Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture – 10
Quantum Harmonic Oscillators

Hello, welcome to lecture seven of this course .As you know in this course primarily we are going to deal with the interaction of electromagnetic radiation with artificial quantum systems. Mostly the artificial quantum systems in particular in circuit quantum electrodynamics they are going to be modeled as two level atoms. On the other hand when we are going to deal with quantum optomechanics the quantum system would be modeled as harmonic oscillator.

This electromagnetic radiations when quantized they will be also modeled as harmonic oscillator. As we have already learned about two level atoms fundamentals of two level atoms. Now let us discuss harmonic oscillators.

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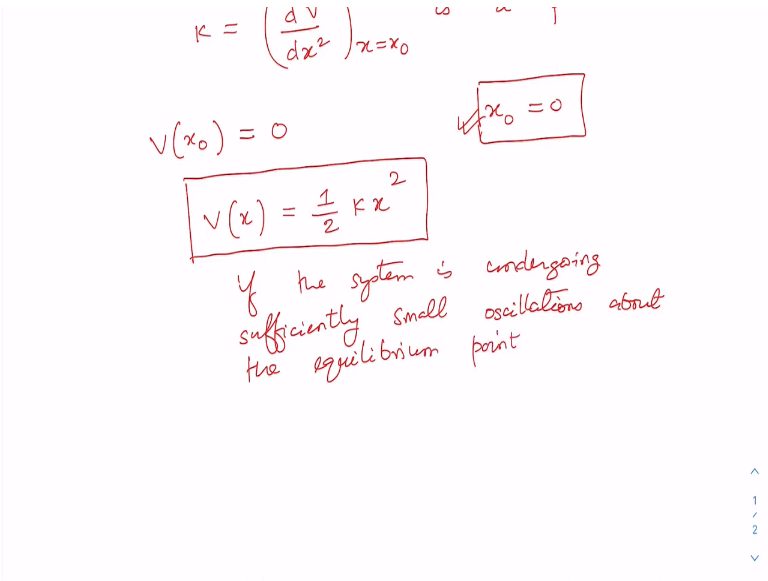

The harmonic oscillator is one of the most basic model in physics. It describes the dynamics of systems close to their equilibrium state. Say, consider a potential energy function say V of x ,let me plot it. Suppose this potential function V as a function of x has a minimum at some position say x 0 then, we can expand this potential function one dimensional potential function V of x into a Taylor series. Taylor expansion would be like this V of x0 around this minimum equilibrium point plus here it will be x - x 0 dv of dx evaluated at x is equal to x 0

+ 1 by 2 factorial d 2 V dx 2 evaluated at x is equal to x 0 and we have x - x0 whole square and. So, on.

Now since x 0 is the location of the minimum of the potential energy ,its first derivative is obviously going to be 0. So, therefore we'll be left out with V of x is equal to V of x 0 plus half k x - x 0 square and there will be other terms as well ,where this k is defined as the second order derivative of this potential function evaluated at x is equal to x 0 and it is a positive constant. Now since it is only the difference in potenial potential energy that matter physically we can choose 0 of the potential energy such that V of x 0 we can take it to be 0.

And if now the position ah of the origin of our coordinate is at say x 0 ,if we shift the origin to the coordinate x 0 ,then we can write V of x is equal to half k x square okay. We are taking our coordinate position of the origin of the coordinate at x 0 is equal to 0 and if we save the coordinate origin of the coordinate system to x 0 then we can write this potential energy function to be as half k x square ,provided the system is undergoing if the system is undergoing very small or sufficiently small oscillations about the equilibrium point about the equilibrium point .Okay.

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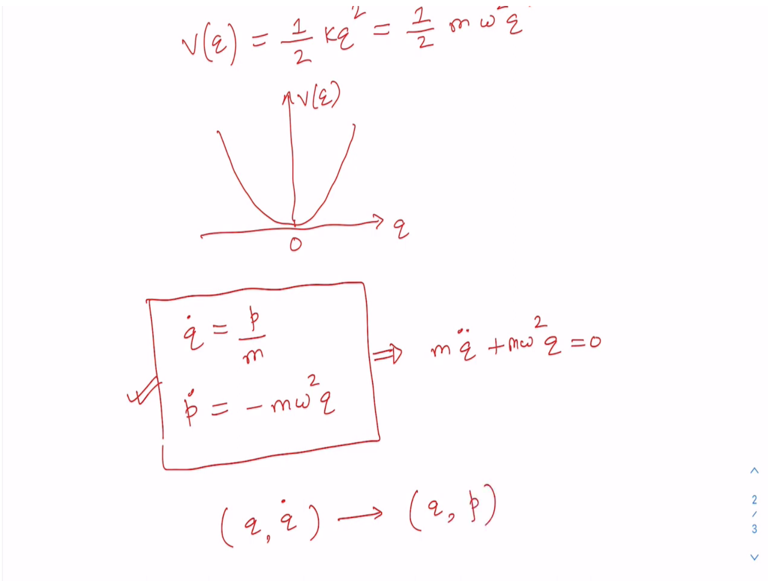

So, in that case we can neglect all the higher order terms in the Taylor series ,then we can neglect all higher order higher order terms in the Taylor series and the effective potential energy is that of a harmonic oscillator. So, this is the harmonic oscillator potential ,in fact you see if I write down the equation of motion uh here for in one dimensional case. So, here the force is equal to mass into acceleration. So, mass into acceleration x double dot and that is equal to force, force is given as uh dv of dx ,right.

If we take it then we are going to get -kx. So, this is what the Newton's second law gives us. In fact, let me use the coordinate q instead of x and then I can write down the equation of motion like this mq double dot is equal to –kq. One typical example of a harmonic oscillator this is the equation of a harmonic oscillator and one typical system is of this type say, a mass a mass of m attached to a spring of spring constant k. So, q is the displacement from its equilibrium point ,then this is the equation of motion.

Alright here we can also write this equation of motion as q dot plus omega square q is equal to 0 where omega square as you can see is simply k by m and the potential energy function I can write it as V of q is equal to a half k q square or half m omega square q square. And Now it looks like because I have shifted my origin to 0 here, okay. So, this is the typical plot of the harmonic oscillator potential energy function.

Now we can express this Newton's equation of motion which is a second order differential equation uh in the form of two first order differential equation, if I write say q dot is equal to p by m where p is the momentum and p dot is equal to -m omega square q uh these two first order differential equations are equivalent to this Newton's equation of motion q double dot omega square q in fact ,you can very easily get it from this system of first order differential equation and the you see that means we are actually.

Now going from when you are in the Newtonian mechanics, we dealt with this variable q and q dot and when we go over to uh this two first order differential equations ,we are going over to the variable q and p.

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configuration space

phase - space

$$[\hat{q}, \hat{p}] = i\hbar$$

$$L = T - V$$

$$L = L(q, \dot{q})$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \quad \text{Langrange Eq}^n \text{ of motion}$$

And this is actually known as going from the configuration space configuration space to the so-called phase space. In fact this face space description of classical system is very useful because we know that as regards quantum mechanics is concerned in quantum mechanics these phase space variables q and p ,they, in quantum mechanics they would be represented by operators and they satisfy this commutation relation q p is equal to i h cross where q and p are canonically conjugate variables.

As you know that when we want to discuss the quantum mechanics of a system ,generally we look for the Hamiltonian of the system. And in fact when we go from the classical regime to quantum regime ,then we must know the Hamiltonian of the classical system. But to know the Hamiltonian of the classical system first of all we need to know the lagrangian of the system.

So, in the case of harmonic oscillator also if we want to discuss the quantum regime of the harmonic oscillator or quantum harmonic oscillator, we first need to know the Hamiltonian of the harmonic oscillator or even before that we need to know the Lagrangian of the harmonic oscillator. So, let us find it out. We know that the Lagrangian of a classical system is given as kinetic energy minus the potential energy and this for one dimensional case it is Lagrangian is a function of the variables q the position and the velocity.

And this Lagrangian satisfy this so-called Lagrange equation that is del l del q - d of dt del L del Q dot is equal to 0. So, this is known as the Lagrange equation of motion equation of motion.

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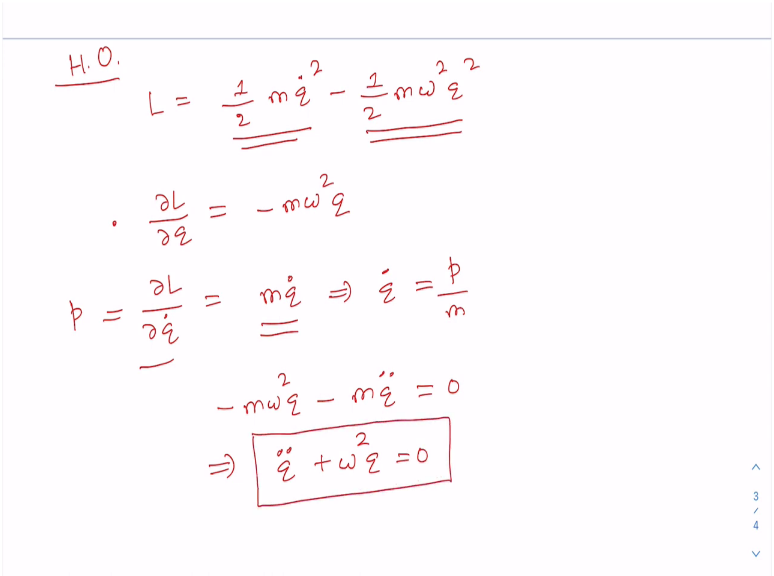

So, uh in our case in the case of harmonic oscillator, the Lagrangian would be, for the harmonic oscillator for the classical harmonic oscillator one dimensional classical harmonic oscillator the Lagrangian would be half m q dot square - half m omega square q square. So, this is the potential energy part, this is the kinetic energy part and it satisfies this Lagrange equation of motion which is nothing but the so-called Newton's equation of motion that is what we are going to get.

You see you have del l del q is equal to you will get from here it will be - m omega square q and del l del q dot is equal to m q dot and in fact del L del q dot this quantity is known as the conjugate momentum, momentum of the harmonic oscillator mass times velocity or I can write q dot is equal to p by m. So, um if I put del l del q and this quantity in the Lagrange equation ,here then you will you see from the Lagrange equation I will get - m omega square q - mq double dot is equal to 0 and which is is nothing but q double dot + omega square q is equal to 0.

So, we rederive the so-called Newton's equation of motion for the one dimensional harmonic oscillator.

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$$\Rightarrow \boxed{\ddot{q} + \omega^2 q = 0}$$

$$H = \underline{\dot{q} p} - L \quad ; \quad H = H(q, p)$$

H.O.

$$H = \frac{p}{m} p - \left(\frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \right)$$

$$\left(\frac{p}{m}\right)^2$$

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2}$$

Now coming to the Hamiltonian of the harmonic oscillator for one dimensional case, the Hamiltonian of a classical system is given by this one q dot p - the Lagrangian. So, in the case of the harmonic oscillator, the Hamiltonian would be uh we you see this Hamiltonian is a function of position and the momentum variable. So, therefore I have to express everything in terms of position and momentum only, q and p variable. So, I need to get rid of this q dot.

So, here I know that q dot I can write it as p by m. So, I have here p by m and then p from here then the Lagrangian is half m q dot square - half m omega square q square but again I have to replace this q dot by um p by m square. So, if I do the maths. So, it is very easy to see that I am going to get this familiar form of the Hamiltonian for one dimensional harmonic oscillator p square by twice m half m omega square k square. So, this Hamiltonian we are now expressing it in terms of this variable q and p.

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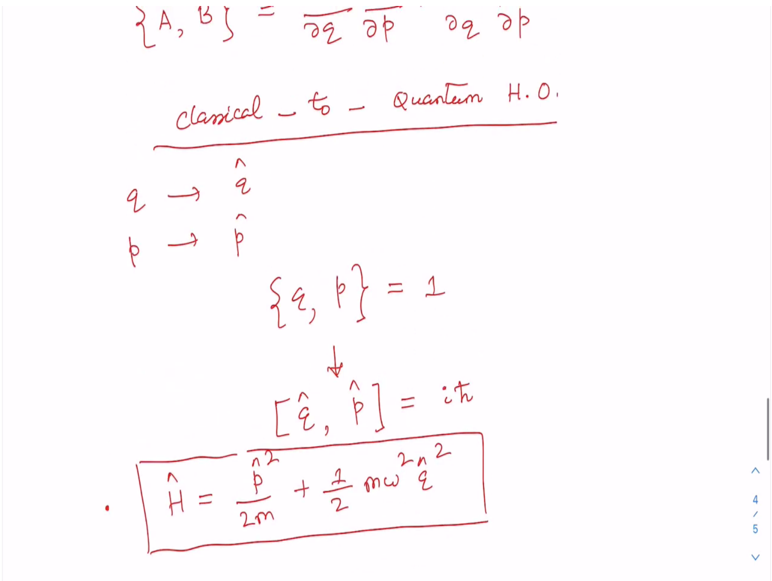

Now this Hamiltonian satisfy the so-called Hamilton's equation of motion and these variables q and p are called canonically conjugate variables and they are called canonically conjugate variables .Actually any pair variables are called canonically conjugate variables if they satisfy the so-called Hamilton's canonical equation of motions. So, or simply call Hamilton's equation of motion. So, Hamilton's equations of motion are ,one equation is q dot is equal to del x del p.

And another equation is uh p dot is equal to - del x del q .In fact you can verify that uh indeed uh this uh q dot and in fact let me quickly see for the harmonic oscillator case, for the harmonic oscillator case we know the Hamiltonian. So, del x del p from here you see that this is going to give us the familiar equation p by m. So, this already we know that this is correct and here p dot if I take del x del q from here then you will see Iwill get this force equation here that is m omega square q.

So, this is also we know that this is Newton's equation these are the correct equations ,correct form of equations. So, these variables q and p they satisfy the so-called uh Hamilton's equation of motions and that is why they are called canonically conjugate variables.And apart from that another thing is that this canonically conjugate variables q and p they satisfy the so-called Poisson bracket.

This Poisson bracket is ,equation is satisfied and let me just remind you that the Poisson bracket between two classical quantity A B is defined like this it is del A del q del B del p minus we will have del B del q del A del p and you can easily verify that q p is equal to 1.

And now knowing all these things, we can easily go from the so-called classical to quantum harmonic oscillator and this transition is very simple and this quantum harmonic oscillator we can go and this transition is known as the canonical quantization.

The only thing that we have to do is that this variables canonically conjugate variable q and p they are now going to be replaced by their corresponding operators in quantum mechanics and these Poisson brackets qp these Poisson brackets are going to be replaced by the so-called commutation relations for the corresponding operators between the variables q and p. So, qp we know that that is equal to is cross and this Hamiltonian.

Now would be written in terms of this operator. So, h is equal to p square by twice m here p is now the operator and I have half m omega square q square. So, thereby we basically obtain the so-called uh Hamiltonian for the quantum harmonic oscillator ,quantum mechanical Hamiltonian can be obtained. So, the procedure is very easy and let me just once again tell that.

In any classical system this we can hopefully quantize it ,provided first of all we need to know the Lagrangian of the system it may have only uh if in one dimension this is what we can do extend this argument for any dimensions. So, first thing is that we need to know the Lagrangian, then we need to find out the Hamiltonian.

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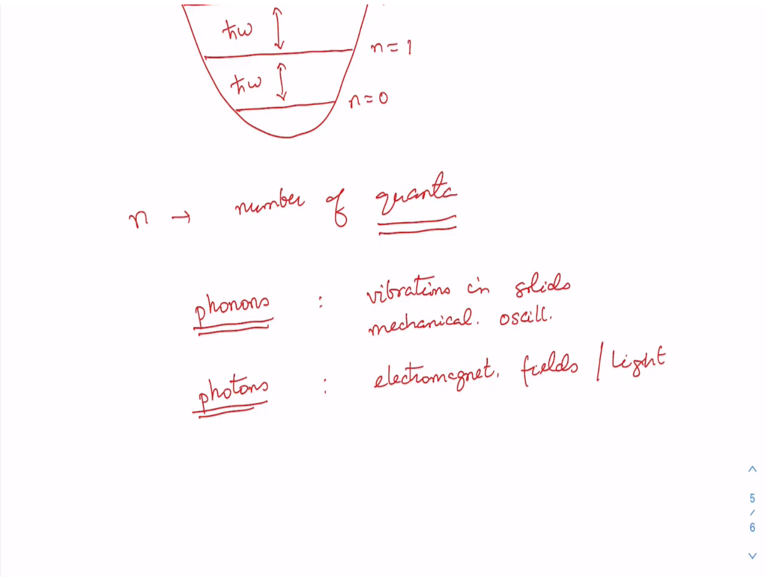

Then we need to uh, sorry, this Lagrangian will be function of q and q dot then we need to find out the Hamiltonian and once we find that the Hamiltonian we have to check whether this variables basically we need to dig out the canonically conjugate variable because these conjugate variables are going to satisfy this Hamilton's canonical equation of motion. So, that is what we have to do and if we can do that if we can find out these canonically conjugate variables.

Then we will also see that they satisfy the so-called Poisson bracket relation and if we have all these things with us ,then the next step is just to you know find out the, replace it by the corresponding operators and they will satisfy this uncertainty relation, this commutation relation and the corresponding operator h everything would be replaced by this operator. And this is the procedure and this is the general procedure that we are going to apply for any classical system.

Later on we will see in our course that we are going to quantize the so-called LC circuit there also uh this would be the kind of procedure that we are going to adopt. Now from your elementary quantum mechanics course you may know that the Hamiltonian of a harmonic oscillator satisfies this eigenvalue equation where psi n is the energy eigenstate and E n is the energy eigenvalue.

And E n is given as n + half h cross omega h cross is the so-called reduced Planck constant defined as h divided by 2 pi and you know that h is equal to 6.626 into 10 to the power – 34 joules second and omega is the angular frequency of the harmonic oscillator and n takes value, this integer value 0 1 2 3 like this. And one thing that you can immediately see is that energy levels in a harmonic oscillator are equally spaced.

Say you have n is equal to 0 n is equal to 1 n is equal to 2 and so on. And the spacing between these energy levels are always equal and that is h cross omega. In fact this is a unique, one of the unique characteristics of harmonic oscillator which sets it apart from other quantum systems and by the way this integer n, here n in this energy eigenvalue expression refers to the number of quanta.

So, n refers to the number of number of quanta and this this quanta has different name in different situations, for example it is called phonons. These quanta's are called phonons when we discuss vibrations in uh solids or is a mechanical oscillator if we model a mechanical oscillator as an harmonic oscillator then these quanta's would be called phonons.

On the other hand if we talk about electromagnetic fields or light these quanta quanta's are called photons. So, for electromagnetic field or light these quanta's are known as photons.

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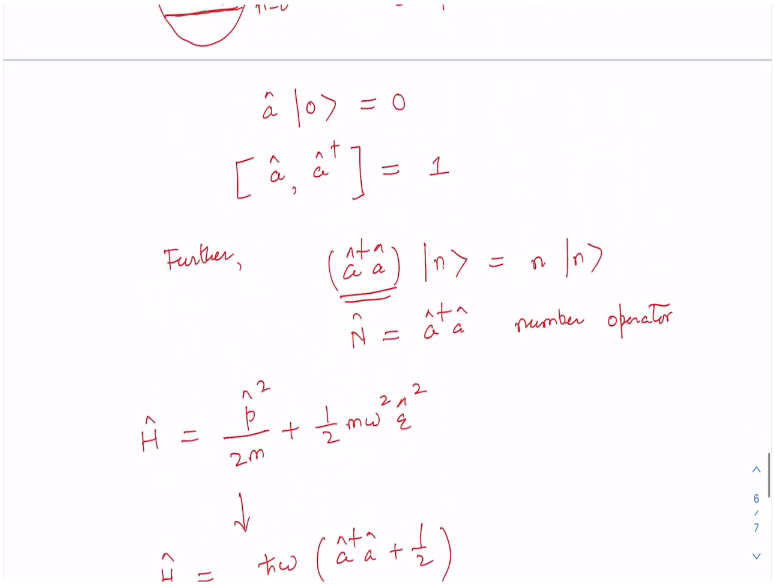

Now uh instead of psi n actually most of the time we can write it by this ket n here and in that case our energy eigenvalue equation would become like this. So, here n is also known as Fock State or it is also called or it is also called number state and this ket n or this Fock state here ,it is, basically a refers to the state ,it refers to refers to the state where there is n number of quanta. Now uh there is a very useful formalism uh called operator formalism of harmonic oscillator and which is going to be very useful for us.

The idea here is that we can introduce operators which may take us from one energy state to another energy state. Say if we introduce operators where one we can go from say this ground state to the these operators may take us from the ground state to the excited state or we may go from the excited state to one another lower state and so on. So, these operators are called creation and annihilation operators.

Say we have, we want to go from ah say, the from the energy state n to n - 1 lower energy state then we have to operate it by an operator that is called the annihilation operator, you are going from a energy state n to n - 1 and this is what we will achieve by this annihilation operator and maybe some of you already know and then there is a creation operator where it is going to take us from n to n + 1 and this effectors here would be square root of n + 1.

And these operators are defined such that when you operate on the ground state because there is no energy state below this ground one. So, you have to ket 0 .It is going to be defined in that way and also uh a and this any this creation, this is annihilation operator and the creation operator satisfies this uh commutation relation that a a dagger has to be equal to 1. Further what turns out that a dagger a ,this bilinear combination when it operates on this energy eigenstate or Fock state n then it is going to give you a number and this operator is term called it is uh symbolized as this n cap and this is called the number operator a dagger a.

So, I am sure most of you know all these things but I am just recapitulating and reminding you. Now in terms of this uh creation and annihilation operator this our original harmonic oscillator Hamiltonian p square by twice m + half m omega square q square can be expressed in terms of annihilation and creation operator in this useful form h cross omega a dagger a + half.

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$$\hat{q} = q_0 \left(\hat{a} + \hat{a}^{\dagger}\right)$$

$$\hat{p} = i m \omega q_0 \left(\hat{a}^{\dagger} - \hat{a}\right)$$

$$\hat{a} = \frac{1}{2q_0}\left(\hat{q} + i\frac{\hat{p}}{m\omega}\right)$$

$$\hat{a}^{\dagger} = \frac{1}{2q_0}\left(\hat{q} - i\frac{\hat{p}}{m\omega}\right)$$

In fact uh these all these things that I have written here, would be satisfied if we define this annihilation and creation operator this way say this position operator is written as a combination of say a + a dagger and this is dimensionless and it has to be multiplied by a dimensional quantity say q 0 which says the dimension of length. In fact q 0 refers to this as the dimension of length refers to the special width, width of the ground state wave function it refers to the ground state wave function of a harmonic oscillator.

What I mean by that is that you know that the ground state for the ground state from your elementary quantum mechanics you know the ground state uh wave function of the harmonic oscillator is Gaussian and if you calculate the width of this uh Gaussian in the ground state you are going to get q 0. So, if you can see the width is del q if you work it out you are going to q 0.

What I mean to say is that you have to just calculate the standard deviation or first you calculate the variance and variance in the ground state with respect to the ground state if you calculate ,that is your q square 0 - 0 q 0 whole square and if you want to find out the width you just have to take the square root and if you take the square root you are going to get uh simply q 0 and in fact it would turn out that this guy is nothing but h cross, if you do the calculation ,this would be h cross by twice m omega.

And so, therefore what we have basically is that this position operator is now defined in terms of the annihilation and creation an operator like this and the corresponding momentum operator is defined as by definition it is i m omega q 0 a dagger – a. In fact you can write from these two equations, you can write annihilation and creation operator like this, annihilation operator would be one by two q q 0 uh you will have q + i p by m omega.

And a dagger is basically the Hermitian conjugate of this annihilation operator, this is the creation operator that would be 1 by twice q 0 q – i p by m omega. So, these are extremely useful relations and you should remember it all the time.

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$$\sum_n p(n) = 1 \Rightarrow C \sum_{n=0} \ldots$$

$$\Rightarrow C\left[1 + e^{-\hbar\omega/k_BT} + e^{-2\hbar\omega/k_BT} + \cdots\right] = 1$$

$$\Rightarrow C \frac{1}{1 - e^{-\hbar\omega/k_BT}} = 1$$

$$\Rightarrow C = \left(1 - e^{-\hbar\omega/k_BT}\right)$$

$$p(n) = \left(1 - e^{-\hbar\omega/k_BT}\right) e^{-n\hbar\omega/k_BT}$$

Now let me briefly talk about the thermal equilibrium statistics of a harmonic oscillator. You know that photons or phonons these are bosons and when a harmonic oscillator is in equilibrium with its environment at some temperature say T, Bose Einstein statistics basically determine the occupation probability of its energy level its level and this is given by occupation probability of an energy level in a harmonic oscillator is given by this expression. So, this is h cross n h cross omega by K B Tor 1 - exponential - h cross omega K B T.

I hope you have uh get this expression or learn this expression in your statistical physics course actually let me quickly show you how this expression can be obtained very easily because if you can start from the very basic statistical mechanics. Because you know that if I, as per this Gibbs canonical ensemble or distributions Gibbs canonical distribution ,Gibbs canonical distribution you know that occupation probability is dependent on ,isgiven by this expression this is proportional to e to the power minus say energy divided by K B Tis the Boltzmann constant T is the temperature.

And for harmonic oscillator okay, for n e n and for harmonic oscillator, obviously you have uh it would be n + half h cross omega by K B T and I can actually this half s cross omega is a constant term. So, I can take it into this constant, I can write it again include it in the constant C then I have e to the power - n h cross h cross omega by K B T. So, this is my starting expression and I know that the total probability has to be equal to 1 and from here you can easily find out the this constant C.

If you take it out c and this is a you will get the you will get a series actually K B T is equal to 1 n goes from 0 to infinity and this I can now write it as uh 1 + e to the power h cross omega by K B T+ e to the power twice s cross omega by K B T and. So, on and it is 1,. So, this one is 1 divided by e to the power h cross omega by K B T. So, you get the constant from here that is simply 1 - e to the power s cross omega by K B T and so, you see you finally got this expressions here.

Let me just complete it. So, you will get pn is equal to this constant ,1 - e to the power h cross omega by K B T that is what you have this one and then e to the power - n h cross omega by K B T. So, that is how you get the expression, finally you can if you can just rearrange it, okay, you get it actually this is I can. Now take it this side and then you will get what I have written uh originally.

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Mean occupancy of the H.O.

$$\bar{n} = \langle \hat{n} \rangle$$

$$= \sum_{n=0}^{\infty} n\, p(n)$$

$$\boxed{\bar{n} = \frac{1}{e^{\hbar\omega/k_B T} - 1}}$$

Now I hope all of you by the way know that this Boltzmann constant K B is useful to remember that has a value 1.38 into 10 to the power -23 joule per Kelvin. Now using this I can easily uh find out the mean occupancy ,the mean occupancy uh of the oscillator, of the harmonic oscillator that is I can just have to find out the average number of quanta and we just have to calculate the expectation below this number operator or I can just calculate this quantity n into the probability pn.

And if you I am just leaving it as an exercise to you can very quickly do it, you know you pn expression is already it is there with you, just have to expand it then please. So, that the expression that you are going to get is 1 divided by h cross e to the power h cross omega by K B T- 1 and this is a very useful expression and this is one of the quantity which is, by which I can actually tell whether a system is quantum or not to characterize the system.

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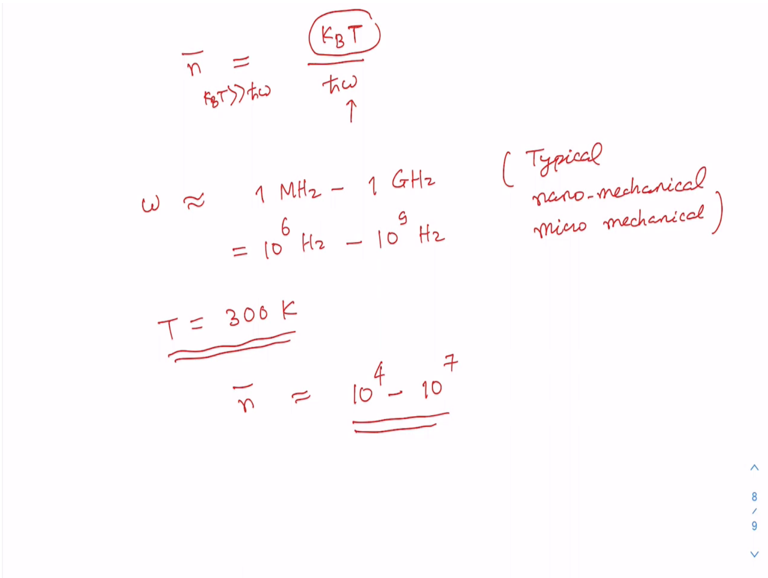

For example in the classical limit ,in the classical limit ,in the classical limit your this quantity K B T is much greater than h cross omega. What I mean by that is in the context of harmonic oscillator that means uh this harmonic oscillator is going to behave classically because the energy spacing the thermal energy would be much larger than the energy spacing between two energy levels.

So, the whole system will actually behave like, you will not be able to see the this discrete nature actually. So, in that case this harmonic oscillator is going to behave classically and in the classical limit if you look at from this expression then you will get this average number of quanta would turn out to be when K B T is much greater than h cross omega you can easily see that that would be K B T divided by h cross omega.

So, uh in a to give some examples say in a typical micro or nano mechanical oscillators where this uh resonance frequency or this frequency omega is basically in the range of one to say one megahertz to say one gigahertz 1 megahertz means say 10 to the power 6 hertz and 1 gigahertz refers to 10 to the power 9 hertz. So, this is for a typical nano mechanical oscillator nano mechanical or say micro mechanical oscillators.

So, this is the frequency, resonance frequency then if you put the numbers suppose at room temperature t is equal to 300 Kelvin then you will find that the mean number of quanta would turn out to be if you put K B you put t is equal to 300 Kelvin okay. So, on then you will find that that would be around 10 to the power 4 the mean number of quantum beta this much.

So, this is a very huge number and that is why uh you cannot uh observe the quantum behaviour of this kind of oscillators at say room temperatures ,right. To see the quantumness of this because of this quantity is very high. So, you have to if you can reduce this number and that you can do, provided you do some kind of cooling, okay, you have to cool these oscillators and then only you will be able to see the quantum behaviour.

In fact uh in this context let me talk about there is another kind of oscillators that we are going to encounter and that is the so-called electromagnetic oscillators or electromagnetic fields and electromagnetic oscillators are light basically.

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$$= \hbar\omega \left[\hat{a}, \hat{a}^{\dagger}\hat{a}\right]$$

$$= \hbar\omega \underline{\left[\hat{a}, \hat{a}^{\dagger}\right]}\hat{a}$$

$$\Rightarrow \boxed{i\hbar \frac{d\hat{a}}{dt} = \hbar\omega\hat{a}}$$

$$\Rightarrow \boxed{\hat{a}(t) = \hat{a}(0)\, e^{-i\omega t}}$$

This has typical frequency of light, if I can consider them as some kind of oscillator and they have a frequency on the order of 10 to the power 14 hertz and if even it had room temperature if you try to calculate their mean occupation number of the quanta or basically mean number of photons at 300 Kelvin ,then just use this expression. If you use this expression you know all the parameters then it would turn out to be around 10 to the power -35 which is a you see that means it is a very small quantity it is nearly 0.

So, that is the reason that optical fields this is refers to optical fields or fields. So, optical fields at room temperature, optical fields at room temperature can be can be ,because this is nearly equal to 0 ,can be safely considered to be considered to be in their ground state in their ground state ,okay. And this issue is going to be very useful why I am saying that because if you have mechanical oscillators say, you have a oscillator like this and if you make it in contact with an optical field.

Suppose you have an optical field, even if it is at room temperature, the optical field is or at some finite temperature you can consider it to be at temperature T is equal to 0 because it is in the ground state and this mechanical oscillator ,this is the mechanical oscillator this is at some uh finite temperature optical field at 0 temperature. So, therefore if you keep the mechanical oscillator with this optical oscillator or optical field and then thereby you can get you can actually cool the mechanical oscillator .This is actually very elementary way I can explain cooling of a mechanical oscillator.

Now one useful information that we'll require later in this course is the evolution of this annihilation operator ,evolution of a annihilation operator. This can be easily found out by using the so-called Heisenberg equation of motion which we learned in the last class ,annihilation operator satisfies this Heisenberg equation ,here h is the, let us work it out ,x is the harmonic oscillator Hamiltonian.

So, a h cross omega a dagger a + half this h cross omega half is a constant term. So, that is not going to contribute. So, we can now write this as h cross omega a a dagger a who is we can further write it as h cross omega a a dagger a and you know that commutation relation between a and a dagger is equal to 1. So, we will have h cross omega a. So, that means we get this simple differential equation and the solution of this differential equation is trivial.

And you can easily see that the solution will be a of t is equal to a of 0 e to the power - i omega t. So, this is a very very useful relation and we are going to encounter it many times later on. In fact I can depict the evolution of this annihilation operator in a picture, just remember that a as per our definition this is the 1 by twice q 0 q + i p by m omega.

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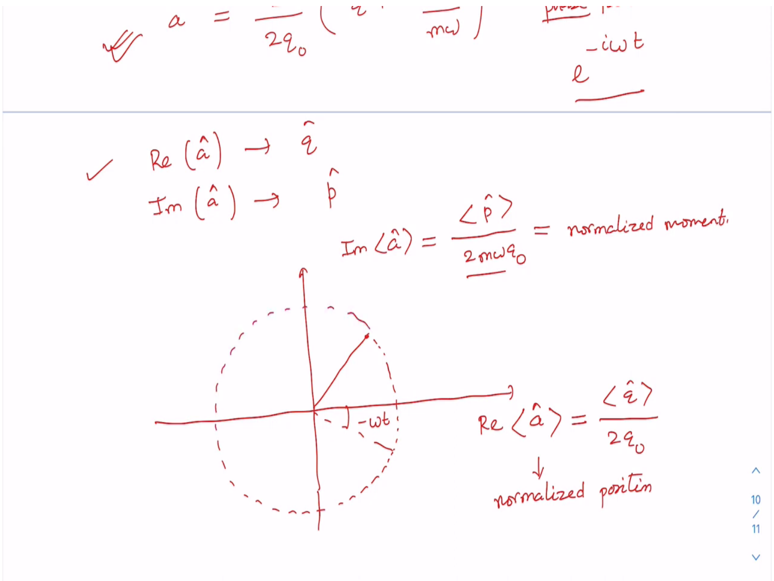

So, the real part of this annihilation operator is associated with position and the imaginary part of the annihilation operator is associated with momentum and taking this as the hint. What I can do I can plot it if I take my x-axis as the real part of any, sorry I cannot actually plot an operator ,what I have to do I have to take the expectation below the operator. If I take the expectation value of the operator ,annihilation operator the real part would be you can see from this expression that would be simply expectation value of q position operator divided by twice q 0.

So, this has a dimensionless. So, we can term it as normalized position. So, x axis is our normalized position on the other hand the y axis is imaginary part of a annihilation operator and who is you can see from here that would be the expectation below the momentum operator divided by twice m omega q 0 which has the dimension of momentum. So, this is again normalized momentum.

Now to plot it because of the fact that here we have right a of t if you look at this expression it is very clear that the magnitude of a of t remains constant magnitude of a of t because of this exponential factor. So, this remains constant in time. So, only it is multiplied by this phase factor here and phase varies linearly in time .The phase phase factor if you look at it, it is the phase factor is e to the power -i omega t and this phase varies linearly in time.

And it rotates here actually because of this minus sign this rotates, in the in the anticlockwise direction, it rotates in the anticlockwise direction and evolution basically takes place this is a

- omega t at time t and it is in pollution is happening in a circle because the magnitude is constant.

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$$\hat{H} = \hbar\omega_1 \hat{a}_1^{\dagger}\hat{a}_1 + \hbar\omega_2 \hat{a}_2^{\dagger}\hat{a}_2 - \underbrace{k q_{01} q_{02}}_{\hbar g} (\hat{a}_1 + \hat{a}_1^{\dagger})(\hat{a}_2 + \hat{a}_2^{\dagger})$$

$$\boxed{\hat{H} = \hbar\omega_1 \hat{a}_1^{\dagger}\hat{a}_1 + \hbar\omega_2 \hat{a}_2^{\dagger}\hat{a}_2 + \hbar g (\hat{a}_1 + \hat{a}_1^{\dagger})(\hat{a}_2 + \hat{a}_2^{\dagger})}$$

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As a final topic of this particular lecture let us discuss how to model a system of coupled harmonic oscillator quantum mechanically. So, say we have a two spring mass system like this ,I have a spring of spring constant k 0 1 and mass m 1 is attached to it and another spring or spring constant k 0 2 and a mass having m 2 is attached to it and they are coupled by spring of spring constant k say this mass oscillator is also um displaced from the equilibrium position by x 1 and then this is displaced by x 2 then you know that the Hamiltonian of this whole system this couple system would be as follows.

So, you will have p1 square by twice m 1 + a half k 0 1 x 1 square, this is the energy of the first oscillator then for the second oscillator you will have p 2 square by twice m 2 + half k 0 2 x 2 square that is the energy of the second oscillator if that is not coupled, these are the individual energies of the oscillators when they are not coupled. But because of coupling they would have another term that would be the coupling energy between them that is a half k x 1 - x 2 square. So, this term is very important.

So, let me analyze it little bit further. So, this coupling term or coupling energy term for the two oscillators that is half k x one - x two square if I open it up then I will get half k x 1 square + half k x 2 square - k x 1 x 2. Now you if you look at these terms this term and this term assess qualitatively speaking it is not going to change the physics of this couple system because these two terms are actually changing the spring constant of the individual oscillators.

But interesting term is this cross term which is actually refers to the fact that energy is getting transferred, transferred from oscillator 1 to oscillator 2. So, let us analyze this particular term. So, this term here if I now want to study quantum mechanics if I go to the quantum regime then I can write x 1 and x 2 in terms of creation in any relation operator. So, let me do that.

So, you will have x 1 would be ,you will have as per our definition, it would be say q 01 a 1 + a 1 dagger just to remind you that we defined earlier that this position operator we can define as q 0 a + a dagger ,right, and therefore here I have x 1 would be represented by this one and x 2 would be again q 02 8 a 2 + a 2 dagger and here q 01 is h cross divided by twice m omega 1.

So, that is for the first oscillator this is the 0 point fluctuation and q 02 is h cross divided by twice m 2 omega 2 it will be m1 here. So, this is what we have. Now if I open it up then i can actually rewrite it like this k q 01 q 02 and I will have a 1 + a 1 dagger and a 2 + a 2 dagger. So, this is what I will have and if I talk about the quantum mechanical version of this part of the Hamiltonian then I will have it would be represented by say h cross omega 1 a 1 dagger + a 1 + half.

And similarly here this part would be h cross omega 2 a 2 dagger + a 2 dagger + half .We are going to neglect these two terms because they are just basically changing the spring constant of the individual springs. So, therefore the total quantum mechanical Hamiltonian for this classical system if we go to the quantum regime. Then the quantum Hamiltonian would be h cross omega 1 a 1 dagger + a 1 + half term is there but let me ignore that because that is a constant term.

And similarly for the second oscillator I will have a two dagger a 2 and we have this uh this coupling term here that is that I have as - k q 01 q 02 a 1 + a 1 dagger + a 2 + a 2 dagger, these are operators, okay, but this one let me write it in because this has to has the dimension of energy. So, let me write it as h cross g ,g is the coupling coefficient between the two oscillators. So, finally the quantum mechanical Hamiltonian I can write in this particular form that would be for the second oscillator I have this one.

And when they are coupled this is the coupling part of this two oscillator a 1 + a 1 dagger and a 2 + a 2 dagger.

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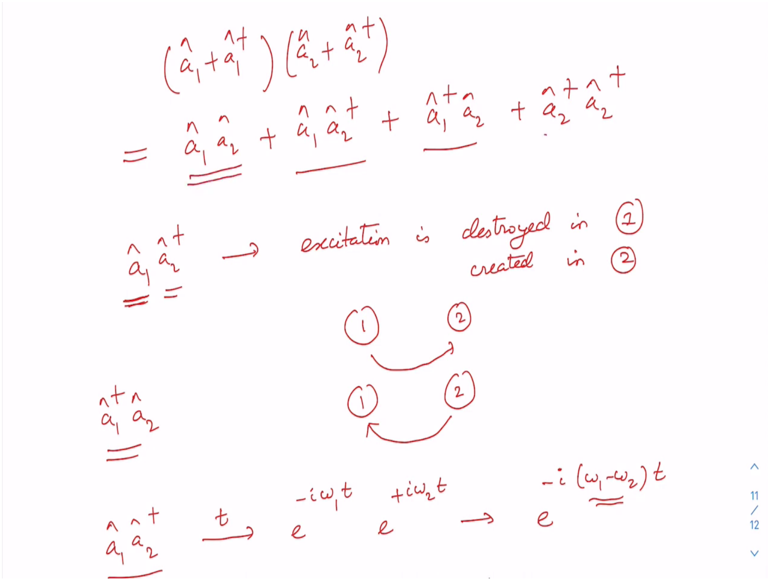

Now let me analyze a little bit further this particular coupling term, if I look at this coupling term you will see that I have ,okay just let me take the product here ,that is a 1 + a 1 dagger and a 2 + a 2 dagger if I take the if I open it up then I will have 4 terms that would be a 1 a 2 a 1 a 2 dagger + a 1 dagger a and I will have a one dagger a two dagger ,alright. So, for what these terms physically mean ,if I look at these two terms.

So, a 1 a 2 dagger this basically means that in an excitation is getting destroyed in oscillator one and that is one excitation is created in the oscillator two. So, and similarly ,okay, let me write here excitation here it means that excitation is destroyed in oscillator one and excitation is created in oscillator two. So, basically now from oscillator one to oscillator one excitation is getting transferred on the other hand the other term that is a1 dagger a 2 here the opposite things happen excitation is.

Now destroyed in two and it is transferred to 1. So, these two terms are called resonant term and because of the fact that if you we already know that how a 1 this annihilation operator or the creation operator evolves in time if you look at it a1 evolves like in time it will evolve like e to the power – i omega 1 t and the other one will it is because it is a dagger that will be + i omega 2t.

So, overall this whole combination would evolve as i - i omega 1 - omega 2t. Now if omega 1 is nearly equal to omega 2 ,you see that that basically refers to the fact that we had energy h cross omega 1 in oscillator 1 and that is getting transferred to oscillator 2 which is helping energy h cross omega 2 which is if omega 2 is nearly equal to omega 1 then uh that is quite physical means simply energy is getting transferred from one oscillator to the other oscillator and that is why they are called resonant term.

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$$\left.\begin{array}{l} \hat{a}_1\hat{a}_2 \xrightarrow{t} e^{-i(\omega_1+\omega_2)} \\ \hat{a}_1^{\dagger}\hat{a}_2^{\dagger} \xrightarrow{t} e^{i(\omega_1+\omega_2)} \end{array}\right] \text{non-Resonant}$$

RWA

$$\hat{H} = \hbar\omega_1 \hat{a}_1^{\dagger}\hat{a}_1 + \hbar\omega_2 \hat{a}_2^{\dagger}\hat{a}_2 + \hbar g\left(\hat{a}_1^{\dagger}\hat{a}_2 + \hat{a}_1\hat{a}_2^{\dagger}\right)$$

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On the other hand if you look at these two terms a one dagger and a and a two dagger ,sorry , this would be a one dagger a two dagger if you look at this term this implies this a 1 and a 2 this means that excitation is destroyed in oscillator 1 and excitation is destroyed in oscillator 2 as well. So, simultaneously excitation is destroyed in the two oscillators or um let me write here that is simultaneous destruction of actually quanta or excitations in oscillators in both the oscillators or opposite things is happening at that is creation of uh excitations in I can just let me just copy it here.

So, the same thing I can instead of this one I have simultaneous creation of excitation in oscillators. So, actually these two events are very unlikely and it is not physical also initially you have nothing and now we have destructed or created two excitations. So, this is not physical and this can be actually neglected, these two terms can be neglected ,these two terms can be neglected on physical ground .An another way to look at it these are actually called non-resonant terms.

So, if you again look at the time evolution of these two operators or these two combinations you will see that this one a 1 and a 2 this would, this, it would be e to the power i omega 1 + omega 2, okay, and similarly for the other one a 1 dagger a 2 dagger this in the time evolution will go like this omega 1 + omega 2 .That is why these are called non-resonant terms ,they are called non-resonant terms and they can be neglected if and when you neglect this is actually another form of the so-called RWA order rotating wave approximation that we discussed earlier.

So, under RWA this Hamiltonian finally I can write the for the coupled oscillators it would be h cross omega 1 a 1 dagger a 1 + h cross omega 2 a 2 degree a 2 and we have this coupling term s cross z a 1 dagger a 2 + a 1 a 2 dagger. Let me stop for today in this lecture we have learned about the fundamentals of harmonic oscillators. In the next lecture we are going to see how to quantize electromagnetic radiations.

It turns out that when quantized electromagnetic radiation behave like a collection of infinite collection of harmonic oscillators and the quanta of these harmonic oscillators are known as photons. So, see you in the next lecture, thank you.