate, that means a Fourier transform the complex conjugate of this function, this is equal to the Fourier transform of the function delta alpha evaluated at frequency - omega and then if you take the complex conjugate.

So, this thing you can exploit because alpha is a complex quantity, on the other hand, there is no issue with the displacement parameter because displacement parameter is a real quantity here. And using these facts, we can rewrite our equation of motion for the mechanics and we will be write this delta alpha omega we can write in terms of no mechanics. So, if we it is very straightforward calculations, if you have pen and paper with you we can do it very easily.

Let me now write it so what you are going to get it is as follows you will get - m omega square + m capital this omega square and you have i m omega this parameter gamma here dissipation parameter gamma and I have delta x of omega, this is basically the same thing I am writing now.

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$$\delta\alpha^*(\omega) = [\delta\alpha(-\omega)]^*$$

$$\delta x^*(\omega) = \delta x(\omega)$$

$$\left(-m\omega^2 + m\Omega^2 - im\omega\Gamma\right)\delta x(\omega)$$

$$= \hbar\left(\frac{\omega_{opt}}{L}\right)^2 i\,|\bar{\alpha}|^2\left[\chi_c(\omega) - \chi_c^*(-\omega)\right]\delta x(\omega) + F(\omega)$$

On the right hand side using these facts, I will have h cross omega optical / L verify yourself these are straightforward calculations, but we may make a mistake while doing it, please verify it yourself. So, this is what you are going to get you will have chi c of omega - chi c this complex conjugate evaluated at - omega and you will have delta x of omega + F of omega.

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$$\underbrace{\qquad\qquad\qquad}_{K(\omega)} + F(\omega)$$

$$= K(\omega)\,\delta x(\omega) + F(\omega)$$

$$K(\omega) = \hbar\left(\frac{\omega_{opt}}{L}\right)^2 i\,|\alpha|^2\left[\chi_c(\omega) - \chi_c^*(-\omega)\right]$$

$$\rightarrow \quad \chi_{xx}(\omega) = \frac{1}{m(\Omega^2 - \omega^2) - im\omega\Gamma}$$

So, this is what we will have rather than writing this lengthy expression all the time let me name it as a function a name let me say denoted by K omega then I will have this term as K omega delta x of omega + F of omega. Now, by the way, this expression is very important and it has its physical meaning we are going to discuss K of omega = h cross omega optical / L whole square i into mod alpha square I have chi c of omega - chi c star evaluated at - omega so this is what we have?

Now, previously, if you recall in our discussion related to fluctuation dissipation theorem we define this mechanical susceptibility parameter chi xx of omega = 1 divided by m into capital omega square - small omega square - i m omega gamma. Now, this chi is the same as this one.

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$$\underline{\chi_{xx}^{-1}(\omega)}\,\delta x(\omega) = K(\omega)\,\delta x(\omega) + F(\omega)$$

$$\Rightarrow \quad \delta x(\omega) = \frac{F(\omega)}{\chi_{xx}^{-1}(\omega) - K(\omega)}$$

So therefore, I can further write the equation for a mechanics says chi xx of this inverse here, omega delta x of omega what I am doing here? I am just rewriting this expression I am replacing it by chi this one. And on the right hand side I have K of omega delta x of omega + F of omega. And from here I can now read this displacement parameter deviation from the steady state for the mechanics, we have delta x of omega = F of omega divided by chi xx, this is inverse omega - K of omega.

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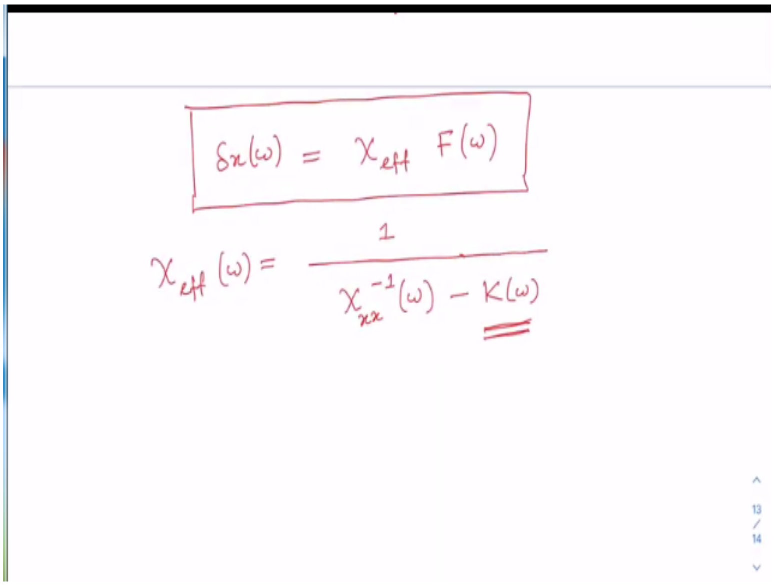

This thing I can write as chi effective, this is effective mechanical susceptibility into F of omega, where this chi effective of omega, this effective mechanical susceptibility parameter is simply it is equal to 1 divided by mechanical susceptibility when it is not coupled to the light and I have this K of omega. So, this expression that I have got here is extremely important and we are going to derive the physics quantitatively out of it.

Now you see clearly if there is no light field, then we will go back to the linear response of the mechanical oscillator, because if there is no light field and K of omega would be equal to 0 because as you see K of omega, this mod alpha squared term will go to 0 because there is no light field. And in that case, we will go over to the usual damped mechanical harmonic oscillator now, because of the fact that the mechanics is now getting coupled to the light field.

This susceptibility parameter is basically getting modified and we are having an effective mechanical susceptibility. Now, we are going to understand the consequence of this equation, particularly due to the presence of this parameter K of omega.

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Meaning of $K(\omega)$

$$\delta x(\omega) = \frac{1}{\chi_{xx}^{-1} - K(\omega)} F(\omega) = \chi_{eff} F(\omega)$$

- Assume that $|\alpha|^2$ is small

$\Rightarrow$ coupling is also small

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Now we will discuss the physical significance or meaning of this parameter K of omega, let me write once again that this term is entering into our discussion via this expression, where delta x of omega = 1 divided by susceptibility of the mechanical system, when it is not getting coupled to the light in fact inverse of the mechanical susceptibility minus this capital K of omega, and we have this external force and overall whole thing can be written as effective susceptibility parameter into F of omega.

Let us assume that the intensity of the light is small, and that means mod alpha square is small. And it means that the coupling is also small coupling between mechanics and light is also small. And in fact, this will easily give us the meaning of the parameter K of omega, because in that case K of omega is a small correction term only.

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$\Rightarrow$ coupling is also small

- Look at $\omega \approx \Omega$
- compare $K(\omega)$ with χ_{xx}^{-1} term by term
- $\chi_{xx}^{-1}(\omega) = m(\Omega^2 - \omega^2) - im\omega\Gamma$
- $\text{Im}\ \chi_{xx}^{-1}(\omega) = -m\omega\Gamma$
- $\text{Im}\ \chi_{xx}^{-1}(\omega \approx \Omega) = -m\Omega\Gamma$

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Let us look at the vicinity of the resonance frequency that means we will analyze the thing near the resonance frequency, say small omega is nearly equal to capital omega. Now, if we look at the denominator of this expression here we will compare K of omega just focus on the denominator of the expression here, this expression and we will compare K of omega with this inverse of the mechanical susceptibility parameter term by term.

And you will see that this is going to give us the physical significance of this term and we will be able to get the meaning. Now, this mechanical susceptibility parameter the inverse is simple expression who is already we know that is m into capital omega square - omega square - i m omega capital this is the gamma parameter, that is the dissipation parameter. Now, if we look at the imaginary part of this inverse susceptibility parameter.

Then you will see that this is very simple this is simply equal to - i m actually because I am just taking the inverse imaginary part. So, I will just write - m omega gamma. Now, near the resonance frequency, so, imaginary part of this inverse susceptibility parameter near the resonance frequency is equal to - m capital omega this parameter gamma. So, this means that again this is critical to look at the term in the denominator here. So, if you compare this, what it means that if K of omega has an imaginary part because of this minus sign you will see this will imply that there is an extra damping term of course, up to some pre factor.

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$$\mathrm{Im}\ \chi_m^{-1}(\omega \approx \Omega) = -m\Omega\Gamma$$

$$\mathrm{Im}\ K(\omega \approx \Omega) = +m\Omega\,\delta\Gamma$$

$$\boxed{\delta\Gamma = \Gamma_{opt} = \frac{1}{m\Omega}\,\mathrm{Im}\ K(\omega \approx \Omega)}$$

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So, therefore, from this logically we see that if this term K has an imaginary part nonzero term near this resonance frequency we can write here plus because you see in the denominator here, this is the minus term is there so I will write it plus term and + m of omega and then we will have delta gamma parameter. So, let me repeat again if you are not clear what I am saying is that because, we see that inverse susceptibility parameter it is imaginary part is related to the damping parameter gamma.

And now, if the parameter K has an imaginary part, then what it basically means is that there is an extra damping because of the imaginary part of the K term the damping is going to further enhance and this enhancement is given by this factor delta gamma of course, it is multiplied by say m omega, then what we have is this? So, if we compare both sides that means, we have delta gamma and this is the extra damping parameter that is coming due to the coupling to the light fields.

So, we can write it as optomechanical damping gamma optical and this is equal to 1 divided by m into omega and imaginary part of K evaluated at near the resonance frequency. So, what we see is that physically speaking the imaginary part of this term K is related to the extra damping parameter and we says induced due to the coupling to the mechanics. So, the mechanical system is going to get damped or its temperature is going to get reduced because of its coupling to the light that is what it mean.

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$$\mathrm{Re}\left[\chi_{xx}^{-1}\right] = m\left(\Omega^2 - \omega^2\right) \qquad \Omega^2 - \omega^2 = (\Omega-\omega)(\Omega+\omega)$$

$$\mathrm{Re}\,\chi_{xx}^{-1}(\omega \approx \Omega) = 2m\Omega\,(\Omega - \omega)$$

$$\Omega + \delta\Omega - \omega \approx \delta\Omega$$

$$\mathrm{Re}\,K(\omega \approx \Omega) = -\,2m\Omega\,\delta\Omega$$

Now, let us look at a real part of again in the similar way if we now look at the real part of this susceptibility parameter the inverse parameter that is simply m into capital omega square - small omega square. Now, near the resonance we can write the real part of the inverse susceptibility parameter and the resonance this susceptibility near the resonance frequency we can write it as twice m omega into omega capital omega - small omega.

Because omega square - omega square I can write it as capital omega - omega into omega + omega because this is nearly equal, so I am writing it as twice omega, that is this 2 omega term is coming multiplied by m is there and capital omega - small omega. So, if there is a change in the resonance due to the coupling, so, that means we have say, resonance frequency scenes from omega to delta omega, then I can write this as simply delta omega or this term, I can simply write it as delta omega.

If I do that, this means that if there is a real part in K of omega, this will lead to extra frequencies shift up to some prefactor. So again, we have that real part of this parameter K near the resonance frequency, we can give it a meaning, and I will write it as minus twice m omega. And this extra frequencies shift and why I am writing it - because of the fact that we are in the denominator here we are having this sign minus is there in that why I am now putting it this is the reason I am putting the minus sign here.

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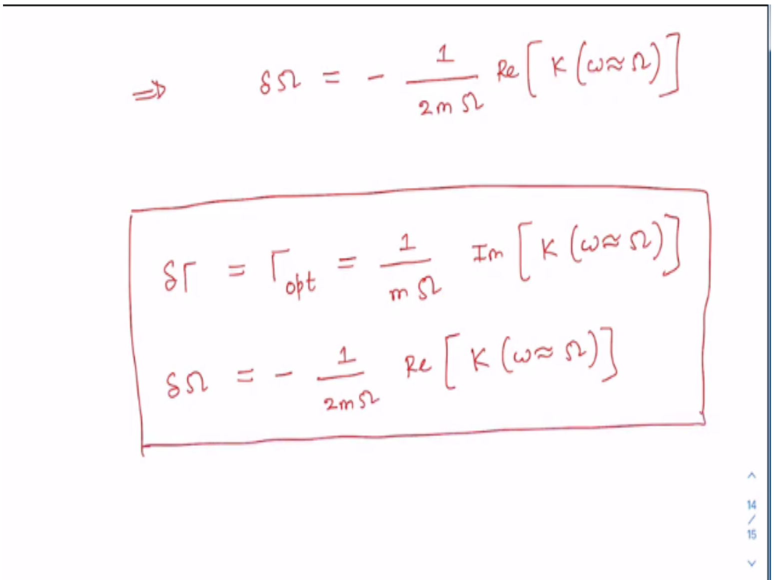

And this implies that delta omega this frequency shift = - 1 divided by twice m omega and this would be real part of K evaluated near the resonance frequency. So, what you see that the real part of K is related to the frequency shift on the other hand imaginary part is related to the extra damping so we get essentially 2 important results out of it we get an extra damping term, because of this analysis in fact the imaginary part of the parameter K is related to extra damping parameter and this is one expression we get.

This is 1 / m omega into imaginary part of K evaluate near resonance frequency. On the other hand, we get delta omega the extra frequency shift that is - 1 by twice m omega real part of K evaluated at resonance frequency. So, these are the 2 important results we obtained. And in the next class, we are going to analyze the expression for K omega in details. Let me stop here for today. In this lecture, we have discussed the classical regime of cavity quantum optomechanics.

And we saw that coupling between the light and mechanics when the coupling is weak, we can go over to the linearize optomechanics regime, our classical analysis let us necessarily to the fact that we will end up with having a extra frequency shift of the mechanical oscillator as well as an extra damping. In the next lecture, we will build up on this issue we will discuss more about it. And also we are going to discuss a very important topic needed for cavity optomechanics. That is the so called Langevin equation formalism and we will first start with the so called Classical Langevin equations. So, see you in the next lecture. Thank you.