ntum Technology and Quantum Phenomena in Macroscopic Systems

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Lecture – 25
Problem Solving Session-6

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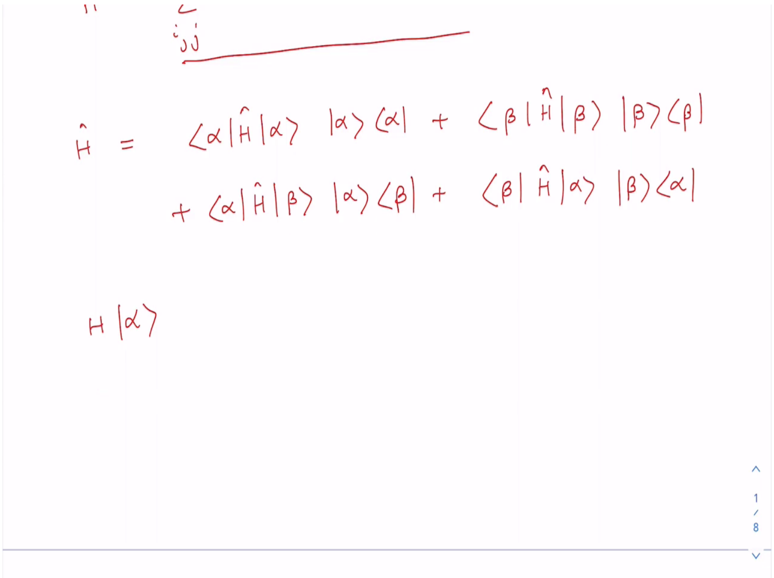

Welcome to this problem solving session number six .In this problem solving session we are going to solve problems related to Jaynes Cummings model primarily. So, problem one says you have to work out the full Jaynes Cummings RWA Hamiltonian in the matrix form. We have done it in the class but we skipped certain steps let me now fill up the gaps here.

So, in lecture number 15 of the course which is lecture 5 of module 2 we wrote down the Jaynes Cummings Hamiltonian as follows H is equal to h cross omega a dagger a + h cross omega atom sigma z + 1 by 2 + h cross z sigma x a + a dagger. Now under the rotating wave approximation we wrote this Jaynes Cummings Hamiltonian as follows H is equal to first two terms remains unaltered we have h cross omega a dagger a + h cross omega atom sigma z + 1 by 2.

However the last term we wrote as h cross g sigma this is plus that is the atomic raising operator and this annihilation operator a + sigma - a dagger sigma - is the atomic lowering operator. Now to express this Hamiltonian in the matrix form let us make the basis states take the basis states as atom in the ground state and n number of atoms n number of photons in the field and the other one atom in the excited state and there is one less photon in the field.

This we are going to take as our basis to simplify the notation I will take it as ket I will denote it as ket alpha and this one I will denote as ket beta all right. Now using the fact that x is equal to we can write it as we have done it so many times .So, we have i i say H j j. Now because we have only two kets here alpha and beta the Jaynes Cummings Hamiltonian we can write first let me write it expand this term.

We will have alpha h alpha ket alpha bra alpha oh sorry that will be bra alpha then we have beta h beta ket beta beta then we have say alpha H beta then alpha beta and we will have beta H alpha ket beta bra alpha. So, this is what we will have let us work out various terms one by one however even before that let me first work out H alpha and H beta but first let me work out H of alpha.

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$$\boxed{\begin{aligned}\hat{H}|\alpha\rangle &= \hat{H}|\downarrow, n\rangle = \hbar\omega n\,|\downarrow, n\rangle + \hbar g\sqrt{n}\,|\uparrow, n-1\rangle \\ \hat{H}|\beta\rangle &= \hat{H}|\uparrow, n-1\rangle = \hbar\omega(n-1)\,|\uparrow, n-1\rangle + \hbar\omega_{at}\,|\uparrow, n-1\rangle \\ &\qquad + \hbar g\sqrt{n}\,|\downarrow, n\rangle\end{aligned}}$$

$$\langle\alpha|\hat{H}|\alpha\rangle = \langle\downarrow, n|\hat{H}|\downarrow, n\rangle = \hbar\omega n$$

$$\langle\downarrow, n|\downarrow, n\rangle = 1$$
$$\langle\downarrow, n|\uparrow, n-1\rangle = 0$$

So, that would be H alpha h is h cross omega a dagger a + h cross omega atom sigma z + 1 by 2 h cross z sigma + a + sigma - a dagger all these are operators and then this ket alpha is this ground state and field state. So, this is what we have let me work out the term separately say a dagger a when it operates on this state a dagger a is associated with the field mode.

So, therefore it will act only on the field state that is your ket n it will act. So, because this is a number operator will have n and this will remain as it is. Because you know that a dagger a when it operates on the number operator you will get this is the eigenvalue equation you know. Now what we have the second term to work out this term first let me see what these guys do.

Sigma z you see when it operates on this atom in the ground state in fact you know this atom in the ground state we can write it in the represent it in a column vector as 0 1 this state and sigma z is the matrix that is your 1 0 0 - 1. So, if you multiply these two matrices you will get 0 - 1 which you can write it as - 0 1. So, this implies that sigma z you can write it as -1 when we have when we have the ground state of the atom. Therefore when h cross omega atom we have sigma z + 1 by 2 it operates on this state you will get because this would now become -1 therefore this whole thing would become equal to 0.

On the other hand you can similarly saw that sigma z is equal to + 1 when we have the atom mixed it to be in the excited one therefore we can write if h cross omega atom it if it operates on the other basis that is your atom in the up state and field is n - 1 field mode has one less photon then you will see that because now sigma z is equal to +1. So, you will have h cross omega atom and here you will have atom in the up state and n – 1.

Again we can write this work out this one sigma + a when it operates on this state now sigma + is the atomic raising operator. So, it will raise the atomic state to the up one and a is the annihilation operator of the field mode it will act on the field mode and it will reduce it by 1 and here the factor you know this root over n just recall that when in a operates on this number ket you will get square root of n n – 1.

And when sigma - a dagger operates on this state because already the atom is in the ground state it cannot go lower than that. So, therefore sigma minus which is the atomic lowering operator you it cannot lower it further. So, you will get simply zero here. On the other hand if you look at the other basis sigma + a when it act on the atom in the excited state and the field mode has n - 1 number of photons then sigma plus it cannot because you have only two energy levels only atomic energy level.

Already it is in the excited state it cannot increase it further it cannot raise it further. So, therefore here you will get zero on the other hand when sigma minus it a dagger operates on this state here you will this lowering atomic loading operator is going to lower the atomic state to ground state and a dagger when it applied on n - 1 it will make it n and here the factor would be this again you recall that a dagger n when it acts on you know that this would be square root of n + 1 and here and it would be n + 1.

So, we now get all the terms. So, we can now work out what is H of alpha. H of alpha is alpha is this atomic down state ground state and n this is the field mode. So, we have worked out all the terms. So, let me just write down the results so you just look at this one here. So, we have everything at our disposal. So, let me write down the final result you are going to get h cross omega n down state of the atom and n here and here you have another term h cross the square root of n upstate of the atom and n – 1.

Again you can work out H beta everything all the terms we know. So, this also you can work out this would be beta ket beta is atom in the excited state and n - 1 number of photons in the field mode and here you are going to get h cross omega n - 1 atom upstate n - 1 photons in the field mode and you will have h cross omega atom, atom in the up state n - 1 number of atoms in the field mode and the last term would be h cross g square root of n atom in the down state and here n number of photons in the field mode.

So, because now we have this we have worked out. So, we can easily find out all the elements. So, alpha it is alpha that is you have to look at the expressions and you can immediately make it out that this would be simply if you look at this one you will get it would be h cross omega n. Because you know that these states are normalized on the other hand if you have inner product with like this then you know that this would be equal to zero. So, utilizing these relations you can immediately work out all the terms.

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$$= \hbar\omega(n-1) + \hbar\omega_{at}$$

$$\hat{H} = \begin{pmatrix} \langle\alpha|\hat{H}|\alpha\rangle & \langle\alpha|\hat{H}|\beta\rangle \\ \langle\beta|\hat{H}|\alpha\rangle & \langle\beta|\hat{H}|\beta\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \hbar\omega n & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & \hbar\omega(n-1) + \hbar\omega_{at} \end{pmatrix}$$

Similarly we have alpha H beta that would be this one here it is upstate n - 1. So, again if you look at this term you can find out that this would be equal to h cross g square root of n and beta H alpha that is beta is atom in the upstate n – 1 photons in the field mode H atom in the ground state and n number of photons in the field mode if you work it out you will again get h cross g square root of n and finally beta H beta that would be equal to.

So, let me write down it explicitly you will have this one here. So, you just have to take the inner product with the state then you are going to get it as h cross omega n - 1 + h cross omega atom. So, thus we can write the Hamiltonian in the matrix form as follows. Now we have all the elements at our disposal. So, we have this matrix remain alpha H alpha alpha H beta beta H alpha and beta H beta we have evaluated all of them.

So, if we put them we will have here h cross omega n here we will have h cross g square root of n here we will have h cross the square root of n and here we have h cross omega n - 1 + h cross omega atom. So, this is the Jaynes Cummings Hamiltonian in the matrix form under rotating wave approximation.

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$$\tilde{H} = \hbar \begin{pmatrix} 0 & \Omega/2 \\ \Omega/2 & -\Delta \end{pmatrix}$$

$$E_{\pm} = -\frac{\hbar\Delta}{2} \pm \sqrt{\left(\frac{\hbar\Delta}{2}\right)^2 + \left(\frac{\hbar\Omega}{2}\right)^2}$$

$$|+\rangle = \sin\theta\,|g\rangle + \cos\theta\,|e\rangle$$

$$|-\rangle = \cos\theta\,|g\rangle - \sin\theta\,|e\rangle$$

with $\tan 2\theta = -\frac{\Omega}{\Delta}$

Now let us work out this particular problem .You are asked to discuss the drastic picture of the JC model under rotating wave approximation. So, let us first of all write down the Jaynes Cummings Hamiltonian under RWA we just we have worked out that is h cross omega n here h cross g square root of n h cross g square root of n and h cross omega n + h cross delta here the detuning parameter delta is equal to omega atom – omega.

Now earlier we have actually worked out the eigenvalues also in the class. So, eigenvalues are two eigenvalues you we have E + that would be h cross omega n + h cross delta by 2 + h cross square root of g square n + delta by 2 whole square and E - is equal to h cross omega n + h cross delta by 2 - h cross g square n + delta by 2 whole square. So, the eigen state of this Hamiltonian are the dressed states.

So, let us denote their eigen states corresponding to the eigenvalue E + to be + ket and corresponding to E - the eigen state is say this ket minus and these are the dressed state and they satisfy this equation it is if you apply on + ket you will get E + + ket and H when it operates on this ket state you will get E - this - ket state. Now actually we can do this address this problem very easily.

If you recall what we learned in lecture five of module one in the context of two level atoms just let me quickly remind you about that quick reminder about two level atom in semi classical approximation we did that because there the atom was considered to be quantized it has quantized energy level. And the electromagnetic field was considered to be classical and there we got these results if you recall the Hamiltonian we wrote there it was lecture 5 it was h cross 0 omega by 2 omega is the Rabi frequency.

And here you have minus delta, delta is the tuning parameter and the eigenvalues where E of this Hamiltonian we worked out as E + - is equal to - h cross delta by 2 + - square root of h cross delta by 2 whole square + h cross omega by 2 whole square. And the dress states we worked out as we defined it such that ket + is equal to sine theta the ground state of the atom the ket state ket g and it is cos theta ket e while here we had - ket is equal to cos theta ket c - sine theta ket e.

And with this angle theta is the Stackelberg angle and that we worked out as tan 2 theta is equal to – omega, omega is the Rabi frequency divided by the detuning parameter. So, this is what we have already learned in module one.

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$$\boxed{\begin{aligned} P_\downarrow(t) &= \frac{1}{2}\left(1 + \cos 2g\sqrt{n}\,t\right), \\ |\psi(t=0)\rangle &= |\downarrow, n\rangle \end{aligned}}$$

$$\Omega \longrightarrow 2g\sqrt{n}$$

$$\tan 2\theta_n = -\frac{2g\sqrt{n}}{\Delta}, \quad \left(0 < \theta_n < \frac{\pi}{2} \right.$$

So, in the similar spirit in the similar spirit then we can avoid doing detail calculation ket state of this Jaynes Cummings Hamiltonian to be let me write it as sine theta we have these two basis states atom in the ground state field n number of photons in the field mode + cos theta atom in the excited state and there are n - 1 number of photons in the field mode or to distinguish it from theta let me just put a suffix n.

So, that you understand that we are dealing with Jaynes Cummings Hamiltonian or that system now and your this other dressed state is cos theta n this particular state - sine theta n atom in the up state and n - 1 photons in the field mode. So, this is what we have. Again without doing calculation we can address this we can find out this Stuckelberg angle for theta here.

Because if you recall from lecture 16 that we have got the expression for the probability of finding the atom in the ground state at a instant of time say t as one half into one + cos of 2 g square root of n t and when we have started from the initial state at time t is equal to 0 the atom was assumed to be in the ground state and there are n number of photons in the field mode then this expression we obtained in lecture 16.

This shows that the Rabi frequency that we had in the context of two level atom in lecture 5. Here the Rabi frequency is for this completely quantized problem is 2 g square root of n. So, therefore we can now define the Stackelberg angle by this definition that is 10 to theta n is equal to minus the Rabi frequency here is 2 g square root of n divided by delta, delta is the tuning parameter and this angle lies between 0 to pi by 2.

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$$+ \hbar\left(\omega_{at} + \frac{g}{\Delta}\right)\frac{\sigma_z}{2}$$

$$\Big\downarrow \text{re-arranging}$$

$$\hat{H}_{eff} = \hat{\sigma}_z\left(\frac{\hbar g^2}{\Delta}\hat{a}^\dagger\hat{a} + \frac{\hbar}{2}\left(\omega_{at} + \frac{g^2}{\Delta}\right)\right) + \hbar\omega\,\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\left(\omega_{at} + \frac{g^2}{\Delta}\right)$$

$$\boxed{\hat{H}_{eff} = \hat{\sigma}_z\left(\frac{\hbar\omega_{at}}{2} + \frac{\hbar g^2}{\Delta}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right) + \hbar\omega\,\hat{a}^\dagger\hat{a} + \frac{\hbar}{2}\left(\omega_{at} + \frac{g^2}{\Delta}\right)}$$

Now let us work out this particular problem you are asked to show that in the dispersive regime that is when the magnitude of the detuning parameter is much larger than the coupling the effective Hamiltonian for the Jaynes Cummings model could be written in this particular form. So, let us do it in fact we have discussed the dispersive regime of Jaynes Cummings model in lecture number 16.

And there if you recall we have written down the Hamiltonian and in fact I request you to refer to that particular lecture, lecture number 16 there we wrote down the effective

Hamiltonian in this form it was h cross omega + g square by delta sigma z a dagger a + h cross omega atom + g square by delta sigma z + 1 by 2. Now various terms can be rearranged. So, if we rearrange the terms then we can write this effective Hamiltonian in this form.

Let me take sigma z common then I have terms like s cross g square by delta a a + h cross by 2 omega atom + g square by delta and we have term s cross omega a dagger a + a half h cross omega atom + g square by delta this I can write little bit in a more cleaner form I have sigma z h cross omega atom by 2 + h cross z square by delta a dagger a + half and we have the other two terms h cross omega a dagger a + h cross by two omega atom + g square by delta.

So, in fact if you look at this Hamiltonian this is what is asked in the equation this is the required form of the Hamiltonian as you can see from here.

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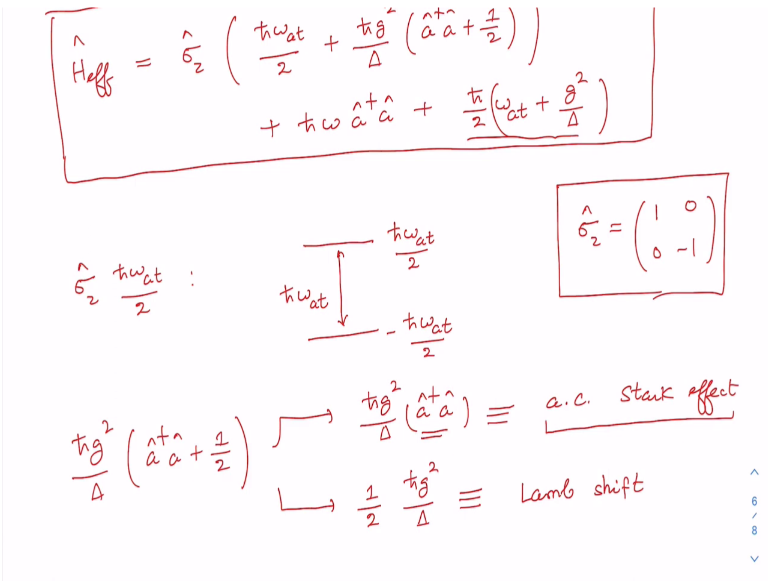

Now let us interpret it if you look at the first term. So, this first term refers to the fact that there is transition between the two atomic energy level or the qubit energy level we have two energy level the excited one is h cross omega atom by 2 and ground one is h cross - h cross omega atom by 2 and they are separated by a frequency omega atom or energy differences h cross omega atom.

In fact you know that sigma z the matrix form is 1 0 0 -1. So, this clearly shows that significance of this particular term. And then let us look at the second term here the second term if you look at it, it is h cross g square by delta a dagger a + half. So, it consists of two terms as if one term is just like in the harmonic oscillator. So, this is basically due to the zero point fluctuates on this particular term is it has two parts one is h cross g square by delta a dagger a a dagger is the a is the number of photons and the other one is due to the zero point fluctuation that is half h cross g square by delta.

Now because of the fact that this particular term you have is proportional to the number of photons and you can say that this term is analogous to the so called ac Stark effect. Because overall this whole term it is related to the shift in the atomic transition frequency and in the presence of the photons that atomic transition frequency is basically getting shifted and photons you know is related to the is the quanta of the electric field.

So, that is why we can term it as ac Stark effect on the other hand this term is shift refers to the shift due to the vacuum fluctuation and this has a name and it is called it is analogous to the so-called Lamb shift. On the other hand this you know that this is familiar to you this i the energy of the film mode. And finally this term is there because it is needed to have our specific definition of zero point energy.

It is not related to any operator this is related to simply you know no pauli metrics nothing is associated with it this is just a you can consider it to be kind of a offset term and it is needed for our step specific definition of zero point energy.

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$$\hat{\phi}(t) = 2\pi \frac{\Phi(t)}{\Phi_0}$$

where, $\Phi(t) = \int v(t')\,dt'$

$$\hat{H}_J = -E_J \cos\hat{\phi}$$

$$= -E_J \cos\left(2\pi \frac{\hat{\Phi}(t)}{\Phi_0}\right)$$

$$\Rightarrow \boxed{E = -E_J \cos\left(\frac{\hat{\Phi}(t)}{\Phi_0}\right)}$$

Let us now work out this particular problem you are asked to show that the potential energy function of the transform could be written in the form E is equal to – E J cosine of phi j - phi zero phi j is the magnetic flux across the junction and phi zero is the so-called flux quantum and E J is the Josephson energy. In fact we discussed about transmon Hamiltonian in lecture number 18 of the course. There we derive the Hamiltonian for the Josephson junction in this form H J is equal to – E J cos phi cap where phi is the phase difference across the junction.

Let me have a give you a quick recap of the derivation we started with the well-known expression for the tunneling energy in the Hamiltonian. So, H J is equal to – E J by 2 sum over N and N running from - infinity to + infinity and we have this ket N and + 1 then ket N + 1 and bra N. So, this is the Hamiltonian we started with this Hamiltonian can be rewritten in terms of phase difference across the junction phi and which can be created as an operator and to do that we defined an new basis that is phi basis where this is defined as summation going from N is equal to - infinity + infinity e to the power i N phi.

And this is the ket N. Conversely we can write this as ket N is equal to 1 by 2 pi integration 0 to 2 pi d phi e to the power – I N phi phi it is very easy to show it that this is what you will get this allows us to write the Hamiltonian in this form H J is equal to - e j by 2 1 by 2 pi integration 0 to 2 pi d phi e to the power i phi + e to the power - i phi and we have this outer product.

So, this is you can see this is an operator. So, that is why what we can do we can introduce a new operator introducing an operator e to the power i phi cap and that is equal to 1 by 2 pi 0 to 2 pi d phi e to the power i phi this ket phi bra phi. So, we therefore write H J is equal to – E J by 2 you can easily write e to the power i phi cap + e to the power - i phi cap and that is equal to - E J cos phi cap.

So, this phase difference phi is related to the magnetic flux via this equation that we discussed in our lecture 18 phi dot is equal to 1 by h cross q into v charge into v here charge is equal to the two electron charge 2e. So, therefore we can write d phi dt is equal to h cross is the reduced Planck's constant that is h by 2 pi. So, therefore I can write it as 2 pi by h cross is the Planck’s constant q is 2e and v is the potential difference of voltage.

And this I can write as 2 pi divided by phi 0 into v where phi 0 is the so-called flux quantum and it is equal to h by 2 e. Now we can write we can now write this phi of t is equal to 2 pi magnetic flux phi of t divided by the flux quantum and where this magnetic flux phi of t is equal to integral of the voltage v of t dt dash. So, therefore we can write the Hamiltonian – E J cos phi cap is equal to – E J cos phi cap is now it is operator.

So, we have 2 pi this flux phi k by phi 0 this i can simply write as - E J cosine of magnetic flux operator divided by the flux quantum phi 0 and in fact this is the required energy function I have in fact this is the part of the Hamiltonian but again this is nothing but the energy function that we are asked to work out.