Quantum Technology and Quantum Phenomena in Macroscopic Systems

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Lecture-21
Problem Solving Session – 5.

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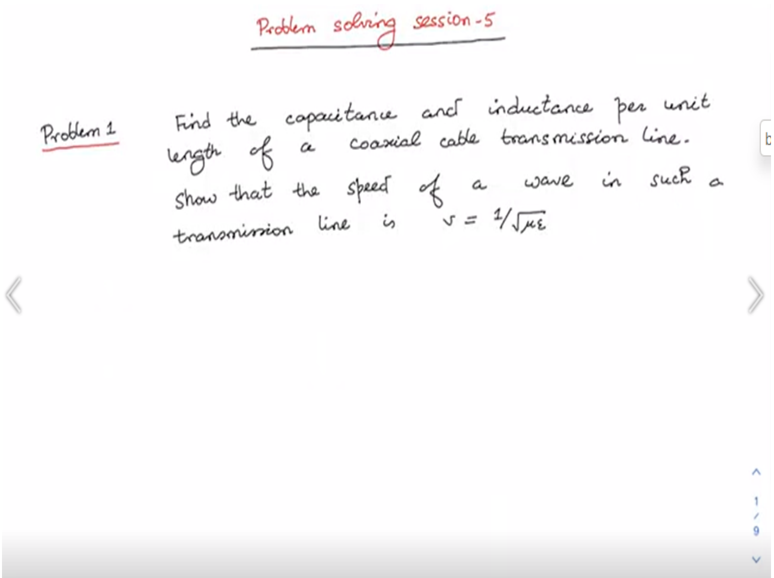

Welcome to this Problem-solving session number 5. In this problem-solving session, we are going to discuss problems related to transmission line. So, in this first problem, you are asked to find the capacitance and inductance per unit length of a coaxial cable transmission line. And you are asked to show that the speed of the wave in such a transmission line is v = 1 / square root of mu epsilon, mu is the permeability of the medium and epsilon is the permittivity of the medium.

Let us do it. First of all, we know that the transmission line basically consist of 2 conductors in parallel and coaxial cable consists of 2 concentric cylindrical conductors.

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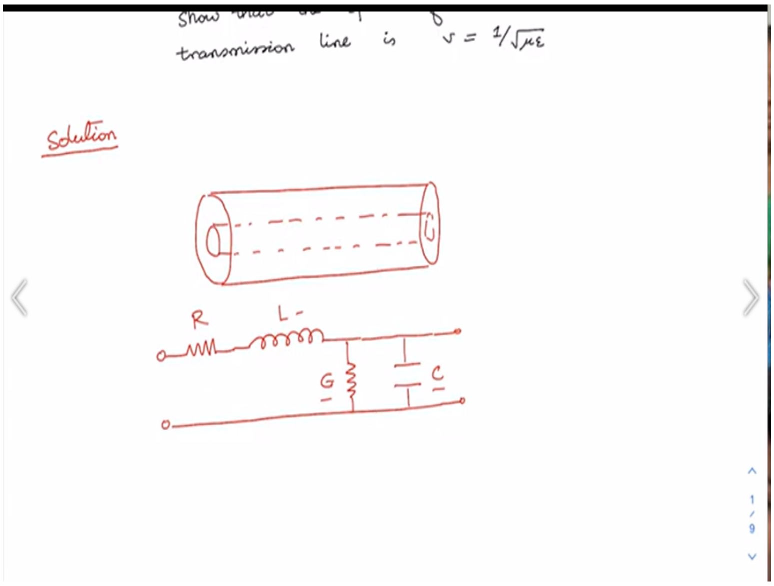

So, you have an inner conductor and you have an outer conductor. So, you have said we have a coaxial cable like this. This is the outer conductor and this is the inner conductor. And this coaxial cable has a circuit representation and in circuit representation it is represented by a resistance in series with this inductor L resistance R and then we have this conductance. In the class, actually, you have taken a very simplified model of the transmission line.

But if I do it rigorously, then that is how we will do it we have a conductance and also, we have this capacitance here. So, here R is the total resistance of the coaxial cables, C is the total capacitance, L is the total inductance, G is the total conductance. In fact, it gives the total G refers to the total leakage resistance.

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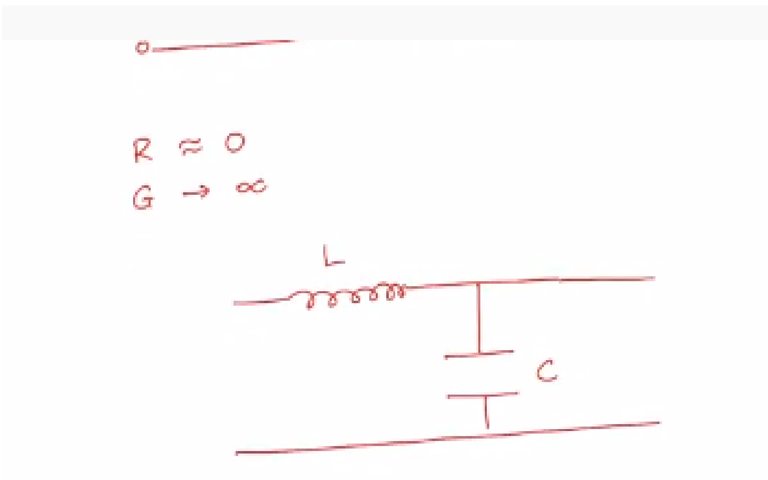

And in general, R is considered to be very small. So, in superconducting circuit we consider the transmission line to be nearly lossless. So, therefore R is 0 at the conductance is taken to be very large, so very huge. So, G this conductance tends to actually infinity. So, under these two approximations the schematic of the coaxial cable would be very simple. So, we will have an inductance as well as we will have this conductance only. So, now, let us go back to the given problem where we are asked to find out the capacitance as well as the inductance of this coaxial cable.

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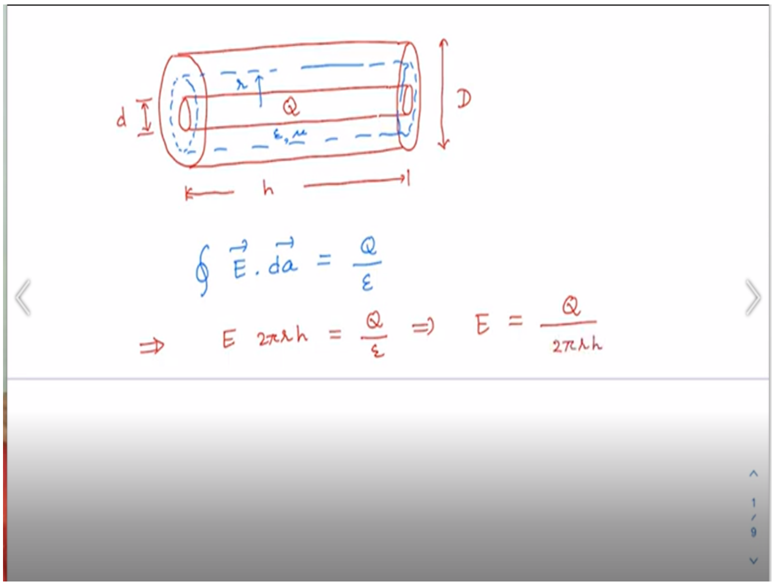

So, to do that let me draw the coaxial cable again. We have this inner cylinder is there and we have this outer cylinder. Let us say this outer cylinder has diameter D, and the inner cylinder has diameter d and the length of the coaxial cable is h. So, let us say charge Q is there inside the inner conductor and if I now draw a Gaussian surface, a Gaussian cylinder like this which has radius r. We know how to solve this kind of typical problem.

So, here this radius is r. There we can find out using Gauss law the electric field. As part of Gauss law, we have E dot da is the area limit that is equal to Q / epsilon. Epsilon is the permittivity and mu is the permeability of the medium that is this coaxial cable is containing. And then here we will have E into 2 pi r h, just look at the Gaussian cylinder here. So, if you look at the Gaussian cylinder here, so, then you have here Q / epsilon and you know that electric field would be Q divided by 2 pi r h epsilon.

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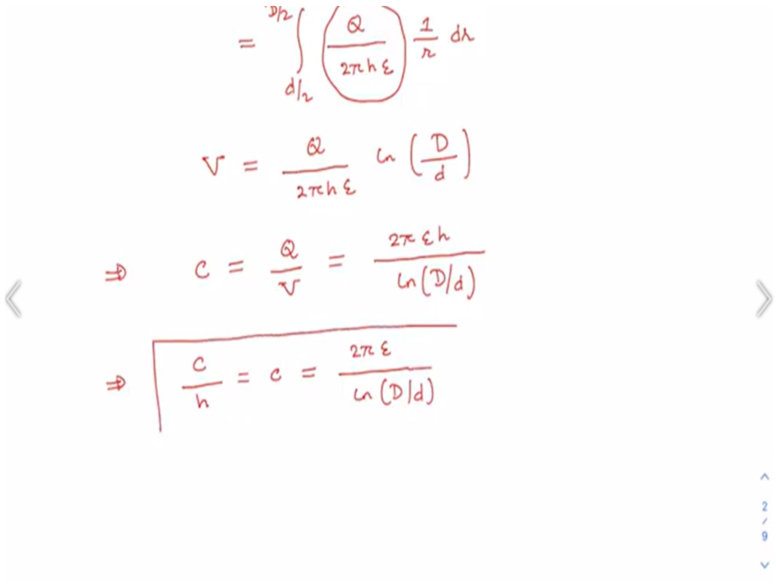

Now, what about the voltage difference. You just have to find out the voltage difference, because, ultimately what is in my mind is to apply this formula that Q = C into V, thereby I will be able to find out the capacitance. So, I need to know the voltage difference. So, voltage difference would be E dr, and let us say you are going from the inner cylinder to the outer cylinder and inner cylinder has radius d / 2 and the outer cylinder has radius D / 2.

And if I put the expression for the electric field, so let us do it. So, I put the electric field expression, so, it is Q / 2 pi h epsilon 1 / r dr. So, this is a constant term. So, I can take it out and then 1 / r dr, that is logarithm of r and if I put the limit then you will get this expression very easily, very straightforwardly you will get 2 pi h epsilon logarithm of D / d. So, this is the voltage and because we know that Q = CV. So, from here I can therefore write that is capacitors would be Q / V and this would give us 2 pi epsilon h divided by logarithm of D / d.

Now, what about the capacitance per unit length. So, that would be C / h. So, I can write it as c, that is the capacitance per unit length there would be 2 pi epsilon divided by logarithm of D / d. So, this is what we have capacitance per unit length of the coaxial cable. Now, let us find out the inductance.

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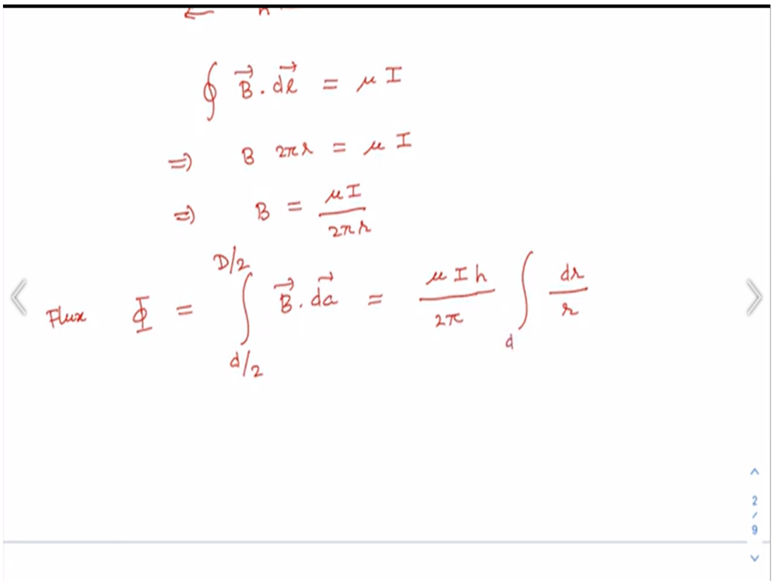

To find the inductance, let us assume that inside this inner cylinder, we have a current I. We have this outer cylinder so, we have to apply the Amperes law to apply the Ampere law, let me draw an Amperian loop of radius r then this has length as again h. So, if I apply the Ampere's law, I know that ampere law is B dot dl = mu I, mu is the permeability. So, this will give the B into 2 pi r = mu into I.

So, therefore, magnetic field B = mu I / 2 pi r and now what about the flux tthrough the imaginary this amperian loop? The flux phi is integration B dot da and you have to take it from the limit, you have to go from the inner cylinder to the outer cylinder so, it is d / 2 to D / 2. If I put the expression for the magnetic field, I can take it out mu I / 2 pi h if it take it out and here I have dr / r. Integration from d / 2 to D / 2 and again the similar way we know that this would be simply mu I h / 2 pi logarithm of D / d.

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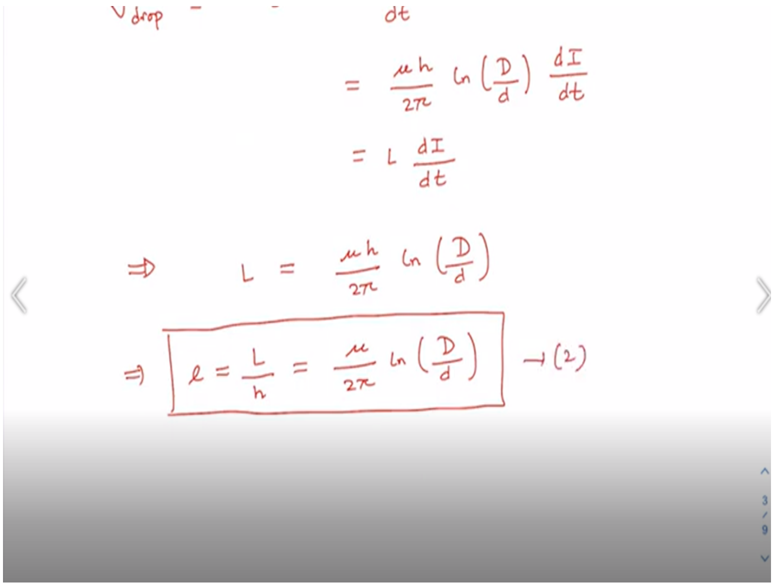

Now, what is the voltage drop? Voltage drop is actually it is the EMF = - d phi dt. And if I know the expression of phi here, so, let me utilize it and I will get it as mu h / 2 pi logarithm of D / d and dL dt, sorry dI dt here, this would be because this the current, so dI dt and this EMF is actually equal to L dI / dt, voltage drop across the inductance is L dI / dt. So, if I compare it, so, these will immediately give me the total inductance that will be L is equal to you will have it as mu h divided by 2 pi logarithm of D / d.

And what about the inductance per unit length? That will be L / h would be equal to mu / 2 pi logarithm of D / d. So, this is the inductance. So, we have both the inductance in the capacitance per unit length we have worked out. So, this is equation 1 and we have this equation 2. Now, finally, we are asked to find out the velocity of the electromagnetic wave there or we already know from our transmission line discussion in our class that of velocity is equal to 1 divided by square root of l c. And if I put this you will immediately get, you please check it. You will get to the l / square root of mu into epsilon.

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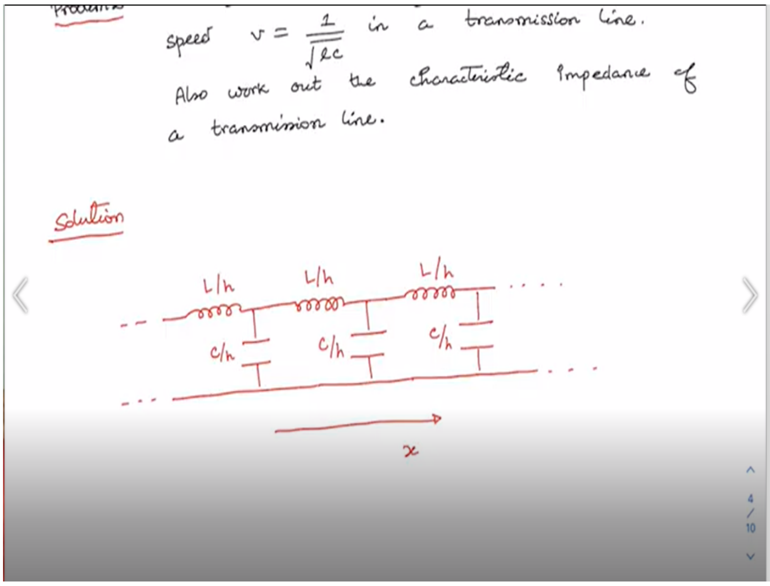

Now, let us do this problem, you are asked to show that the voltage travels with a speed v = 1 / square root of l c in a transmission line. Here l is the inductance per unit length and c is the capacitance per unit length, also work out the characteristic impedance of a transmission line. Let us do it. We learned that a transmission line without loss can be modelled by a series of unit cells comprising of inductance and capacitance like this. So, this is one unit cell. There we have a unit cell like this and so on.

This is loss-less transmission line and this is, say, inductance is L. Inductance per unit length let me take it to be L / h and this is L / h and this is capacitance per unit length this a C / h, L / h and so on. So, we consider that this transmission line is extended we say one-dimensional transmission line and it is extended along the x direction. So, it is infinite transmission line and now, let us consider a unit cell in the length delta x.

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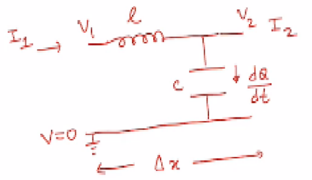
x

$$l = \frac{L}{h} \; ; \quad c = \frac{C}{h}$$



$$V_1 - V_2 = \frac{L}{h} \Delta x \frac{dI}{dt} \rightarrow (1)$$

$$I_1 - I_2 = \frac{dQ}{dt} = \frac{C}{h} \Delta x \frac{dV_2}{dt} \rightarrow (2)$$

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Let me just consider only one unit cell like this. So, say this length is delta x, this is l / h, which let me now write as l and this is c, l is L / h inductance per unit length and c is capacitance per unit length. Now Let us say here current I 1 is flowing through it and this is current I 2 and the voltage here is V 1 and the voltage across here is V 2 and a current is flowing to the capacitance so, that is dQ dt. The capacitor is getting charged and here the voltage is V = 0 and this part is grounded.

So, in that case the potential difference across the inductor would be V 1 – V 2, that would be equal to across the inductance. The inductance L / h and the length is delta x. So, this is the total inductance. So, L dI / dt we know so, this is the voltage difference. Let us say this is my equation number 1. On the other hand, difference in the current is I 1 – I 2 that is across the capacitance we have dQ dt, this is what we have and we know that we have Q = C V. C is the capacitance. So, total capacitance would be C / h into delta x and here we have dv 2 dt 2. So, this is let us say it is equation number 2.

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$$\boxed{V_1 - V_2 = \frac{L}{h}\Delta x \frac{dI}{dt}} \rightarrow (1)$$

$$I_1 - I_2 = \frac{dQ}{dt} = \frac{C}{h}\Delta x \frac{dV_2}{dt} \rightarrow (2)$$

$$\Delta V = V(x_2) - V(x_1)$$

$$\frac{\Delta V}{\Delta x} = -\frac{L}{h}\frac{dI_1}{dt}$$

$$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = \boxed{\frac{\partial V}{\partial x} = -l\frac{\partial I}{\partial t}} \rightarrow (3)$$

So, the potential difference delta V, I can actually write it as V because this is I have considering it is a continuum kind of a transmission line. So, we have very small inductance and capacitances are there. So, I can write the potential difference as V of x 2 – V of x 1, the 2 different locations, these I can write also as delta V / delta x if I divide it by the length. So, from this equation number 1, I can write it as minus because I have taken V x 2 – V x 1. Unlike here it is V 1 where this is V 2. So, the side would get changed.

So, therefore, I have minus I hope you are getting the idea L / h and delta x, I am taking to the other side. So, it will have L / h d this would be I 1 dt. Now, in the limit delta x tends to 0, we will have this we can write del V / del x, we can write as del V del x that will be equal to minus l / h. Let me write it as l now I have del I del t. So, this is let me say equation number 3.

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$$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = \boxed{\frac{\partial V}{\partial x} = -l \frac{\partial I}{\partial t}} \rightarrow (3)$$

$$\text{Again,} \qquad \Delta I = I(x_2) - I(x_1)$$

$$\frac{\Delta I}{\Delta x} = -\frac{c}{h}\frac{dV_2}{dt}$$

$$\lim_{\Delta x \to 0} \frac{\Delta I}{\Delta x} = \boxed{\frac{\partial I}{\partial x} = -c \frac{\partial V}{\partial t}}$$

Similarly, again we can write the current difference that is delta I is equal to again look at this equation delta I, I can write it as Ix 2 - Ix 1. Here I am writing it as I 1 – I 2 again you will get a change in sign. So, therefore, I can write delta I / delta x = - C / h dV 2 dt. Please look at this equation again here. So, again in the limit delta x tends to 0. I can write delta I / delta x in the limit delta x tends to 0 as del I / del x that would be equal to this let me write c capacitance per unit length. Let me here I have del V del t, this is another equation that I get.

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$$\boxed{\begin{aligned} \frac{\partial V}{\partial x} &= -l \frac{\partial I}{\partial t} \\ \frac{\partial I}{\partial x} &= -c \frac{\partial V}{\partial t} \end{aligned}}$$

$$\frac{\partial^2 V}{\partial x^2} = -l \frac{\partial}{\partial t}\left(\frac{\partial I}{\partial x}\right)$$

So, let me write these 2 equations separately, basically we have 2 equations, one for voltage, how the voltage is changing with distance, then this is equal to l del I del t and we have delta I delta x = - c capacitance per unit length del V del t. So, these 2 equations now we are going to exploit. Now, if I take the time derivative of this equation, actually, not time derivative, space derivative with respect to x. If I differentiate on both sides. I will have del 2 V del x 2 that would be equal to minus l del t. Let me write here del I del x and from this equation.

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$$\frac{\partial^2 V}{\partial x^2} = -l \frac{\partial}{\partial t}\left(\frac{\partial I}{\partial x}\right)$$

$$\Rightarrow \boxed{\frac{\partial^2 V}{\partial x^2} = lc \frac{\partial^2 V}{\partial t^2}} \qquad \frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

velocity $v =$

Therefore, I write it as minus and another minus is here so, it would be plus l c delta V del t 2. So, this is the equation for voltage. So, I get del 2 V del x 2. Actually, it is pretty easy to see if you compare this equation with the classical wave equation that is your del 2 f del x 2 = 1 / v square del 2 f del t 2. From a classical wave equation the velocity so this is, do not get confused with this V.

This is let me say V voltage, this is V, then the velocity so, that is a v = 1 / square root of l c. So, you see the voltage travels at the speed of given by 1 / square root of l c. Another way to look at it because we know that this equation has a traveling wave solution.

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$$\# \; V = A e^{i(kx - \omega t)}$$

$$k^2 = \omega^2 lc$$

$$\Rightarrow \frac{\omega}{k} = v = \frac{1}{\sqrt{lc}} \; //$$

$$\frac{\partial I}{\partial t} = -\frac{1}{l}\frac{\partial V}{\partial x} = -\frac{1}{l}\, ik A e^{i(kx-\omega t)}$$

So, V, I can write it as A e to the power i kx - omega t. So, there is a traveling wave along the x direction. So, if I put this solution V = A e to the power ikx - omega t in this equation, then you can see that you will get this dispersion relation that would be k square = omega square lc. If you put it you will see that and from here you can write omega / k the magnitude would be which is basically the velocity and that would be equal to again 1 / square root of lc. Now, we are asked to find out the characteristic impedance as well. So, to do that,

let me go to this current equation here. I can form this relation, I can written del I del t = - 1 / l del V del x. So, now, I know the solution for a potential V = A e to the power ikx - omega t. This one I have so, if I put it there then I will get it as –1 / l ik A e to the power ikx – omega t.

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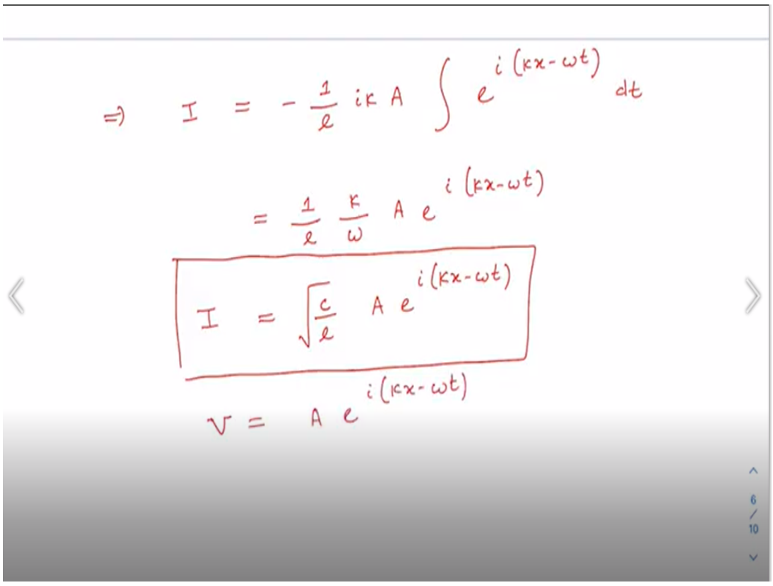

So, from here I can find out the current, if I take the integral. So, I will have minus 1 / l. Let me first write the integral I k A integration e to the power ikx - omega t dt and if I do the integration so you will get it. I will give you a very simplified approach here. So, you will get it as k / omega A e to the power ikx - omega t or from here, because I know what is omega / k that is 1 / square root of lc. This I know, so if I put it there so, I will get c / l square root of l A e to the power ikx - omega t.

So, this is my current and voltage also already I know that is A e to the power ikx - omega t. Now, let us consider the voltage and current at any particular point on the transmission line.

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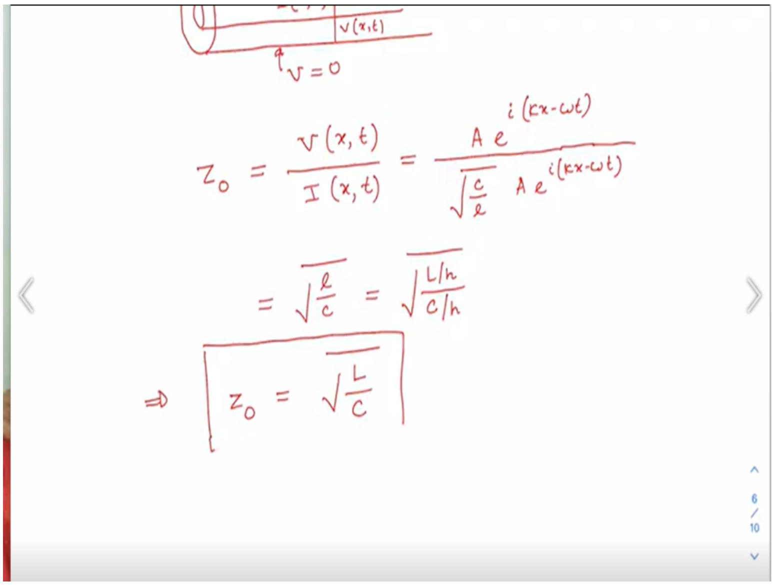

Now, if we look at the picture of the coaxial transmission line which has inner cylinder as well as outer cylinder. So, let us say in the inner cylinder at any given point here the current is I of x t it is going along this direction along x direction and its potential here is V of x of t and if the outer cylinder we consider that its potential is V = 0 then the characteristic impedance is given by the ratio of the voltage given at a point divided by the corresponding current at that point instead of time t.

So, V of x t is A e to the power ikx - omega t and current already this expression we know that is square root of c / l A e to the power ikx - omega t. So, therefore Z 0 I can write as square root of l / c which is also you can write as total inductance per unit length L / h here is total capacitance per unit length. So, this is generally characteristic impedance is written as square root of L / C. So, this is the well-known expression for the characteristic impedance of a transmission line.

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Problem 3 Show the detailed derivation of the dispersion relation obtained in the process of quantization of the infinite transmission line.

Solution

$$\hat{H} = \sum_n \frac{\hat{P}_n^2}{2C} + \frac{\left(\hat{\Phi}_{n-1} - \hat{\Phi}_n\right)^2}{2L} \rightarrow$$

Let us now work out this problem. You are asked to show the detailed derivation of the dispersion relation obtained in the process of quantization of the infinite transmission line. If you remember we have actually done the quantization problem in lecture number 14 where I have showed you the process how to quantize it. Here I will actually fill up the gaps because I skipped, I just give you the steps there, but here I will show you all the steps.

The transmission line Hamiltonian, we wrote it in this form that is summation over n P n square / 2C + phi n – 1 phi is the magnetic flux phi n whole square divided by 2 L. L is the inductance total inductance of the transmission line and C is the total capacitance of the transmission line.

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Solution

$$\hat{H} = \sum_n \frac{\hat{P}_n^2}{2C} + \frac{\left(\hat{\Phi}_{n-1} - \hat{\Phi}_n\right)^2}{2L} \rightarrow (1)$$

where,

$$\hat{\Phi}_n = \sum_k A_k \left(a_k e^{ikx} + a_k^{+} e^{-ikx} \right)$$

$$\hat{P}_n = \sum_k C A_k (-i\omega_k) \left\{ \hat{a}_k e^{ikx} - \hat{a}_k^{+} e^{-ikx} \right\}$$

with, $A_k = A_{-k}$, $\omega_k = \omega_{-k}$, $x = na$

And here phi n, this is equal to the sum over k, k lies between -pi / A to +pi / A amplitude A k into a k that is the annihilation operator e to the power ikx + a k dagger e to the power – ikx and the momentum P n = sum over k C into A k – i omega k. Omega k is the angular frequency of the kth point. a k e to the power ikx – a k dagger e to the power – ikx. Because of transitional invariance we had A k = A –k, omega k = omega -k and x = n into a, n is the number of units cell and a is the size of the unit cell. Now, let us expand the terms in this Hamiltonian one by one.

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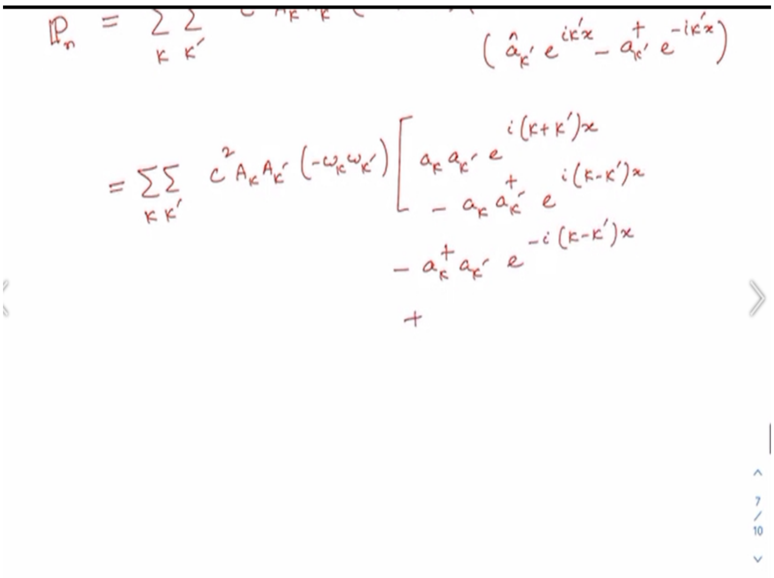

So, first let me expand P n square. So, this would be, we will have now 2 sum over case. Let us say what is k dash and another is k. So, if I square it I will have C square A k, A k dash. I will have i i , minus minus plus, so i square is minus, so therefore, –omega k omega k dash and here I have a k. Let me write it a k e to the power ikx – a k dagger all these are operators. a dagger e to the power – ikx into a k dash e to the power I k dash x – a k dash dagger e to the power – i k dash x.

Now, let me expand it so, here I will get k k dash C square A k A k dash - omega k omega k dash. I will have a k a k dash e to the power i k + k dash x – a k a k dagger dash e to the power i k – k dash x – a k dagger a k dash e to the power –i k – k dash x, then I have plus a k dagger a k dash dagger e to the power -i k + k dash x.

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Orthogonality conditions

$$\sum_n \sum_k \sum_{k'} e^{i(k+k')x} = \frac{W}{a}\,\delta_{k,-k'}$$

$$\sum_n \sum_k \sum_{k'} e^{i(k-k')x} = \frac{W}{a}\,\delta_{k,k'}$$

Now let us use the orthogonality condition, I talked about it in the class, orthogonality condition. I have here as sum over n k k dash e to the power i k + k dash x that would be equal to w / a delta k – k dash that means when k = - k dash we will get 1 here and then sum over n will give you the total number of cells and that is w / a. So, similarly, we will have another condition that will be sum over n k k dash e to the power i k – k dash x that would be equal to w / a delta k k dash. These are the conditions that we are now going to exploit.

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Then,

$$\sum_n \hat{P}_n^2 = \frac{W}{a}\sum_k \sum_{k'} C^2 A_k A_{k'} \left(-\omega_k \omega_{k'}\right) \left[a_k a_{k'}\,\delta_{k,-k'} - a_k a_{k'}^{\dagger}\,\delta_{k,k'} - a_k^{\dagger} a_{k'}\,\delta_{k,k'} + a_k^{\dagger} a_{k'}^{\dagger}\,\delta_{k,-1} \right.$$

So, then what we are going to do? We will get is this. Sum over n now, so P n square this will give. So, you just have to this is difficult maybe I have to make it, scroll it again and again. But if you just write it on a piece of paper that will be more useful. So, let me now write down what you are going to get if you apply this orthogonality condition. You will get w / a summation over k and k dash C square A k A k dash - omega k omega k dash. And here you will get a k a k dash delta k –k dash – a k a k dash dagger delta k k dash – a k dagger a k dash delta k k dash. And we will have finally +a k dagger a k dash dagger delta k – k dash.

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$$- a_k^{\dagger} a_{k'} \delta_{k,k'} + a_k' a_{k'} \delta_{k,-k'} \Big]$$

$$\boxed{\Rightarrow \sum_n \frac{\hat{P}_n^2}{2C} = \frac{1}{2C} \sum_k \frac{W}{a} c^2 A_k^2 (-\omega_k^2) \left(a_k a_{-k} - a_k a_k^{\dagger} - a_k^{\dagger} a_k + a_k^{\dagger} a_{-k}^{\dagger} \right) \longrightarrow (2)}$$

So, this implies that I will get the term, the first term in the Hamiltonian, that would be P n square / 2C that would be equal to 1 / 2C summation over k w / a C square A k square - omega k square, inside the bracket I will have a k a -k – a k a k dagger – a dagger k a k + a k dagger a -k dagger. So, this is what I will get. So, let me say this is my equation number 2. So, I actually worked out, expanded the first term in our Hamiltonian.

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$$\frac{(\phi_{n-1} - \phi_n)}{2L} = ?$$

$$\hat{\phi}_n = \sum_k A_k \left(a_k e^{ikx} + a_k^{\dagger} e^{-ikx} \right)$$

Let me now look at the second term that means, we have to now find out phi n – 1 – phi n whole square divided by 2 L. So, what is this? Let us work it out. Because we have phi n = sum over k A k a k e to the power of ikx + a k dagger e to the power –ikx.

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$$\hat{\Phi}_{n-1} = \sum_k A_k \left(a_k e^{ik(n-1)a} + a_k^\dagger e^{-ik(n-1)a} \right)$$

$$\Rightarrow \hat{\Phi}_{n-1} = \sum_k A_k \left(a_k e^{ikx} e^{-ika} + a_k^\dagger e^{-ikx} e^{ika} \right)$$

Thus, $\hat{\Phi}_{n-1} - \hat{\Phi}_n$

So, we will have phi n - 1, this would be equal to sum over k A k a k e to the power, you see x = n into a. So, therefore let us be exploit that here. Here we will have e to the power ik n – 1 a + a k dagger e to the power – ik n – 1 into a. So, using this I can therefore write it implies that phi n – 1 = sum over k A k a k e to the power ikx e to the power – ika + a k dagger e to the power –ikx e to the power ika. Very simple. So, therefore, we have phi n – 1 – phi n that we can now work out.

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$$= \sum_k \left[A_k a_k e^{ikx} \left(e^{-ika} - 1 \right) + A_k a_k^\dagger e^{-ikx} \left(e^{ika} - 1 \right) \right]$$

$$= \sum_k \Big[A_k a_k e^{ikx} e^{-\frac{ika}{2}} \left(e^{-\frac{ika}{2}} - e^{\frac{ika}{2}} \right) + A_k a_k^\dagger e^{-ikx} e^{\frac{ika}{2}} \left(e^{\frac{ika}{2}} - e^{-\frac{ika}{2}} \right) \Big]$$

e

That would be equal to sum over k you have here A k a k e to the power ikx e to the power – ika – 1 and we have A k a k dagger e to the power – ikx e to the power ika – 1. This I can further write as summation over k A k a k e to the power ikx e to the power –ika / 2 then here I will have e to the power - ika / 2 - e to the power ika / 2 and + A k a k dagger e to the power

- ikx e to the power ika / 2 e to the power ika / 2 - e to the power -ika / 2. And you know that e to the power - theta - i theta + e to the power, you know e to the power i theta – e to the power – i theta equal to twice i sin theta. So, we can use this identity and then put it in the expression and we will get this one

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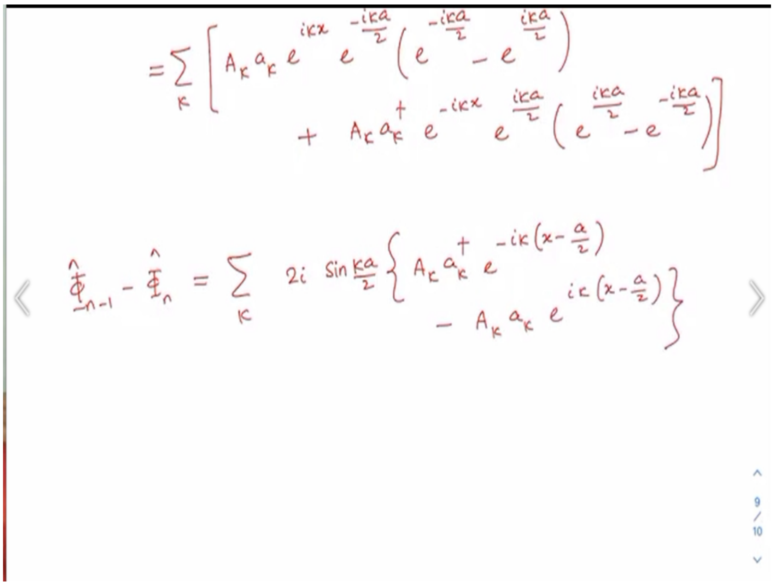

Let me write the final expression. Finally, what you will get if you apply that you will get phi n -1 - phi n = sum over k 2i sin ka / 2 and here we will have A k a k dagger e to the power - ik x - a / 2 – A k a k e to the power ik x - a / 2.

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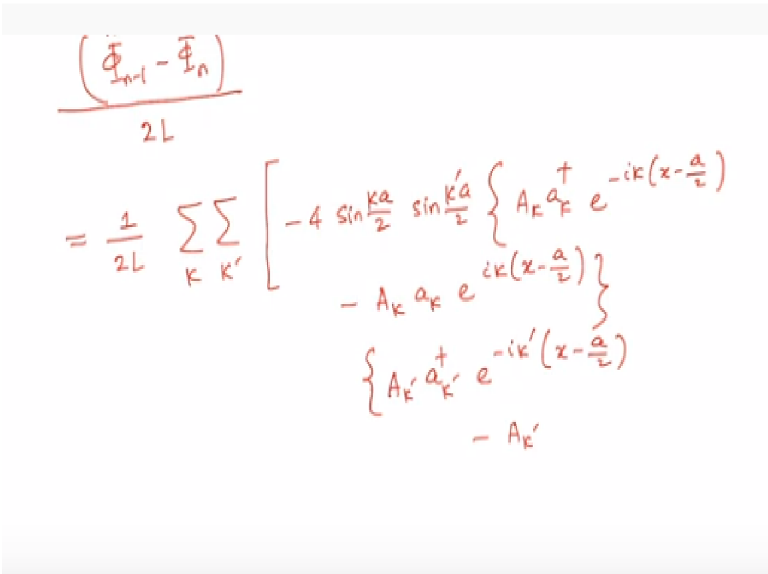

So, this is what I will get. Therefore, I have phi n - 1 - phi n whole square divided by 2 L. So, this would be equal to 1 / 2 L. Now I have to take 2 sum over case, k another one is k prime and here I will have utilizing this expression I will get let me we write the full thing you will have 4 – 4 sin ka / 2 sin k dash a / 2 you will have A k a k dagger e to the power –i k x – a / 2 – A k a k e to the power ik x - a / 2. This is one term and you have to multiply it with A k dash a dagger k dash e to the power - i k dash x – a / 2 – A k dash a k e to the power i k dash x – a / 2. This is basically algebra.

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$$
= -\frac{2}{L}\sum_{k}\sum_{k'} \sin\frac{ka}{2}\sin\frac{k'a}{2}\Big[A_k A_{k'}\, a_k^{\dagger} a_{k'}^{\dagger}\, e^{-i(k+k')x}\, e^{i(k+k')\frac{a}{2}} - A_k A_{k'}\, a_k^{\dagger} a_{k'}\, e^{-i(k-k')x}\, e^{i(k-k')\frac{a}{2}} - A_k A_{k'}\, a_k a_{k'}^{\dagger}\, e^{i(k-k')x}\, e^{-i(k-k')\frac{a}{2}} + A_{k'} A_{k'}\, a_k a_{k'}\, e^{i(k+k')x}\, e^{-i(k+k')\frac{a}{2}}\Big]
$$

Let me expand it. Let me do it. What I will have here? I will have – 2 / L sum over k and k dash sin ka / 2 sin k dash a / 2. In the bracket I will have A k A k dash you will get several terms a k dagger a k dash dagger e to the power -i k + k dash x e to the power i k + k dash a / 2 and you will have – A k A k dash a k dagger a k dash e to the power - i k – k dash x e to the power i k – k dash a / 2 and you will have – A k A k dash a k a k dash dagger e to the power i k – k dash x e to the power - i k – k dash a / 2.

And finally, you will have A k dash A k dash a k a k dash e to the power i k + k dash x e to the power – i k + k dash a / 2. As you can see these are tedious algebra but you have to keep on doing it systematically then you will get the results.

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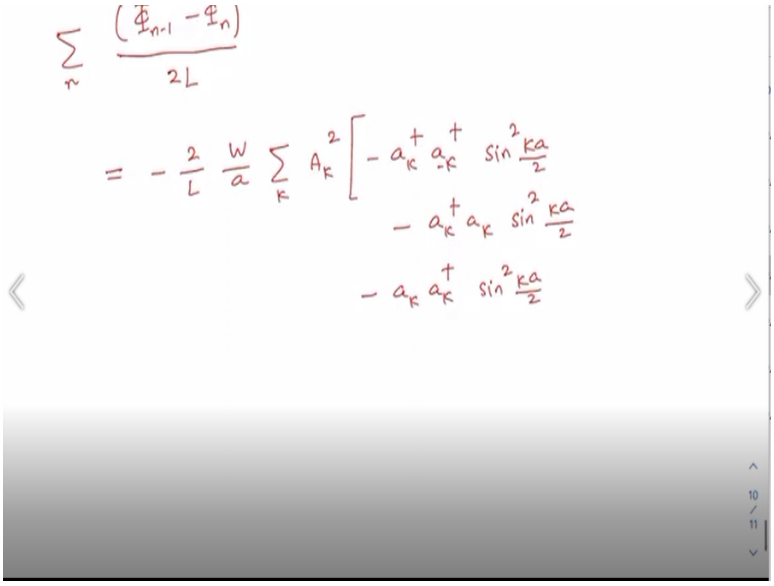

Now apply orthogonality condition as we discussed earlier. So, if I do that you have to sum over n phi n – 1 - phi n whole square / 2L. So, you will get - 2 / L w / a sum over k A k square and here I will have minus a k dagger – a – k dagger sin square ka / 2 minus you will get a k dagger sin square ka / 2.

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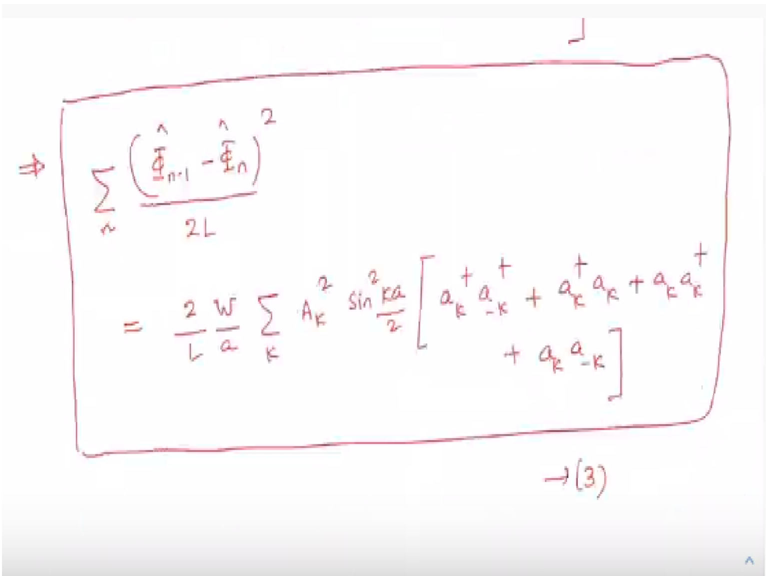

And you will get finally –a k a - k sin square ka / 2. These are the terms you will get. So, let me therefore, finally write that I will get summation over n phi n -1 – phi n the whole square. These are operators divided by 2L = 2 / L W / a sum over k A k square sin square ka / 2. Here in the bracket you will get a k dagger a – k dagger + a k dagger a k + a k a k dagger + a k a -k. So, this is what is the second term in the Hamiltonian. Let me say this is equation number 3. We had equation number 2 also. So, now let me put this expression 2 and 3 in our original Hamiltonian that is here. So, if I put it there I will get my Hamiltonian as this.

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Putting (2) and (3) in (1):

$$\mathcal{H} = \sum_k \frac{W}{a} A_k^2 \Big[\underline{\Big(-\frac{\omega_k^2}{2}C + \frac{2}{L}\sin^2\frac{ka}{2}\Big)} a_k a_{-k} + \left(a_k a_k^\dagger + a_k^\dagger a_k\right)\Big(\frac{\omega_k^2 C}{2} + \frac{2}{L}\sin^2\frac{ka}{2}\Big) + \underline{\Big(-\frac{\omega_k^2}{2}C + \frac{2}{L}\sin^2\frac{ka}{2}\Big)} a_k^\dagger a_{-k}^\dagger \Big]$$

Putting 2 and 3 in 1 we will get the Hamiltonian as, actually let me give you after doing bit of algebra, you will get W / a A k square and you will get terms like this minus omega k square / 2 C + sin square ka / 2, 2 / L is also there. You have a k a - k and you will have terms + a k a k dagger + a k dagger a k that would be multiplied by omega k square C / 2 + 2 / L sin square ka / 2 and you will have a term like - omega k square / 2 into C + 2 / L sin square k a / 2 a k dagger a – k dagger. Now, you see these terms and this term is not we do not want these terms. These are unwanted terms.

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$$-\frac{\omega_k^2}{2}C + \frac{2}{L}\sin^2\frac{ka}{2} = 0$$

$$\Rightarrow \boxed{\omega_k = \frac{2}{\sqrt{LC}}\sin\left|\frac{ka}{2}\right|}$$

And therefore, we take minus omega k square / 2 C + 2 / L sin square ka / 2 = 0 and from this we get the required dispersion relation that is omega k = 2 / square root of L into C sin modulus of k a / 2. So, this was the dispersion relation I mentioned in the lecture classes.

Now, again let me quickly show you because now these 2 terms are 0, these 2 terms are not there.

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Again:

$$\mathcal{H} = \sum_k \frac{W}{a} A_k^2 \omega_k^2 C \left(a_k a_k^{\dagger} + a_k^{\dagger} a_k\right)$$

$$= \sum_k \frac{W}{a} A_k^2 \omega_k^2 C \left(2 a_k^{\dagger} a_k + 1\right)$$

$$\left(\left[a_k, a_k^{\dagger}\right] = 1\right)$$

$$= \sum_k \frac{2W}{a} A_k^2 \omega_k^2 C \left(a_k^{\dagger} a_k + \frac{1}{2}\right)$$

So, we can write our Hamiltonian as sum over k W / a A k square omega k square into C. And here I have a k a k dagger + a k dagger a k who is further I can write is summation over k W / a A k square omega k square C into this would be twice a k dagger a k + 1. I think all of you are getting it because I have just utilize this computation relation a k k dagger = 1. So, this is what I utilized to write it and I can therefore write it as 2 W / a A k square omega k square C a k dagger a k + half.

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$$= \sum_k \frac{W}{a} A_k \ldots$$

$$\left(\left[a_k, a_k^{\dagger}\right] = 1\right)$$

$$= \sum_k \frac{2W}{a} A_k^2 \omega_k^2 C \left(a_k^{\dagger} a_k + \frac{1}{2}\right)$$

Taking: $\hbar\omega_k = 2 W A_k^2 \omega_k^2 \frac{C}{a}$

$$\boxed{\mathcal{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^{\dagger} \hat{a}_k + \frac{1}{2}\right)}$$

Now if we take that is what we did in the class, taking h cross omega k = 2 W A k square omega k square C divided by a. We can write the Hamiltonian in the usual harmonic oscillator form that will be summation over k h cross omega k a k dagger a k + half..