Theoretical Mechanics
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Module No # 02
Lecture No # 08
Conservation Laws

Okay in this section what we are going to do is look at the conservation laws as the conservation laws which we obtain from Newtonian mechanics but of course in case of Lagrangian mechanics you can do little more.

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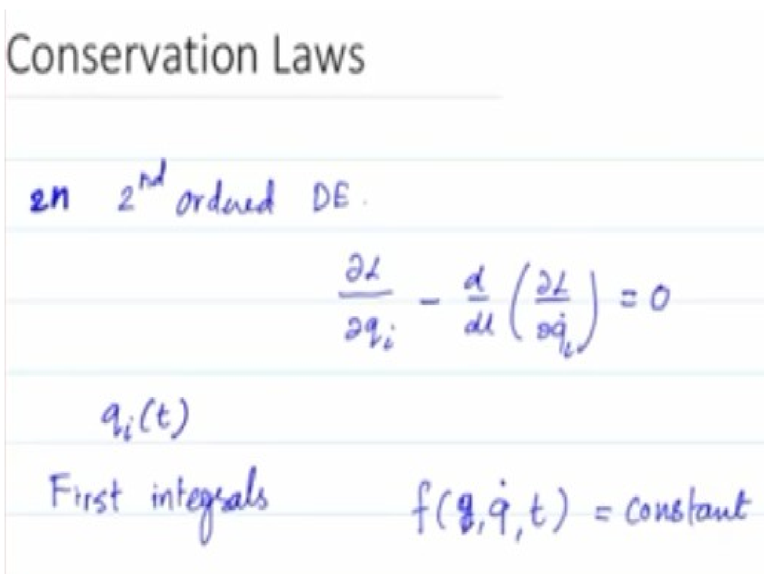

Now what we have here in Lagrangian equation are 2n differential second ordered differential equations and they given by del L / del qi – d / dt = 0. So these are the 2n differential equations now when you find the complete solution to these systems of differential equations you actually find each of these coordinate function are coordinate as a function of time t given some initial conditions. And it is in terms of t and 2n arbitrary constants or constants which are given in terms of either initial conditions or the boundary conditions.

Now what will happen is this most of the time we will not be able find the complete solutions but even if we do not find the complete solutions it is still alright because if we can do at least one integral that helps us lot and in fact in the number of cases what we find is that we can actually do one integral which is also called as first integrals and these first integrals typically can be put in a neat functional form like this.

So some function of y sorry q, q dot and t = constant so what will happen is in this equation there is only q dot so you have first order differential equations in t. So once we get this is also important because this does give us the information about the nature of the actual system. Now the conservation laws actually are built at this level itself.

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Cyclic Co-ordinates : If $\mathcal{L}$ ind of q_j for some j

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = \text{Constant}$$

Ex $\mathcal{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}} = \text{Constant}$

$$\Rightarrow m\dot{x} = \text{constant}$$

So to take an example consider the case of cyclic co-ordinates now in cyclic coordinates if Lagrangian is independent of one of the q variable say independent of qj for some j okay for some specific j. If it is independent of qj then del L / del qj becomes 0 and you immediately get rule so d / dt of del L / del qj dot = 0 which means del L / del qj dot actually is conserved quantity.

It remains constant throughout the motion of and if you look at this quantity here if q's are (()) (04:23) Cartesian coordinates then you immediately identify this quantity to be the linear momentum. So for example for case of a free particle in two dimensions. So lagrangian is half m x dot square + y dot square and this Lagrangian is independent of both x and y if I just look at x then this immediately tell us because Lagrangian is independent of x that del L / del x dot must be constant and that in fact is nothing but mx dot = constant and mx dot remember is nothing but the x component of the linear momentum.

So in this problem of course also y is absent from Lagrangian my double sorry my dot that is linear momentum in y direction also we will be conserved in fact the net linear gets conserved.

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Cyclic Co-ordinates : If L ind of q_j for some j

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_j} = \text{Constant} \leftarrow$$

Ex $\quad L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \Rightarrow \frac{\partial L}{\partial \dot{x}} = \text{Constant}$

$\Rightarrow m\dot{x} = \text{constant}$

Ex $\quad L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$

θ is not present $\Rightarrow$

$$\frac{\partial L}{\partial \dot{\theta}} = \underbrace{m r^2 \dot{\theta}}_{\text{Ang Mom.}} = \text{Constant}$$

Another example would be if I take the same situation but write the Lagrangian in terms of polar coordinate then now you see this particular Lagrangian is independent of theta but it is not independent of r okay. So what gets conserved here because this theta is absent theta is not present this immediately tells us that del L over del theta dot and this in this case will turn out to be mr square theta dot must be constant.

And this we immediate recognize as angler momentum. So in both these cases if a particular coordinate is missing from the Lagrangian or Lagrangian is independent of the coordinate the corresponding quantity del L / del qj qj dot remains constant and by analogy with these two examples if I am taking about Cartesian coordinate then the corresponding quantity del L / del x dot little component of linear momentum if it was theta then it is a angler momentum.

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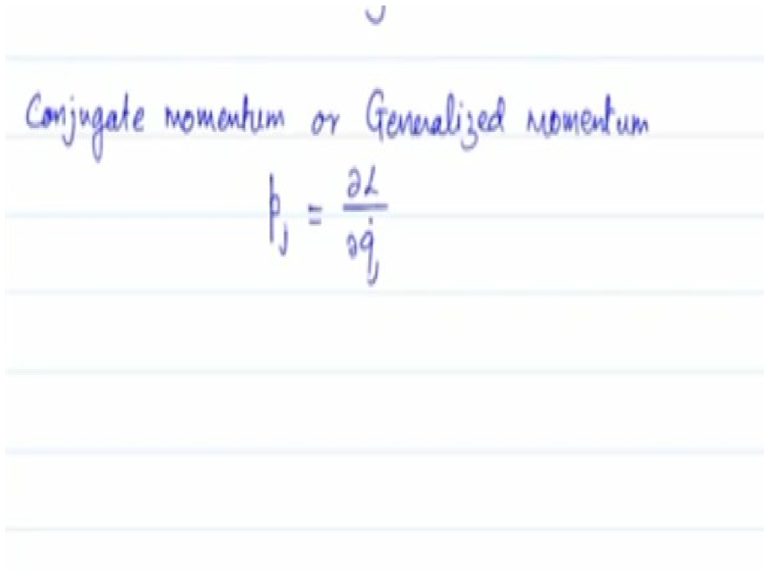

So by analogy with these quantity del L / del q dot is in fact named as conjugate momentum or generalize momentum. And remember this is always conjugate to a given coordinate so I will write it as pj defined as del L over del qj dot and the conservation theorem now says that if L is independent of qj then the corresponding conjugate momentum pj is conserved. And also we have remember that pj is not necessarily a momentum it does not necessarily dimension of linear momentum.

What we have shown here is that in one case it turns out to be linear momentum in an another case it turns out to be angler momentum in more complicated cases which I will do in the next week. For example if you have charge particles in electromagnetic field even then we can write down the Lagrangian for this system and in that case the conserved quantity for a particular coordinate is in fact is not just the mechanical momentum of the mast or massive charges but it actually is the combination of the mechanical momentum + the momentum carried by the electromagnetic field themselves.

See the total of these two is actually conserved so what we are talking about here is not necessarily the momentum of the system or even the momentum itself. But some quantity which we call as conjugate momentum or generalize momentum that is the quantity that is conserved in this case okay. Now in a same way I can also derive from this point on I can also actually derive the normally momentum conservation law which we get form the Newtonian mechanics.

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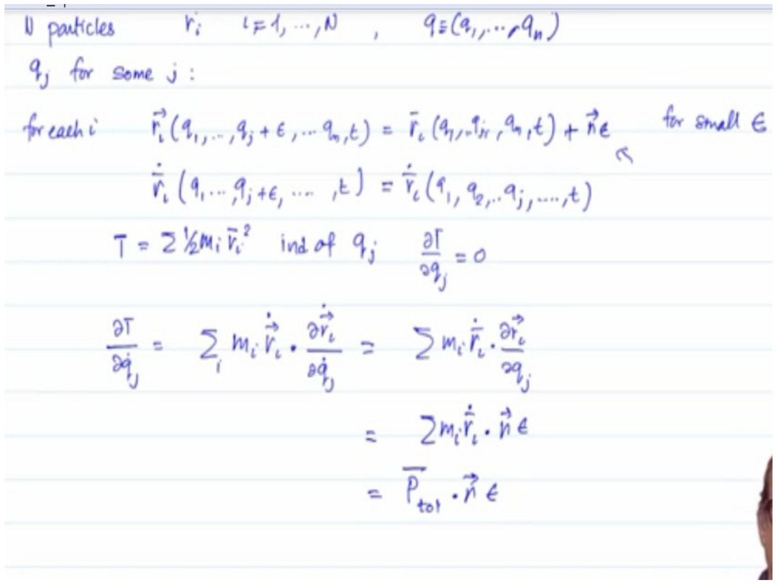

Say take for example we have a mechanical system so there are n particles and the coordinate or position vectors of these n are given by n okay. And we are using generalize coordinate in this case first I will for simplicity use only unconstraint system so I have generalize coordinates which we denote by q but includes q small n these many coordinates okay. And what I will do is this see it is very common for example one of the examples even we were discussing the generalize coordinates it is very common to choose one of the coordinates to be center of mass coordinates.

So let me assume that in this there is one variable qj okay for some specific j and the role of this qj is this when you take the system and if I keep all the q's fixed and only change qj then the entire system is translated by exactly the same amount. So each one of these ri's will translate by exactly the same amount. So what I mean is this so ri at so for each i ri of q1 qj + some small epsilon and qn t this is equal to ri at q1 to qn including qj t + some n vector times epsilon okay.

So all position vector shift in the direction of n and by the same amount which is magnitude of n vector into that small constant epsilon okay. So this is of course for small epsilon okay what happens to the velocities? Velocity of course will remain constant because if it take ri dot here at q qj + epsilon and all the way up to t this also will be equal to ri dot at q1, q2, qj and all the vector t. And because n times epsilon is constant for all of the position vectors its derivative would be 0.

So the all the velocities are unaffected by this translation of the entire system so what must happen is this? If the velocities are unaffected what happens to t? t which is kinetic energy which is sum over half mi ri square this is independent of qj okay or in other words I will write this as del t / del qj is 0. And look at what happens to del t / del j dot so del t / del qj dot this is nothing but sum over i mi ri vector dotted with del ri vector / sorry dot del qj dot.

And this we remember is in fact equal to half mi ri dot into del ri / del qj remember while we derive the Lagrangian equations we had this identity proven. And this of course is equal to half sum over mi ri dot but what is ri/ del qj **ohh** look at this one here that is nothing but dotted with sorry yup the factor half of course will not be there sorry delete the factor of half okay this is nothing but n vector times epsilon.

And what is this? This is nothing but total momentum which sum over mi ri dot vector into epsilon that is it okay and what happens to the potential term. We have already seen that the kinetic energy term will be independent of qj the potential energy term.

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$$\frac{\partial V}{\partial q_j} = \sum_i \vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left(-F_i\right) \cdot \vec{n}\,\epsilon$$

$$= - F_{tot} \cdot \hat{n}\,\epsilon$$

Lagrange Eq for q_j

$$\frac{d}{dt}\left(\vec{P}_{tot} \cdot \vec{n}\right) = F \cdot \vec{n} = 0$$

$$\Rightarrow \vec{P}_{tot} = \text{Constant}$$

So del v / del qj we will of course at least for this case assume that the potentials are independent of velocities okay in which case this is in fact nothing but sum over i and you have gradient of i with respect to v into del ri / del qj and this is nothing but sum over i this is – fi into and again this is nothing but n vector times epsilon. So the Lagrange's equation are actually in this case for qj will be nothing but d / dt of total momentum dotted with n = summation over fi.

I will write this as –f total dotted with n times epsilon of course i can immediately remove the and what happens when you have del v / del qj so if the Lagrangian is independent of qj then we automatically know that this would be equal to 0 and total momentum is a conserved quantity okay. So if Lagrangian is independent of qj corresponding generalize momentum qj dot del l over qj dot will be conserved quantity and in this case it just turns out that quantity is the component of the total momentum along n cap vector and that is the quantity that could be conserved in this case.

So this is the general conservation law for linear momentum of the entire system. We can do exactly the same thing for the angular momentum of the entire system think in terms of you know there is a particular coordinate which causes the entire system to turn okay then we will be able to derive a similar law for the angular momentum. Now I will not do that here but I will ask you people to try that at home and see for yourself that you can actually derive same similar condition for the angler momentum case okay. Now I will do one more conservation theorem which we get from the newton's equations and that is energy conservation.

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Energy Conservation.

$$L(q,\dot{q},t) \;\; \underline{\text{ind of } t} \;\; = \frac{\partial L}{\partial t} = 0$$

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\dot{q}_i \right) + \frac{\partial L}{\partial t}$$

$$= \sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \quad \frac{d}{dt}\underbrace{\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L \right)} + \frac{\partial L}{\partial t} = 0$$

h : Energy Function.

$$\Rightarrow \quad h = \text{Constant}$$

Okay remember you have a system whose Lagrangian is given by q and lagrangian of course you know whenever you have a path q of t Lagrangian is of course is a function of time t and it is a implicit function of time t through variables q and q dot and it also may be explicit function of time t okay. Let me assume that given Lagrangian is independent of t you see we will have identity which is similar to Beltrami identity.

So let me write that here if I calculate dL/dt this is nothing but because there is a implicit dependence del L over del q dot sorry del q into dot + del L over del q dot d / dt. So of qi dot okay and + they will be a partial derivative with respect to at time t. Now this we know how to this can be rewritten as so d / dt remember this is similar to what we did in case of Beltrami identity. So the first factor here del L over del q dot can be replaced by d / dt of del L over del q dot from the Lagrange's equation.

And this of course can be written as q dot into del L over del q dot okay and then summed over i + del t and put it together this will become d / dt of q dot del q dot – L t = 0. And the quantity which goes here in the bracket this is usually denoted by h and is called energy function of the system. And remember if L is independent of time t that means there is no explicit dependence on time t that means partial derivative is 0 this immediately gives us that h is a constant or is a conserved quantity okay.

Now how does this turn out to be a energy conservation that is not immediately clear here but if I look at one example it probably will immediately become clear.

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Ex $\quad \mathcal{L} = \frac{1}{2} m(\dot{x}^2+\dot{y}^2) - V(x,y)$

$$h = \dot{x}\,\partial\mathcal{L}/\partial\dot{x} + \dot{y}\,\partial\mathcal{L}/\partial\dot{y} - \mathcal{L}$$
$$= \tfrac{1}{2} m(\dot{x}^2+\dot{y}^2) + V(x,y) = \text{total Energy}$$

If $\quad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$

$$\mathcal{L}_2 = \sum_{i,j} c_{ij}\,\dot{q}_i\dot{q}_j \qquad c_{ij}(q_1,\dots q_n, t)$$

$$\mathcal{L}_1 = \sum_i b_i\,\dot{q}_i$$

Again take on example suppose the Lagrangian of the system is of a single particle which is travelling in two dimension and the influence of some potential field okay and the potential field here depends only on the sorry only on the coordinates and not on the velocities okay. In this case I can now calculate the h function and h function here would become so this would be x dot I will first write this explicitly + y dot del over del y dot.

Remember there is summation over qi so here it would x dot del L over del x dot +y dot into del L over del y dot and – the Lagrangian and can you immediately see that this simplifies to half m x dot square + y dot square and + v of x y there you go this is nothing but the total energy. This the total energy of this so at least in Cartesian coordinates if v is the only function of coordinates and not of velocities and time then you have a conservative system the total energy is conserved.

So this is the conservation theorem for total energy but remember normally h will not be same as the total energy of the system that can be different. Now if your Lagrangian as a form like this I will write this as L0 + L1 + L2 okay now the piece L2 is say only quadratic in velocities. So I will write this as sum over sum Cij qi dot qj dot okay summation is over i and j okay and there will be double counting which will take care of when we write this Cij. Remember Cij themselves will be function of q1 to qn and probably also time t and similarly L1 is a linear term so I will write this as bi qi dot okay it is only a linear term in qi dot.

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If $L = L_0 + L_1 + L_2$

$L_2 = \sum_{i,j} c_{ij}\,\dot{q}_i \dot{q}_j \qquad c_{ij}(q_1, \dots q_n, t)$

$L_1 = \sum_i b_i \dot{q}_i$

$L_0\,(q,t)$

$h(q,\dot{q},t) = L_2 - L_0$

$\vec{r}_i \rightarrow q \qquad \vec{r}_i = \vec{r}_i(q_1, \dots, q_n)$

$\Rightarrow$ transformations are ind of t

$L = L_2 + L_0$

$h = L_2 - L_0 = T + V$

And L0 this is a term which does not depend on the generalize velocities okay it is just the function of q and t. In this case if I calculate h this is the exercise for you people from the assignment so that this will be equal to L2 – L naught okay. So h function just turns to be this much in this special case what happens is this if I have suppose you have the position vector ri and you make a transition to generalize coordinates q.

So when I write this transformation equation ri as ri function of q1 to qn and in this transformation are independent of time t then what you get is a homogenous so when you take ri dot you get a homogenous function of the generalize velocities and it is linear in generalize velocities. So when you take squares you only get L2 term so this remember if the transformations are independent of t then you get L2 term in Lagrangian and you will get L0 term okay.

The first linear term in generalize velocities will be absent now what would be L0 term? L0 term is your potential term in the potential term if we assume that the potential depends only on coordinates and not on generalize velocities or on time then your h function in fact will simply become L2 – L0 and this would be same as kinetic energy + potential energy. So in this special case the h function looks like the total energy or = total energy and in this case if your Lagrangian is not explicitly dependent on time the total energy is concerned.

So that is how we get the energy conservation rule remember h energy function even though it is called as energy function it is not equal to total energy and it may happen that h is a conserved quantity but out total energy may not be conserved quantity. I will give you one example of this or even it may happen that h is actually = total energy but the total energy is not consent. Remember this h function so Lagrangian even though there is a unique Lagrangian the h function actually depends on the coordinates so which coordinates you choose will keep on changing your h function. Now also the total energy function so here is an example which illustrate this.

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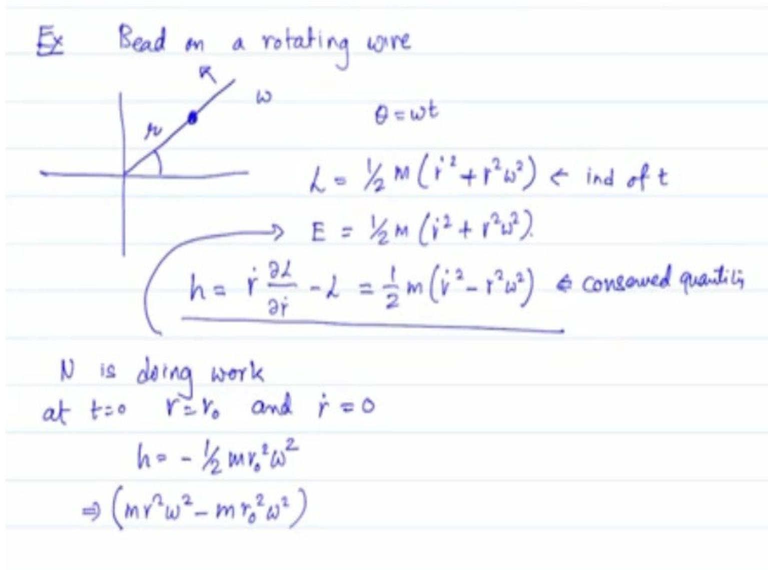

This is a bead on rotating wire okay so you have a wire and there is a bead on the wire and this wire is in fact rotating about origin with the constant angular speed omega okay. So here because there are two coordinates to begin with but there is one constraint and because of this constraints all I really need is one generalize coordinate which is r. The other coordinate theta is in fact just equal to omega times t okay.

Now let us write down the Lagrangian for this will be half m r dot square + r square theta dot square but that we already know is just omega square okay. This is the Lagrangian for the system and what is the total energy of the particle? So total energy of the particle here because there are other than constraint forces they no other forces acting on the particles. So total energy is simply half m r dot square + r square omega square okay and what is the h function?

Let us calculate h function so h function will be energy function will be equal to r dot del L over del r dot – L and this is half m r dot square – r square omega square. See here the h function is not same as energy function now let me ask you a question is this a conservative system? That means is the energy consort quantity here? Surely not see if I had a free rotating rod like a merry go round which is not driven.

And if you sit on it and if you move outwards what would normally happen is the angular speed decreases but in this case even if the particle moves away from the origin the external system constraint system is actually driving the rod or the wire with the same constant angular speed which means the external forces or the normal reaction here is doing work. And that work is actually reflected in the energy e so what remains conserved here?

Remember Lagrangian again independent of t that means this is conserved quantity what does this mean? How do I interpret this? In fact we can immediately see that if I hold the particle at t = 0 at some initial instant and let go that means of the initial instant r dot is 0 and your h function is in fact is nothing but so at t = 0 r = r naught and the speed the radial speed of the particle is 0. See we are letting go at one instant so this is in fact – half m r naught square omega square and what it says is this.

If the particle moves out to some other location then it is new energy will be same as this quantity + the amount of work done by the constraint forces okay that is why the difference between h and e is exactly equal to mr square omega square okay. So the difference the gain in the energy could mr naught square omega square this would be total amount of work done by the constraint forces.

So in this example what we see is at the total energy is not conserved but as long as the Lagrangian is independent of time t there is some conserved quantity which is called as energy function and that energy function is given by this in this case. Okay now these ideas in fact are also related to each one of this conservation principle is related to the symmetries of the Lagrangian and this going to be focus of my next session.

So the conservation laws are intimately connected to symmetry of the Lagrangian and this is going to be focus of this small section today. If you see that if you make a coordinate transformation this is a great facility given by Lagrangian equation for very simple reason that the general form of the Lagrangian equations it remains same what that means is this when I make a transformation the Lagrange’s equation as we write them.

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$$q = (q_1, .., q_n) \longrightarrow s = (s_1 \ldots, s_n)$$

$$q_i = q_i(s_1, \ldots s_n) \quad \forall i$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \Rightarrow \quad \Leftarrow$$

$$\frac{d}{dt}\left(\frac{\partial L_s}{\partial \dot{s}_i}\right) - \frac{\partial L_s}{\partial s_i} = 0 \Leftarrow$$

$$L_s(s, \dot{s}, t) = L(q(s,t), \dot{q}(s, \dot{s}, t), t)$$

$\underline{\text{Ex}}$ $(x,y) \longrightarrow (r,\theta)$ $\qquad L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

$$L_s = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

(x,y) $\qquad m\ddot{x} = 0 \;, \; m\ddot{y} = 0$

(r,θ) $\qquad m(\ddot{r} - r\dot{\theta}^2) = 0 \qquad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$

So if we start from coordinates q which are given by q1 to qn and make a transformation to s which is s1 to sn. So the transformation here would be qi is written in terms of s1 to sn for all for each qi okay under this transformation the form of the Lagrangian equation remains same that means d/ dt of del L / del qi – del L / del qi this is equal to 0 this is if it is Lagrange's equations for q coordinates then for s coordinates it is a same equation except that sorry si dot – del Ls / del si = 0.

The only thing is the Lagrangian is of course changes the Lagrangian will be given by the new Lagrangian which is function of s, s dot and t this must be obtained from the previous Lagrangian which was written in terms of q but remember q is a function of s and possibly also t then q dot will be function of s, s dot and t and the explicit dependence on t. So this is how the new Lagnrangian is calculated but the general form remains same.

This is not true for the coordinates though even though the general form is same the individual equation when you substitute coordinates will be different I mean you cannot simply you know take this equation for qj and substitute sj in that equation to get a new equation here what I mean by this is that if you start with Cartesian coordinate let me take simple example where you go from xy coordinates to r theta coordinates.

And the Lagrangian is that of a free particle so you have half mx dot square and + y dot square so when you make a transformation to the new Hamiltonium sorry new Lagrangian Ls which is function of r theta r dot theta and so on this would come to half I dot square + r square theta dot square. So the Lagrangian looks different what about equation of motion or they also looked very different.

Because in first case the equation simply turns out to be mx double dot = 0 and my double dot = 0 but so these are the equations in x and y but in r and theta we already know that mr double dot – r theta dot square must be equal to 0 and the second equation for the theta becomes m r theta double dot + twice r dot theta dot must be equal to 0. You see I cannot simply take the x equation mx double dot substitute r in it and write down the equation for r that is not true.

So the even though the Lagrange’s equation in the form are same the actual equation in terms of coordinates they will be of course different. Now when will they be same they will be same if the Lagrange’s Lagrangian has the same form.

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Ex $(x,y) \to (r,\theta)$ $\quad L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

$L_S = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$

(x,y) $\quad m\ddot{x} = 0, \quad m\ddot{y} = 0$

(r,θ) $\quad m(\ddot{r} - r\dot{\theta}^2) = 0 \quad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$

Ex $(x,y) \to (x',y')$

$$x' = \cos\alpha\, x - \sin\alpha\, y$$
$$y' = \sin\alpha\, x + \cos\alpha\, y$$

$$L(x,y,\dot{x},\dot{y},t) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$
$$L_S(x',y',\dot{x}',\dot{y}',t) = \frac{1}{2} m(\dot{x}'^2 + \dot{y}'^2) \quad ?$$
$$\Rightarrow m\ddot{x}' = 0, \quad m\ddot{y}' = 0$$

Now that also I will explain using an example so you start with xy coordinates and go to new coordinates which are x prime and y prime and these x prime and y prime are basically obtained by rotating x y prime by an angle alpha. So the connection would be x prime will be equal to Cos alpha times x + - sin alpha times y and y prime will be equal to sin alpha times x + Cos alpha times y and the original Lagrangian here which is function of xy, x dot y dot and t this is half m x dot square + y dot square and what is the new Lagrangian?

Great now the new Lagrangian also has exactly the same form which is equal to half m x prime dot square + y prime dot square and you see this is what is called as the invariance of the Lagrangian under coordinate transformation. Even if you go from here to here the Lagrangian form of the Lagrangian is same. And in that case of course because the form of general form of Lagrange's equation is same here I can find out the equation for x prime simply by taking equation for x and substitute by x prime and that immediately gives you this equation.

And similarly m y prime double dot will be equal to 0 so these are the new equations obtain new Lagrangian so I will call this as Ls so this was the new Lagrangian. So this is the idea that we want to follow under which transformation the Lagrangian remains invariant.

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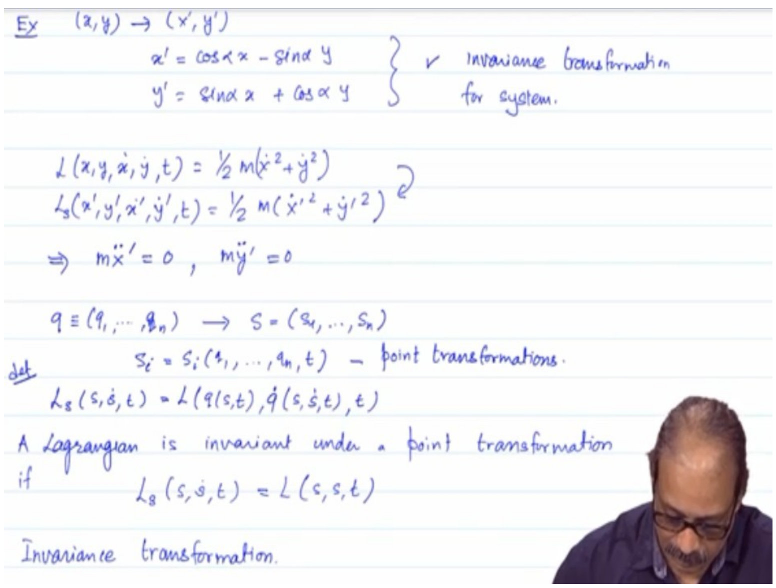

So make a general case we have a system with n degrees of freedom with one set of variables which are q1 to qn and we making transformation to another system another set of coordinate which is s1 to sn and the transformations here is si are given in terms of q1 to qn and possibly t. Now of course when you make this coordinate transformation all the other conditions of you know differentiability and the invertibility must hold here and will assume that.

These transformation are called as point transformation okay and then we get new Lagrangian Ls which is function of s, s dot and t so remember this is in some sense definition which must be equal to L at Q but now written as function of s and t q dot which is written as function of s, s dot and t and t so you have to be basically take this transformation find the inverse transformations and in the Lagrangian wherever there is a q you substitute the form in terms of s.

So this is what you get a new Lagrangian and then we will say the Lagrangian is invariant under a point transformation if the new Lagrangian has exactly same from as old Lagrangian remember this means you take the form of old Lagrangian wherever there is q is substitute by s where is this q dot is substitute by s dot okay. This is called as and then the point transformation is called as invariance transformation.

So when you have invariance transformation the Lagrangian remains form invariant where is the example so there it is. In the previous two examples this one here this transformation is not a invariance transformation because if you take a new Lagrangian function of r and theta wherever there is a r substitute by x whereas there is a theta substituted by y will not work okay. So this one is not an invariance transformation but this one is here this one is a invariance transformation for this particular system okay.

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Cyclic Co-ordinates: q_j Cyclic if L is ind of q_j

$$s_i = q_i \quad i \neq j$$

$$s_j = q_j + \alpha$$

$$L_s(s_1, \ldots s_n, \dot{s}_1, \ldots \dot{s}_n, t) = L(s_1, \ldots, s_n, \dot{s}_1, \ldots \dot{s}_n, t)$$

if q_j is cyclic then L is invariant under translation along q_j. $\Rightarrow$ p_j = Conserved.

Now what is the connection with the conservation laws so there it so you get conservation laws I will first consider the simplest possible case of cyclic coordinates remember coordinate qj is cyclic if L is independent of qj and we already know that if this happens then quite obviously the corresponding generalize momentum or conjugate momentum is conserved. Now we will see what is connection of this statement with the symmetry of the Lagrangian think of this the transformation where si = qi if i is not equal to j and sj is qj + sum alpha okay.

And you can immediately see that the new Lagrangian here which is s1 to sn s1 dot to sn dot t this is in fact exactly = Lagrangian the whole Lagrangian at s1 to sn, s1 dot to sn dot and t. That is because each si is in fact same a qi so that is what go into the Lagrangian and sj is different from qj but wait your Lagrangian does not even have the term qj so it would not have sj. So this in that case holds that means if qj is cyclic then L is invariant under translation along qj.

See what we have done is in fact keeping all other coordinate same we have just translated the system along the coordinate qj okay. So under the translation transformation or the system here the Lagrangian here is invariant under the translation transformation for or along the direction of qj or along the coordinate qj. And this results in the conservation law that pj is conserved will see exactly which quantity is conserved in this case okay.

The connection right now we have this segment qj cyclic pj is conserved qj is cyclic Lagrangian is invariant now we want to put these two together and say if the Lagrangian is invariant under a particular transformation there is conservation law okay alright so let me do that here.

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$$\mathcal{L}(q, \dot{q}, t)$$

$$q_i \rightarrow q_i + \delta q_i$$

$$\dot{q}_i \rightarrow \dot{q}_i + \delta \dot{q}_i$$

Lagrangian is invariant $\rightarrow$

$$\mathcal{L}(q, \dot{q}, t) = \mathcal{L}(q_i + \delta q_i, \dot{q} + \delta \dot{q}_i, t) \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$

$$= \mathcal{L}(q_i, \dot{q}_i, t) + \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} \delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i\right)$$

$$\Rightarrow \frac{d}{dt}\Big(\underbrace{\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta q_i}_{\text{Conserved quantity}}\Big) = 0$$

So let me start by taking a Lagrangian of q q dot and t okay and we will say that we have this infinitesimal transformation which takes each qi to qi + some delta qi okay. With this transformation what would happen to the Lagrangian if qi is change like this then qi dot are also qi dot + delta qi dot okay and let us put this together. So if under this transformation Lagrangian is invariant implies that the Lagrangian as a function of coordinate q, q dot and t also must be equal to Lagrangian substituting the new coordinates now qi + delta qi, qi dot + delta qi dot and t.

And you can of course expand this so this would become Lagrangian at qi, qi dot and t + sum over i and then you have del L over del qi into delta qi + partial of L with respect to qi dot in to delta qi dot okay. And then of course we can immediately say that you remember the trick that we did in case of Beltrami identity this is nothing but replace del L / del qi dot here by d / dt of del L / del qi dot and then this just looks like a product rule for differentiation and I will put this.

So Lagrangian of course will cancel on both the sides and what you get is d / dt of sum over qi dot into delta qi and this is a conserved quantity. So there you go for every infinitesimal transformation which more accurately are called as continuous transformation for infinitesimal transformation there is a conserved quantity will immediately look at the examples of this.

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Ex. $L = \frac{1}{2} m(\dot{x}^2+\dot{y}^2) + \frac{1}{2} m\omega^2(x^2+y^2)$ | 2D Harmonic Oscillator

$$x' = x - \alpha y \Rightarrow \delta x = -\alpha y \qquad \alpha \text{ : infinitesimal}$$

$$y' = \alpha x + y \Rightarrow \delta y = \alpha x$$

$$L = \frac{1}{2} m(\dot{x}^2+\dot{y}^2) + \frac{1}{2} m\omega^2(x^2+y^2) + O(\alpha^2)$$

$$\frac{\partial L}{\partial \dot{x}}\delta x + \frac{\partial L}{\partial \dot{y}}\delta y = \text{Const}$$

$$-\alpha m\dot{x}y + \alpha m\dot{y}x = \text{Const}$$

$$\Rightarrow \alpha m\underbrace{(x\dot{y} - y\dot{x})}_{L_z} = \text{Const}$$

So let me take a example of 2D Harmonic oscillator here the Lagrangian is y dot square + half m omega square x square + y square okay. And now I am going to do a transformation to new coordinates. So if I take x prime which is remember this is same as the rotation of the system by an angle alpha but now I am going to assume that the angle alpha is infinitesimal. So if angle alpha is infinitesimal then I can write x prime as x – alpha times y and y prime will be equal to alpha x + y okay.

This immediately gives us delta x to be equal to – alpha y and this gives us delta y = alpha x and if is this particular Lagrangian invariant under such transformation yes if you substitute all this there then the new Lagrangian will be equal to half m x prime or x dot square + y dot square + half m omega square x square + y square and + the term which are quadratic in alpha square. So all the linear terms in alpha actually have vanished leaving the Lagrangian invariant at least infinitesimal invariant under this transformation.

And what is this conserved quantity then so conserved quantity will be del L / del x dot into delta x + del L / del y dot into delta y and this is the conserved quantity and we can immediately identify this. What is del L over del x dot that is nothing but mx dot and what is delta x? Delta x is nothing but – alpha times y and for the second term this is mx sorry my dot into alpha times x and this must be constant.

And this immediately gives you alpha times m times so this will be x y dot – y x dot is constant and what does this immediately tell you that this quantity also must be constant of motion and this is in fact nothing but z component of the angular momentum. So in case very obvious that there is a symmetry circular symmetry or symmetrical symmetry in xy place in which case the z component of the angler momentum is conserved. So this is how the conservation principles emerge from the invariance of Lagrangian under point transformation.

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Ex $\quad L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

$\theta' = \theta + \alpha \qquad$ small α.

$\delta\theta = \alpha$

$r' = r \Rightarrow \delta r = 0$

$$\Rightarrow \quad \delta r \cdot \frac{\partial L}{\partial \dot{r}} + \delta\theta \cdot \frac{\partial L}{\partial \dot{\theta}} = \text{Constant}$$

$$\Rightarrow \quad 0 + \alpha \underbrace{(mr^2\dot{\theta})}_{L_z} = \text{Const}$$

And another example in this example I will take a simpler case or here the Lagrangian is half m r dot square + r square theta dot square and let met assume that the potential only depends on r. This situation is similar to the previous one okay situation is exactly same because in the previous case the potential I had explicitly given it as the potential of the Harmonic oscillator was proportional to x square + y square and that is that means it is proportional to r square.

Here what we are saying is that the potential here does not depend on the angle theta which also makes theta a cyclic coordinate here remember and the transformation that we are going to is theta prime is theta + small alpha okay. Small alpha this is like a translation along the theta direction which eventually means a rotation which is exactly same as the previous example but here the transformation looks like when you look at it from the coordinate theta point of you it is like a translation by small amount alpha.

And in this case again the conserved quantity will be so here delta theta is alpha what is delta r we are not change in r okay so the new coordinate r prime is exactly equal to r which gives us delta r to be 0. So this is the translation where all the coordinates are fixed except one which is being translated the conserved quantity then would be equal to delta r into del L over del r dot + delta theta into theta dot and in this case this will be equal to constant but this of course is 0 + delta theta is nothing but alpha and del / del theta dot is nothing but mr square theta dot = constant.

This is again the z component of angular momentum so there is a intimate connection between the invariance of the Lagrangian or symmetry of Lagrangian to the conservation principles. So every continuous invariance transformation there is conservation law and the conserved quantity is also given by.

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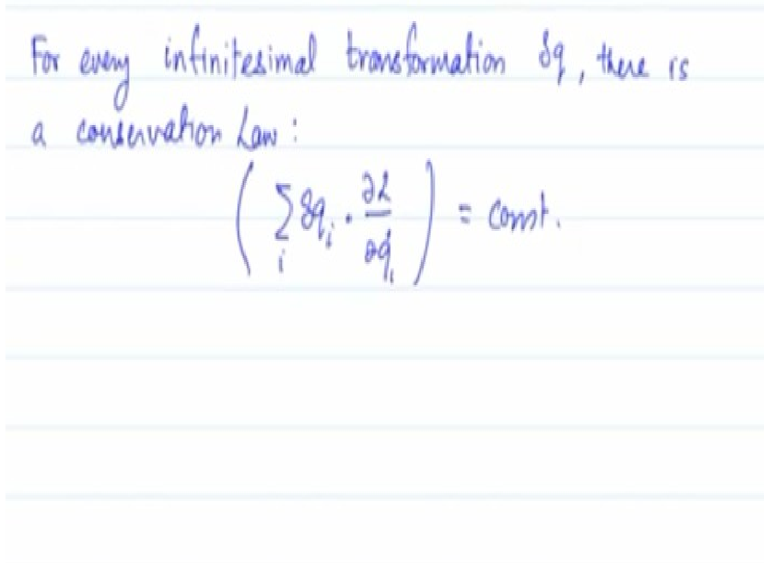

So for every infinitesimal transformation given by delta q okay there is so conservation law which status that some over I delta qi into del L over del qi dot is a conserved quantity. This theorem is called as Noether's theorem this is by a mathematician Emmy Neother's.

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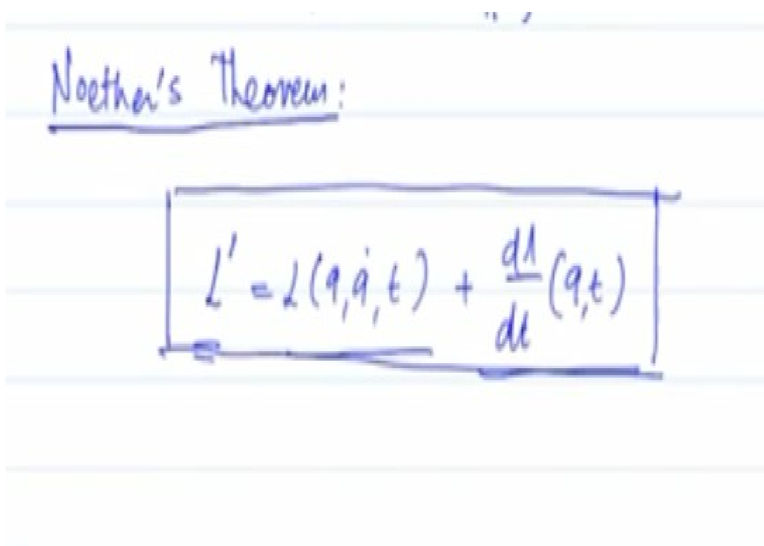

So it is in honor of her work this is called as Neother's theorem right. Now only one last point before i stop here remember the invariance of Lagrangian the way we posed it here. In fact it is little bit restrictive if you notice and this is also problem in assignment is that if you take a Lagrangian you get equations of motion. Now if I have set of equation motion do I have a unique Lagrangian for it may be possible that different Lagrangian may give you the same set of equation.

And that actually happens we can immediately or easily prove that if you have a Lagrangian L you construct new Lagrangian L prime by taking L which is function of q, q dot and t and act to it t of q and t both L and L prime they in fact will give you exactly same set of equations in coordinates to. What that means? Is when I was definition my invariance earlier when I make a point transformation I get a new Lagrangian and as long as new Lagrangian is equal to old Lagrangian + a total derivative of sum function of q and t I still get a invariance okay.

So the new definition of invariance should also include this. So L prime at q, q dot and t if it is equal to old L old Lagrangian at q, q dot and t + another total different derivative of sum function then also your Lagrangian is considered as invariant.