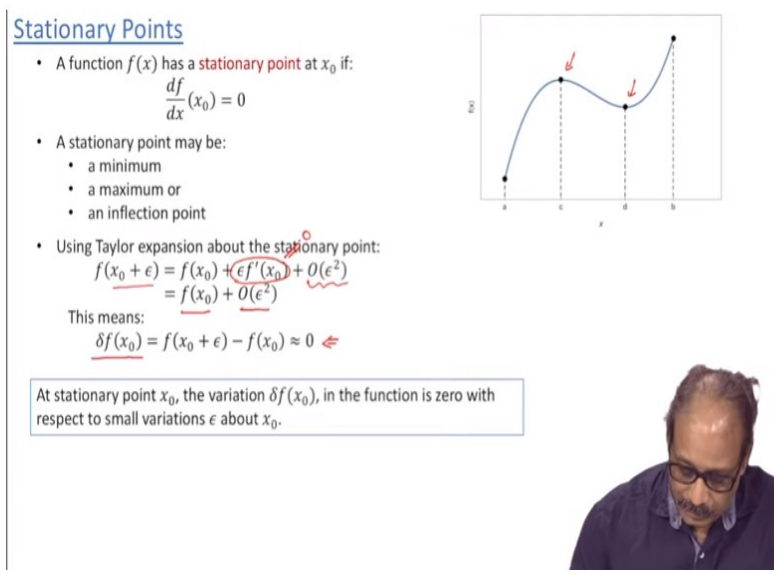
Theoretical Mechanics
Prof. Charudatt Kadolkar
Department of Physics
Indian Institute of Technology – Guwahati

Module No # 02
Lecture No # 07
Variational Calculus

Okay so we want to find the extremum points of the functional those which we discuss in the last section. The problem is little more difficult because the domain of our functional is a space of all paths. Now this is a infinite dimensional space. We can put some sort of a vector structure on it but when we put vector space structure it is a infinite dimensional vector space. And we need to set of do some sort of calculus to find the minimum of the function or maximum or extremum whichever case that may be.

(Refer Slide Time: 01:17)



Now what I will is I will begin with simple ideas from the ordinary calculus. Is a in case of ordinary calculus if you have a real valued smooth function defined over some segment of real line. Then within that domain the function as a stationary point if it is first derivative is 0 at that point okay. So for an example the stationary point can either be minimum or maximum or an inflection point.

In this example we have a minimum at point D and we have a minimum at point sorry maximum at point C. Or there is also a minimum at point a or it is a boundary of the domain okay we are of course not interested in the points like A and like B where there is a minimum and maximum respectively. Our interest is in the stationary points where the derivative the first derivative of the function becomes 0. And which is what happens at point c here and point d here.

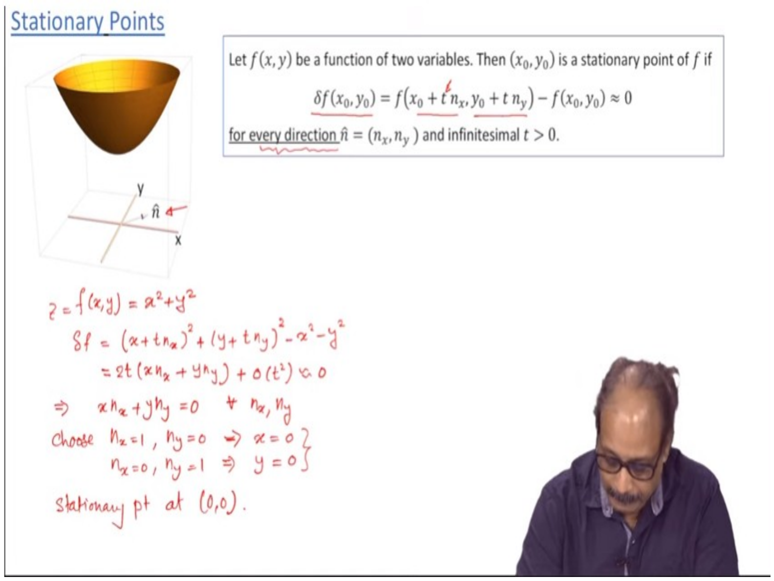
And in this case when we find a point which is stationary then of course we look at the second derivative to decide what kind of point it is whether is a minimum or a maximum or a inflection point by inflection point I mean take a example of xq and the derivative of xq which is 3x square will become 0 at x = 0. But the second derivative is also 0 and the function here neither as a minimum nor as a maximum okay it temporarily stops raising or lowering at that point.

But then it continuous to rise in the positive side okay now what we are going to do is look at the same idea in the slightly different way. So if I look at the Taylor expansion of this function about x naught then I will get f at x + epsilon will be equal fx naught + epsilon times f prime at x naught and + all the terms which have epsilon square and higher order okay higher power of epsilon.

But because x naught is a stationary point the derivative of function is 0 at this point and hence this term here is in fact 0. In that case I am only left with the constant term plus terms which are quadratic in epsilon or higher. And then I will define a variation or delta fx delta f calculated at x naught as f at x + epsilon – fx. Now if epsilon is small enough then this would be approximately 0.

So at x naught if you take small epsilon this small change in the function is 0 in other words the function just does not change and that in fact is the meaning of the word stationary that it sees us to either raise or lower at that point. So we will put that in a neat statement so at stationary point x naught the variation delta f is 0 if you vary the point by small infinite decimal amount epsilon about x naught about the stationary point. And this is the idea that we want to generalize to the other dimension or to higher dimensions.

(Refer Slide Time: 05:42)



So look at the case of two dimensions in case of two dimensions I have taken a simple functions here which is x square so z = x square + y square. So the plot is that of z = and it is a function of two variables we want to use the same idea we know that at x = 0 and y = 0 and that is the origin we have the minimum of the function which you can easily see here okay we can visually inspect. But if want to generalize the previous idea what I need to do is I need to go away from that stationary point by infinite decimal amount.

And look at what happens the function if does not raise or lower it remains stationary or the variation is 0 then I would call this as stationary point. But which direction so here I have math one direction which I called as n cap see if I go in the direction of n cap the value of function would not change. But why this particular n cap no it must happen for all possible n cap and there are infinitely many directions.

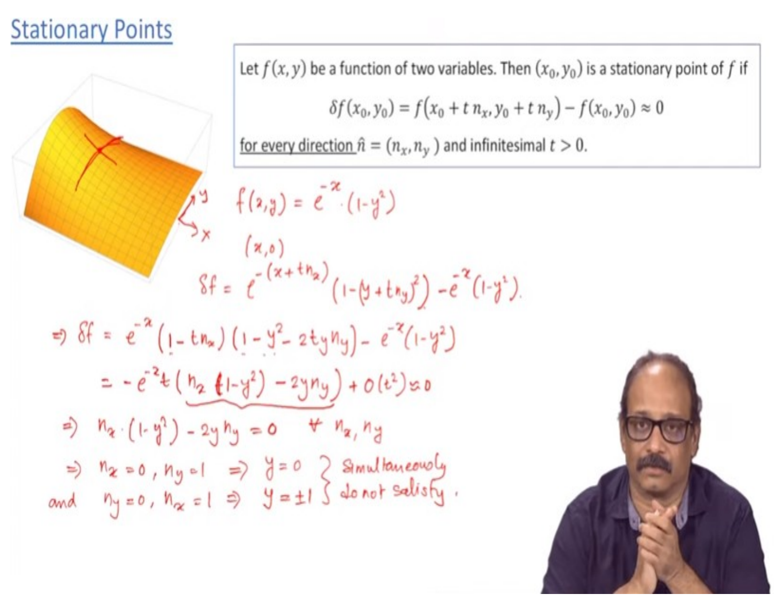
So you have check in all directions whether the function remains stationary or not so I will define the stationary points or two dimensional case as if the variation delta f at x naught y naught which is of course defined now like this. So the nx and ny here in this expression are the two components of n cap vector. So what you are doing is your actually travelling along the direction of n cap and I multiply it by an infinite decimal number t and then calculate the variation by taking the difference.

And this must be 0 but the key point here is that it must be 0 for all possible directions there so all possible nx and ny the function must remain stationary then we will call x naught y naught is the stationary point of the function okay. Let me apply that to the problem that we have here see if I wanted to calculate so this procedure of course we will use to also find the minimum points. So suppose I take a variation here if I take variation then delta f will be equal to so we will take x + t time nx whole square + y + t time ny whole square okay.

And then do a simple expansion here of course that Taylor expansion is straight forward so sorry – x square – y square that is f at xy. And this immediately will come to t times so 2t x nx + y ny + all the terms of the (()) (09:29) t square and this must be equal to 0 okay when will this be 0? Since is some arbitrary infinite decimal number this must be 0 for all values of nx and ny. So I will write that condition so condition for stationary point here will be x nx + y ny must be equal to 0 but must be true for all nx and all ny and now you see what happens here.

Because they are true for all values of nx and ny I can of course in particular choose nx to be 1 and ny to be 0 and what does that immediately give us that tells us that x must be 0 and similarly if I choose nx = 0 and ny = 1 that gives us y = 0 there you go. We have found the point at which the function is stationary so function as a stationary point at 0, 0 and this you know true for every nx and every ny is crucial part of this argument there.

(Refer Slide Time: 11:05)



Let me take one more example now this function is so I have the same definition here but this function here is in fact e to the power –x into 1 – y square if you visually inspect this graph here you can immediately see that there is no minimum anywhere or no maximum is there a stationary point no there is no stationary point either. So let us look at whether we can find so let me take one point which is on the so here is the x axis and this one here is y axis.

So let me choose one point which is on x axis so if I choose a point which is x comma 0 okay and we want to find out whether there is a stationary point here. Now do the same thing calculate delta f so your delta f becomes so it is e to the power – x + epsilon then + sorry multiplied by 1 – y sorry. So let us calculate the delta f again so delta f here will be equal to e to the power – x + t times nx and into 1 – y square.

So here y will become y + t times ny whole square and we want to retain only the linear terms here. So I will write this as delta f = this is e to the power –x and the e to the power –t times –nx because t is infinite decimal I will retain the linear terms which is 1 – t times nx and then on this side you have 1 – so – e to the power – x into 1 – y square. And you have 1 – y square – 2 times t times y times ny and + of course the t square terms which I am going to ignore.

So – e to the power –x into 1 – y square and this will simplify to so out of this the first bracket 1 into 1 – y square this of course will cancel with and what you are left with is e to the power – x then because we are looking for linear term in t so there is t and there is nx and + you have so nx into 1 – y square – 2 times y ny okay. And of course there are all the other terms which are in t square.

And now look at this must be equal to 0 this immediately tell us e to the power – x is never 0 t is not 0. So this bracket here must be equal to 0 so nx for xy to be a stationary point nx multiplied 1 – y square – 2 times ny must be equal to 0 and remember this must be true for all nx and all ny and in particular I can of course choose nx = 0 and ny = 1 this immediately tell us y must be equal to 0 y coordinate definitely must be equal to 0 and if I choose ny to be sorry ny to be 0 and nx to be 1.

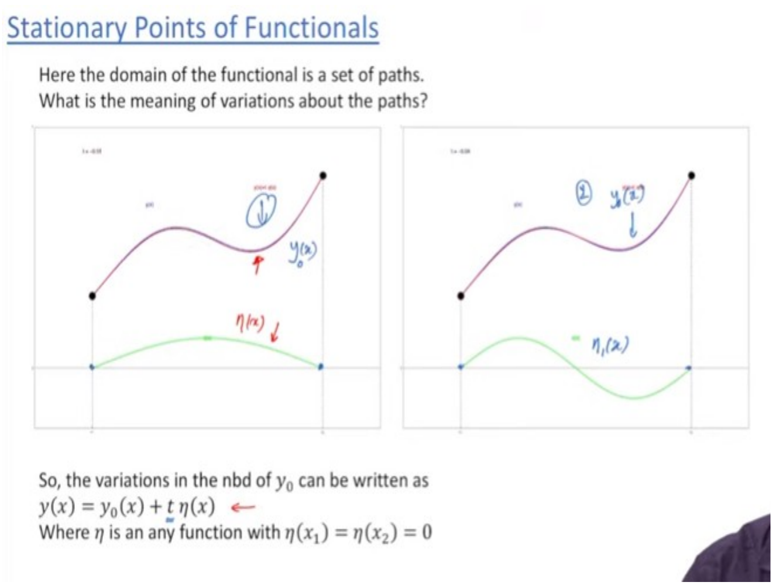
This immediately tell us that y must be equal to either + or -1 and these two will never simultaneously satisfy (()) (16:28) do not satisfy. Which means any point in this entire plane actually is not a stationary point if you choose y = 0 which comes to a point like this you can see that if I travel along the y direction along with contour then on that contour along that path the function is stationary.

But passing through the same point but now take this path and the function is not stationary okay which means the function of course is not stationary in this case. Now in two dimensions of course you can have variety of stationary points one is of course minimum one is maximum and you can also have what is also called as saddle points. And saddle points are very easy to visualize if you have seen a horse saddle on the horse saddle you can have maximum on one side and along the axis or the along the direction of the body of the horse you have the minimum okay this is called as the saddle.

And also the points which have this kind of the behavior remember this also a stationary point but this particular point will be called as a saddle point okay. Now what we want to do is take this idea to further more dimensional spaces. If you have n dimensional space or a domain what would happen the situation will be similar to 2 dimension except in 2 dimensions I had x axis and y axis in n dimensions may be I have n axis x1, x2 and so on and then I will vary nx1, nx2, nx3 and so on.

And if the point has to be stationary then in the n dimensional path you would be travelling along an arbitrary direction the definition will look same except nx and ny would now become n1, n2 and so on and then the variation of the function as you travel along any direction but by infinite decimal amount the variation in the function should be 0 okay alright. This is the idea I want to use in our functional minimization program.

(Refer Slide Time: 19:03)



(Video Starts: 19:03)

So here I have these two cases in the functional case the domain is the infinite dimensional and every point in the domain is actually a path it is the function of function remember. So what is the meaning of varying or going little bit away from the given path so here in this diagram I have a blue path there and I want to look at all possible paths which differ from this blue path infinite decimally.

And how do I do that exactly the same way I did in the earlier case so I will take this y naught which is a blue path then add to it some Eta x so your blue path there is y of x sorry y naught. And I want to take the variations about y naught. So what I will do is I will take a infinite decimal t multiply it to arbitrary function Eta and add it to y and you can see in this animation that as you change the value of t you get paths which are nearby okay.

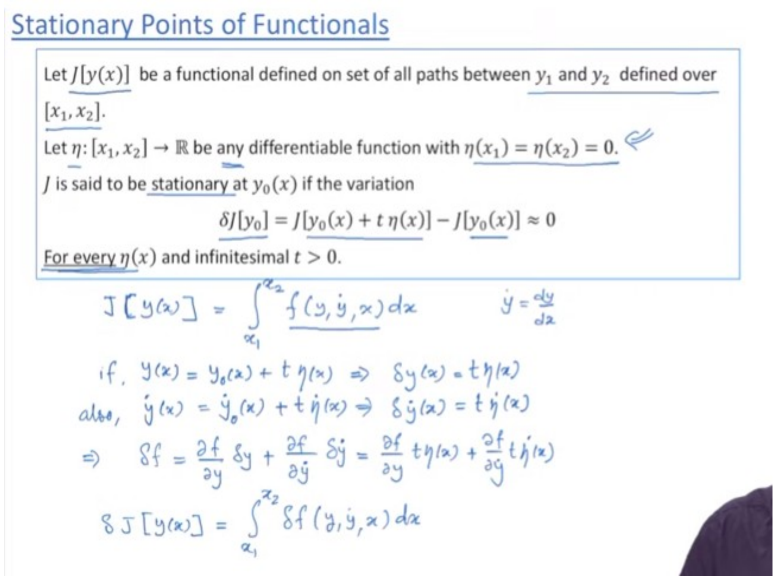
As long as t is infinite decimal you will get path which are nearby now this is like going it one direction how do I go in another direction away from y naught here. Now you just change your function Eta x to something else so this is Eta 1 x and what we are doing is to the same y naught we are adding Eta 1 x and see what happens. Now we also get paths which are nearby y naught they are not the same paths as covered in this diagram and this diagram both have very essence which are very different from each other.

And how many ways you can do this you can do that in infinitely many ways because you see if I want to get a new path all I have to do is add arbitrary Eta x to it and this arbitrary Eta x because the n points of x whatever Eta that we add that of course must be 0 at end points. So Eta must be 0 at the end points as long as Eta is 0 at end points I can multiply it by infinite decimal value and add it to y naught either a infinite decimal path which is near y naught.

And this is the idea of variation that I am going to use in finding the stationary points of the functional okay.

(Video ends: 22:10)

(Refer Slide Time: 22:11)



So let us extend this definition here I have written the definition already so the functional is already defined as j and which is between all the paths which pass through x1 y1 and x2 y2 and we will say that so Eta is some arbitrary remember some arbitrary function with n points which is equal to 0 or value of Eta at n points 0 and then we define j is set to be stationary at y naught at given path if the variation that is delta j at y naught and how do I define the variation?

Variation would be calculate action at nearby path which is y naught + theta n theta x – the functional at y naught and the different of this two should be 0. And remember again this must be true importantly for any arbitrary Eta that satisfies this conditions as long as that happens then we call it as a stationary point. See eventually what we were interested in was to find a minimum of a function or in probably some cases maximum of the functional some extremum but for the functional to be extremum the necessary condition is that it must be stationary point it is not a sufficient condition remember.

But it is a necessary condition if you find stationary point then we will go ahead and check whether it is minimum or maximum or none of them and whether it fits the requirement that we had so we might find a we are looking for a minimum we find a stationary point but it turns to be maximum of course we have discard that and search for the minimum again okay. So now let us apply this if I want to apply this to the functional of our interest and remember the functionals of our interest where of this kind.

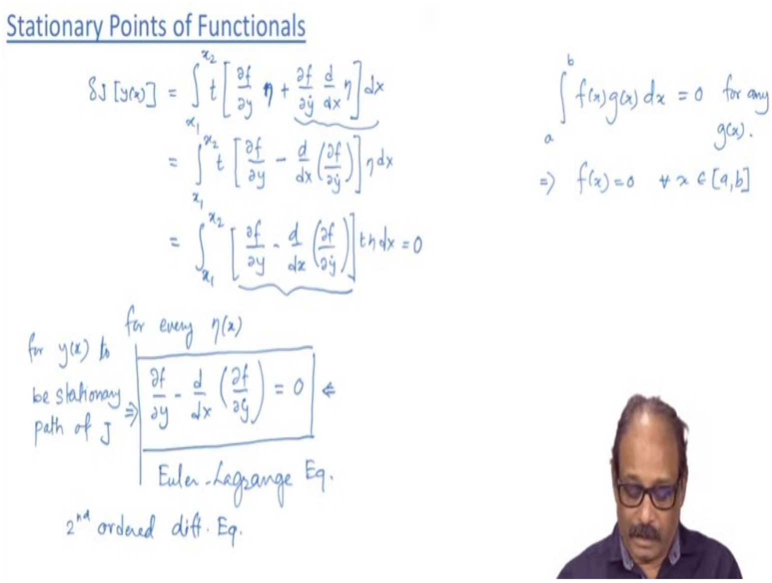
So J of y of x we defined it as integral over x from x1 to x2 of sum integral which depends on y y dot and x. Remember here y dot is dy / dx so when I take a small variation then what happens to the integral let us look at that first. So if I take y of x which is equal to y naught of x + t times some Eta x then that means the variation in the path at each point then variation of the path at each x if you want I will go back to the previous one.

Remember at each x at each value of x I have this delta y okay and this is what we are going to call it as delta y of x. So delta y of x is nothing but t times Eta x and I can also take the derivative of this with respect to x then y dot of x is going to y0 of x dot + t times Eta dot of x and this immediately means that for the function y dot the variation in y dot is in fact simply equal to t times Eta dot of x.

And let us look at now what happens and when I change the path the y dot also changes and then eventually the integral would change so the change in the integrant would be delta f will be equal to the partial derivative of f with respect to y into delta y + partial derivative of f with respect to y dot into delta y dot. Of course we are looking at the variation at each x so basically the third term because it depends on x will not appear here similar to the virtual displacement that we talk in our first week okay.

So let us substitute various quantity is there so this would be equal to del f over del y into t times Eta and + del f / del y dot into t times Eta dot of x okay. Now let us go ahead and put this back into the so your delta j x this will be equal to integral from x1 to x2 and all have to do is put delta f there which still the function of this dx okay done let me do that.

(Refer Slide Time: 28:32)



So delta j at y of x will be equal to integral from x1 to x2 and now I am going to write this slightly differently so there is t which can be taken out and there is del f / del y and + sorry into Eta + del f / del y dot into d / dx of Eta dx. And this term here can be integrated by paths so this will be integral from x1 to x2 t times and del y I am going to write Eta outside so I will leave it for the time being and write the second term of this integration by paths that would be equal to integral of.

So this would be d / dx of del f / del y dot into Eta now I will put that Eta outside and write it like this okay. Now what we get these is integral from x1 to x2 and remember this bracket here is sum function of x del y dot and then there is a t Eta dx. And I am going to use the identity here sorry I am going to use an identity here from the real analysis if I have integral from sum a to b of sum smooth function and if this is equal to 0 for any function g then f of x must be equal to 0 identically over the domain.

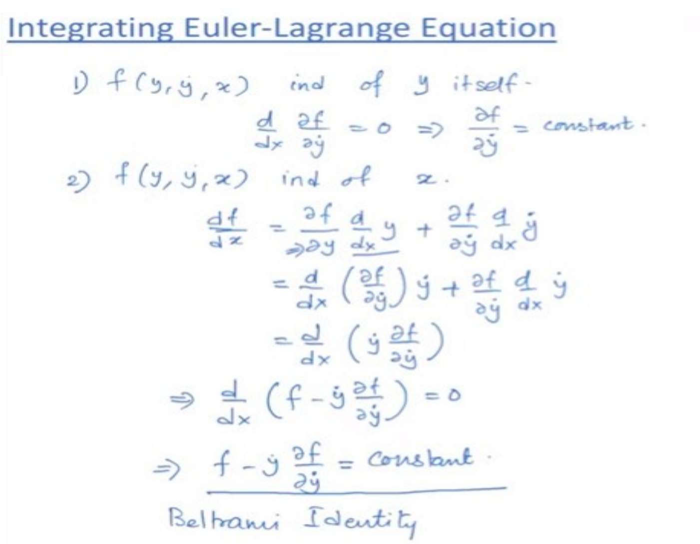
I you see here remember this for y naught to be a stationary point this must be equal to 0 and if this is 0 this must be 0 for every Eta x and that immediately tell us that the quantity in the bracket here like this one this must be equal to 0. So del f over del y – d / dx of del f over del y dot must be equal to 0 so for y of x to be stationary point or stationary path of j the condition is this equal here and this is the equation which is called as Euler – Lagrange's equation.

Or you can immediately start noticing the similarity between these and the Lagrange's equation that we derived in the first semester sorry first week and not surprisingly so when we are going to define functional as a action for a mechanical system the integrand is l so all that will happen is that this equation will in fact actually look like Lagrange's equation for a mechanical system okay. So but here of course they are in a variational calculus these equations are called as Euler's Lagrange's equation.

And note that these or this equation here is in fact second ordered differential equation so what we have done here is this just like you know ordinary calculus to find the minimum of the function we actually get a condition in terms of the derivative here we have got of condition for the path which become stationary or at which the functional become stationary the condition is in the form of differential equation okay.

Now because this is second order differential equation there are of course few special cases where integration this equation.

(Refer Slide Time: 34:05)



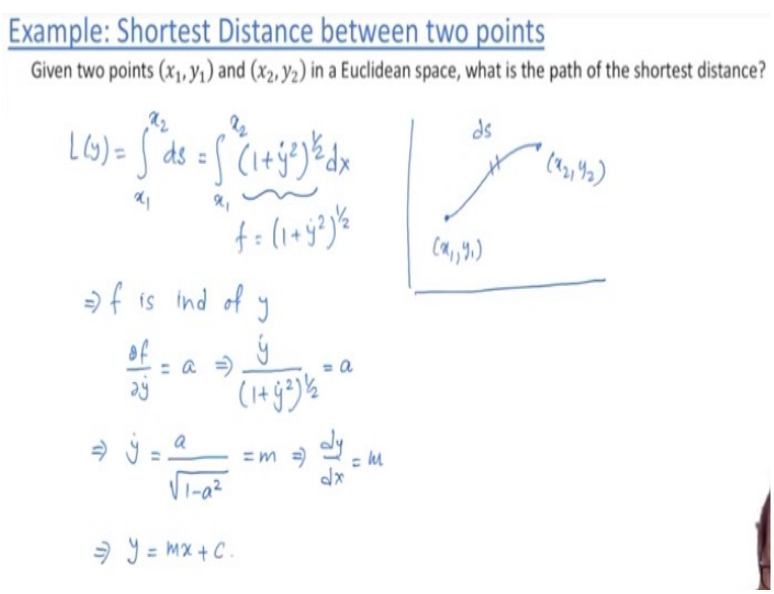
Is actually quite easy one of the two conditions is one condition if your integrant in fact independent of y itself but then from the equation itself we immediately see that d / dx of del f over del y dot then must be equal to 0 on the right hand side you have del f / del y and because f is independent of y this would be the case and this immediately tells us that del f / del y dot must be equal to some constant then this is the first order differential equation this is equivalent to integrating your second order differential equation once okay.

And this quantity then would be called as first integral or the second case where thing are easy is when your function integrant f is independent of x okay. If this is independent of x then clearly d / df / dx remember this is the total derivative of f with respect to x at the stationary point will be equal to partial derivative of f with respect to y into d / dx of y + del f over del y dot into d / dx of y dot and because it is independent of x the partial derivative of f with respect to x is 0.

Now for this stationary path I immediately know that del f over del y this term is in fact nothing but d / dx of del f over del y dot and then you have y dot this is y dot and + del f over del y dot into d / dx of y dot and you now see that this just looks like the product formula for derivatives and we will write it as d / dx of y dot and this gives us d / dx of f – y dot into del f / del y dot = 0 that immediately gives us the first integral of this equation and that would be f – y dot into del f / del y dot must be equal to constant okay.

So there are this two special cases in which actually find or doing one integral is (()) (37:39) and this second case or this identity here is called as Beltrami identity okay. So now we are (()) (37:56) with procedure for finding minimum or maximum or first stationary point of the functional let us look at some of the examples.

(Refer Slide Time: 38:07)



As our first example I will look at the same problem which we had in our classic concrete signed puzzle but there also we needed to find path which was the path of shortest distance. The path of the shortest distance is that from the point A to point B in that example or in that case it being a Euclidean plane we already know the result we know the result the path must be a straight line path and that path will have the shortest distance between the two points.

Now this problem of course can be more difficult if this place is not Euclidean for example on the surface of the earth when you go from Delhi to London the distance measured over the surface of the earth that is not same as straight line path which will go through the earth okay. So there the in case of the non-Euclidean geometries which gives you the shortest distance is called a geo disc.

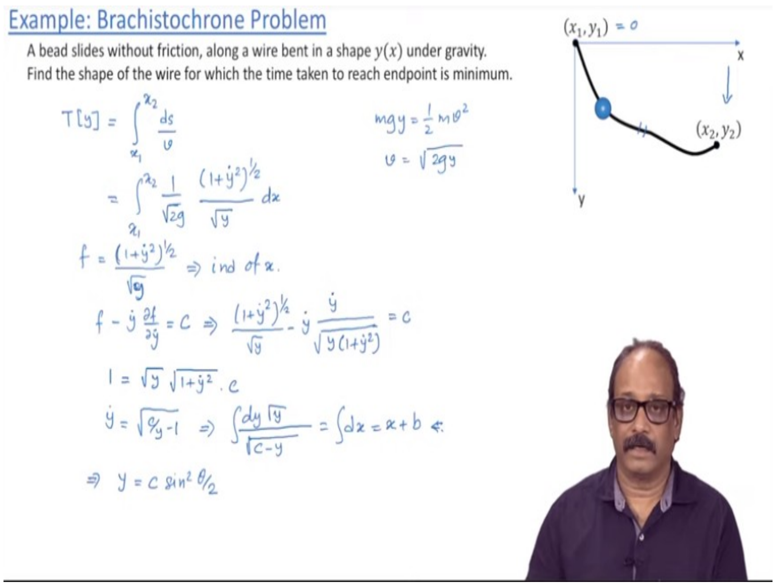
Now I will take this example only for Euclidean space and in the assignment I will ask you to do the example of the earth surface. So here if I take two points this is x1 y1 this is x2 y2 and consider any path this is the infinite decimal paths segment then we know that the total length of the path is integral from x1 to x2 of ds. So basically for all the small path segments together to get the total length and this we will write as x1 to x2 and remember I had written this as 1 over sorry 1 + y dot square to the power half dx.

Now this functional now is in the form that we want and in this case we identify the integrant function f as 1 + y dot square to the power half and which path will give me the shortest length or shortest value for this functional is the path which satisfies the Euler Lagrange's equation and in this case you can immediately see that f is independent of y and in which case we already know that the del f / del y dot must be equal to some constant.

I will call this constant as a and let us calculate this del f / del y dot that is very easy and that would be equal to y dot divided by 1 + y dot square to the power half and this must be equal to a but a this is easily simplified to y dot is equal to so that would be square so that would be a divided by square root of 1 – a square. But that is also constant I will call this constant as m and that immediately gives us dy / dx must be equal to m.

This is the first integral do one more integral and that immediately gives us y must be equal to ms + that so it is a equation of a straight line. A straight line would give me the shortest distance path now how do I find m and c there are two unknowns in the integration but I also have two conditions that y at x1 must be equal to y1 and y at x2 is y2 okay. So I can immediately find out the two constants there.

(Refer Slide Time: 42:55)



Now let me go to a second but little more involved example and this is also the famous Brachistochrone problem which originally (()) (43:04) has posed as a open challenge to the scientist that type now in this problem what we have is a wire which is bent in some shape okay and the shape is rigid. And then there is b which frictionlessly slides over this wire and what we are going to do is we are going to drop it from x1 y1 with 0 velocity and under the gravity under gravity this simply would fall down okay and as if falls down it gains the velocity and it reaches the other point after some time t.

And that time t we can immediately write down as t as a function of path y is integral from x1 to x2 and it is ds divided by velocity. So you take a small infinite decimal lines segment here at xy and then divided by the velocity at that point that gives you time dt to cover the small distance ds and then we can get the after integrating we get the total time. Now what is the velocity v here but if it reaches the height y then of course we know that mgy must be equal to half mv square because from the top point we had simply dropped it with 0 velocity okay.

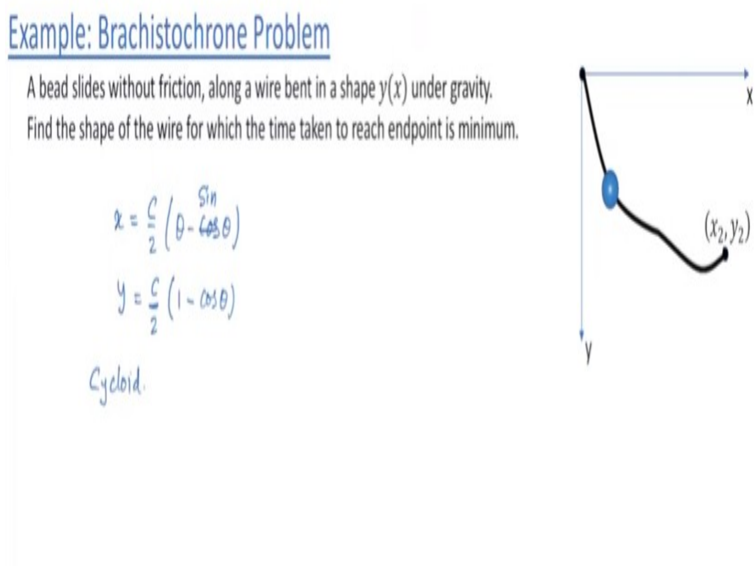
And I am going to assume y1 to be equal to 0 hmm otherwise I would have to write y1 – y here okay. So this gives us v must be equal to square root 2g times y let us put it back in this and then we get integral from x1 to x2 and then you have 1 over square root 2g which is only just a number which we can forget about but in the numerator I have 1 + y dot square to the power half in the denominator I have square root y dx.

So now we can identify our integrate instead of minimizing t I will minimize square root to g times t which is the same thing again the same path will be the minimum for both cases. So here f the integrate function is in fact 1 + y dot square divided by square root y and you can immediately see this one here is independent of x which means I can do one integration immediately using Beltrami identity.

So let us write that down so what we have is f – y dot into del f / del y dot this must be equal to sum constant c okay and this gives us so it is 1 + y dot square to the power half divided by square root y – y dot and I will immediately take the derivative here so that would be y dot divided by square root of into 1 + y dot square okay and this must be equal to constant c okay. So we can of course immediately simplify this will give us 1 must be equal to we can do this calculation at home into square root of 1 + y dot square into c okay.

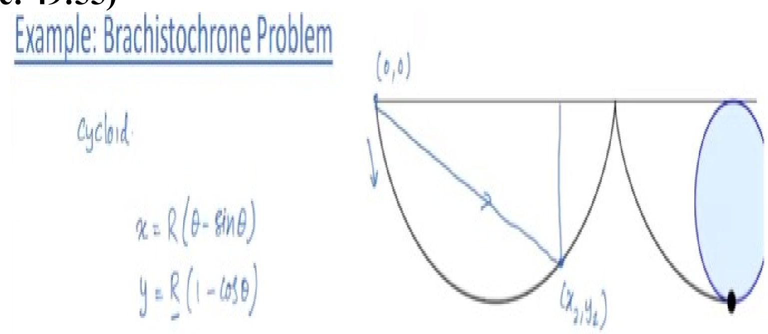
And then square both sides rearrange the terms and once you rearrange the terms you can immediately get y dot = square root of c divided by y – 1 or dy divided by square root y divided by square root c – y integral and this must be equal to integral of dx and I will write it as x + b okay this is the system that we want to solve. Now the integral on the left side this one here it is not very difficult to calculate I am going to again ask you people to do algebra at home but I will give you the substitution.

(Refer Slide Time: 48:58)



So if you put y = c times sin square theta / 2 and carry out the integration in which case you get x = constant c / 2 into theta – Cos theta and y remember was c times sin square theta / 2 but I can also write that in terms of so y would be = c / 2 1 – Cos theta sorry in the sin theta and what is this? This you can recognize is the equation of a cycloid and what does cycloid look like?

(Refer Slide Time: 49:55)



Cycloid looks like this looks like this if I take a disc rolling on a surface and you mark a point on the circumference and as it rolls the point on the circumference will trace a path in two dimensional space like this one here.

(Video starts: 50:15)

So here you can see that the black dot there on the circumference as the disk rolls will form a curve and this curve is called as cycloid and the solution here is in the form of cycloid.

(Video Ends: 50:38)

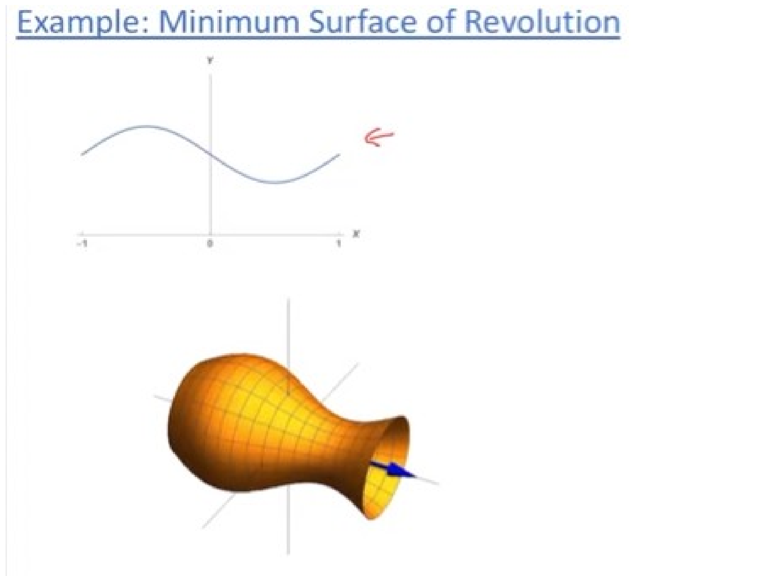
So if I start dropping the bead at x = 0 and y = 0 and I want that particular bead to reach this point here which is x2 y2 then I will rewrite the solutions x remember is constant I will write c / 2 as R into theta – sin theta and y which is r times 1 – Cos theta okay in this equations I have already used the fact that the initial value of x that is x1 is 0 y1 is 0 which also we have already used.

Now in the solutions there is one unknown constant which is R and we have one condition here at this point which can be used to calculate the value of R okay. Now you can see that quite clearly now if I had a wire in this shape this particular case the bead will actually travel below y2 turn back upwards and go to y2 and this still would be the shorter path sorry shorter time path compare to the straight line path.

Straight line path remember again is the shortest distance path but that is not the case here and that is very obvious here the gain in the speed is much slower than the gain in the speed in this case and in which case of course the ball travels the distance much faster than the shortest distance path. Okay after this I will do one more example and that is again a very surprising and nice example but that is also tells you that this particular method when we were deriving the condition for extremum values of functional we automatically assume that the path are continuous path.

What if the path is not continuous? In which case we actually do not get the solution that we what and that is the next example which is minimum surface area of surface of revolution.

(Refer Slide Time: 53:23)



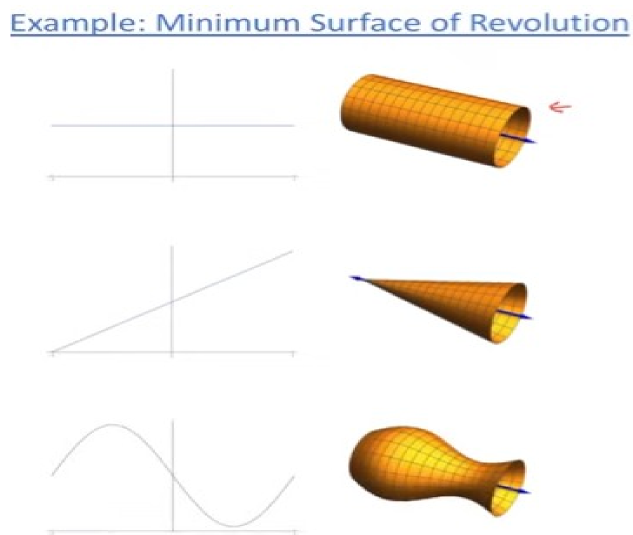
In this example what we want to find is the minimum surface or minimum area surface of revolution. By surface of revolution what I mean is this suppose you are given a curve like this in two dimensions and then you take this curve and turn it above x axis to make a 3 dimension figure and then you get a surface and we want the surface area of that figure. So when I rotate that figure it would like this.

(Video Starts: 54:01)

So in this case the shape that this curve generates is that of a some kind of flower pot.

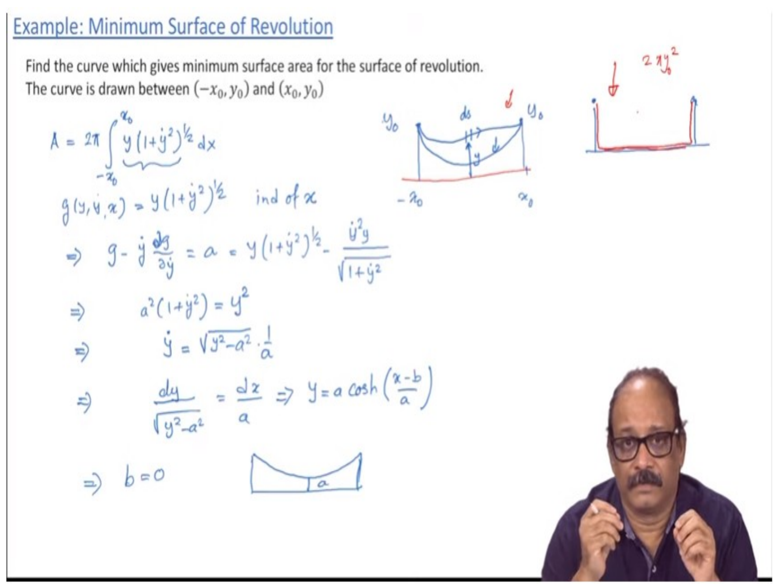
(Video Starts: 54:12)

(Refer Slide Time: 54:14)



Now if you change the curve you will get a different surface for example if I just take a straight line then I would get a cylinder if this straight line is inclined and one of its point touches the x axis then you will get a cone okay and the third figure that we have already seen is you can generate you know some kind of flower pot or vase.

(Refer Slide Time: 54:45)



Now the problem posed is like this. So we have x axis and on this x axis we have this two point – x naught and + x naught and will fix these height to y naught here and we want to find the curve between these two okay such that when I construct a surface of revolution it is area should be minimum so what would be that curve is the question. Now this is a we can start by you know instinctively thinking about the solution. One of the solution is of course the surface area will be length of the curve at one point.

Say if I take a small elemental curve ds which is at a height of y then it is that particular ds area would turn around and make a strip and the area of that strip will be 2 / y times ds. Now what must happen to get a lower area is if I start with a cylindrical solution. That means I just joint – x naught y naught and + x naught y naught by a straight line I would get a cylinder. But we know if I sort of you know curve that cylinder inward then area is likely to draw but then if I take so if I take a curve like this the area is likely to be smaller than that of the cylinder.

But I should not go too far below because as I go down on this curve the length of the curve is also increasing which means there will be some tradeoff between of the curve and the area generated. So probably the solution is somewhere in the middle. Now there is one more solution would be something like this which I will draw separately here so this is your y naught the height and in this solutions what I will do is I will straight away drop down and take this curve.

This curve is passing along the x axis here which means when you rotate this and construct a surface revolution but I will simply get is two disc's on the side on the radius y naught and they connected by extremely thin or cylinder of 0 surface area. And what would be the net surface area in this case that will be each of the disc's on the side will have Pie times y naught square as the area. So this will be 2 times Pie y naught square and see what happens this solutions where you have square graph like this.

This solutions does not depend on the separation between the two point that means it does not depend x naught at all which means if I increase x naught keeping y naught fixed quite obviously this solution is going to win in the end this area of any of these curves will definitely grow with x naught however the area of this curve here does not grow in which case that will remain a minimum surface fx naught is very large.

Now let us try finding the solution here so what is the area of the curve area of the surface so area of the surface will be given by 2Pie integral from – x naught to + x naught and remember the area of the strip was 2 Pie y times ds and by now we know how to write ds that is nothing but 1 + y dot square to the power half dx and this immediately allow us to identify the integral so the integrant function which is function of y, y dot and x y dot and x this = y times 1 + y dot square to the power half.

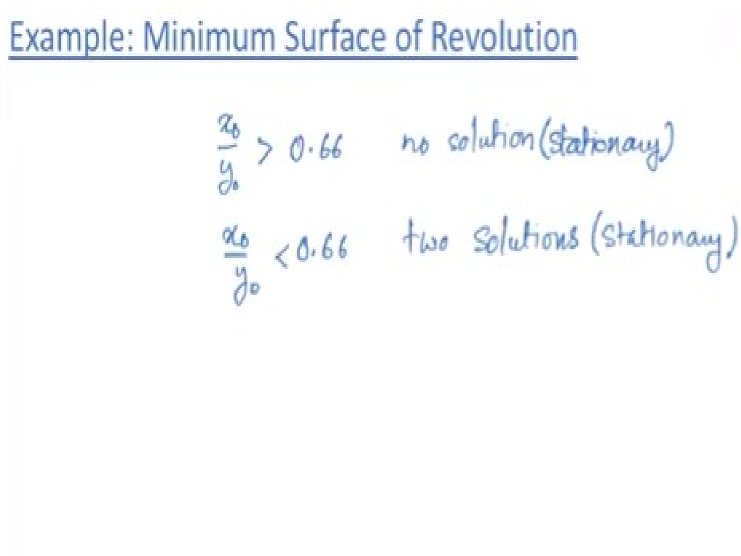
And you can immediately see that this independent of x which means we can immediately use Beltrami identity and that means this gives g – y dot times dg / dy naught this must be equals to some constants and I will put that constant as a. And we can do this calculation so this will be y times 1+ y dot square to the power half minus this will be y dot square into y divided by square root of y dt square and of course you can immediately simply this and the simplification would be this would be a square times 1 + y dot square must be equal to.

So here y times y dot square will get cancelled and what you are left with is just y okay so that would become y square okay. And this gives us so y dot = square root of y square – a square times 1 over a and we will write this as so this is dy / square of y square – a square and this must be equal to dx / a and this immediately gives us solution and that solution is. So y must be equal to a times Cos h into x – b / a okay.

This solution is very clear here the integration is straight found now if you look at initial conditions we immediately see that b must be clearly 0 because the diagram is very symmetric it is from – x naught to + x naught so the diagram will be symmetric. So what kind of solutions are possible if you remember the curve for Cos h the curve for Cos h looks like this it goes through a minimum and something like this and x = 0 Cos h is 0 which means this value here must be equal to a okay so its drops down.

Now from here what do we do one of the constant d = 0 that is already been decided because there is a symmetric in the figure. The second constant a we will decide by putting the condition of one of the end points now it terms out I will include this in assignment for you people to do. But it turns out that for x naught and y naught for not all values of x naught and y naught you have a solutions.

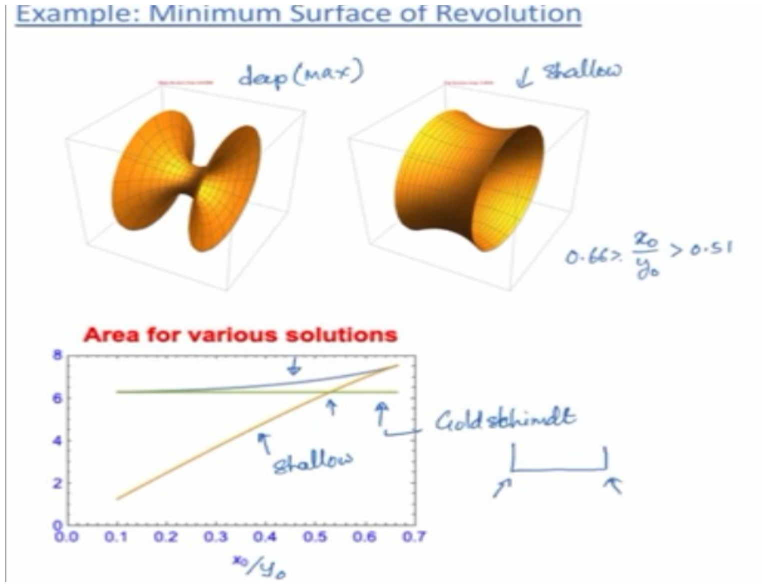
(Refer Slide Time: 10:03:21)



So you have no solution for so if x naught / y naught is greater than roughly 0.66 then there is no solution that means all the curves of course exist but there is no curve at which area is stationary that means if you move that curve a little bit then the area does not change. So there is no stationary solution if you want you can make that very clear here no stationary solution and for x naught / y naught less than 0.66 it turns out at that critical value you will have one solution but as soon as you fall below this critical value of 0.66 then it two solutions okay.

So two stationary solutions which one of them is minimum and which one them is maximum we have to decide by actually calculating either the second derivative or looking at the variation about these stationary points so I will describe these two solutions here and I will ask you people to work these out in the assignment.

(Refer Slide Time: 1:04:48)



So the solution look like this there are two solutions both are causage time that means both of them have this depth one I will call it has shallow solution and the other one is deep solution and you can probably just by looking at the figure it is very easy to see that the shallow figure as a minimum and deep is actually maximum there. And if you look at the comparative solution for all three of them so what I have done here is plotted the area for each one of those solutions at x naught / y naught = so x naught / y naught is plotted on x axis and on the y axis you have area and you can see that the shallow solution actually is the minimum solution.

The deep solution this one is in fact the area is higher and the green line which you see there this one is the Goldsmith sorry solution which we earlier considered this is basically the area as you change x naught / y naught does not change and it turns out that somewhere near 0.5 for x naught / y naught greater than 0.5 I think 0.51 and of course less than 0.66. The Goldsmith solution actually gives the minimum area and the question we are going as is this why is this that our calculation did not give us the Goldsmith solution.

The answer is very clear remember while deriving the or doing the variational calculus we implicitly assume that the paths were not just continuous but differentiable in fact we say sufficiently differentiable. So this solution here Goldsmith solution is not differentiable at this point so these solution of course are not covered by the variational calculus where we assume the path to be differentiable okay.

So these are the three examples we have seen now of course we will return to the mechanical system and look at the Hamilton's principle and from Hamilton's principle we want to show that we can derive Lagrange's equation which means these two formalisms would be equivalent to each other okay. So in Hamilton's principle or before going to Hamilton's principle I will generalize this variational calculus too many variable many dependent variables.

(Refer Slide Time: 01:07:57)

Extending the Functional to many variables

$$J[y_1, y_2, \dots y_n] = \int_{t_1}^{t_2} f(y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n, t)\, dt$$

$$y_i = y_i + \epsilon \eta_i \quad , \quad \dot{y}_i = \dot{y}_i + \epsilon \dot{\eta}_i$$

$$\delta y_i = \epsilon \eta_i \quad , \quad \delta \dot{y}_i = \epsilon \dot{\eta}_i$$

$$\delta J = \int_{t_1}^{t_2} \epsilon \left[\sum_i \left(\frac{\partial f}{\partial y_i} \eta_i + \frac{\partial f}{\partial \dot{y}_i} \frac{d}{dt} \eta_i \right) \right] dt$$

$$= \epsilon \int_{t_1}^{t_2} \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}_i} \right) \right) \eta_i \, dt = 0$$

$$\forall \ \eta_i$$

$$\Rightarrow \text{for each } i, \quad \frac{\partial f}{\partial y_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}_i} \right) = 0$$

$$J = \text{Action} \quad \text{and} \quad f = L(q, \dot{q}, t)$$

$$\forall\ i \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \quad \Big\} \text{ Lagrange' Eqns.}$$

So if you have a functional J which now depends on several yn and this is integral from t1 to t2 but in integrant is now function of y1 up to yn then y1 dot upto yn dot and t dt. So here of course I immediately I have made some changes instead of x now I am writing independent variable as t okay. And this is we want to find the extremum of this functional the trick that we do is exactly same as we did in case of the one variable.

So for each variable yi I will make a small variation which is Epsilon times Eta i remember for each variable i I now have a new function Eta i okay and each of this Eta i can be different from each other okay. So yi dot will be yi dot + epsilon times Eta i dot. So clearly delta yi is epsilon times Eta i and delta yi dot will be equal to epsilon times Eta i dot. And let us calculate the variation in the functional so delta J now = integral from t1 to t2 and what we go what we put here is a delta f and delta f now can be written as.

So summation over i then partial derivative of f with respect to yi into delta yi but delta yi is nothing but Epsilon times Eta i .So I will put that epsilon outside and + partial derivation of f with respect to yi dot into d/ dt of Eta i dt. And we of course will use the same trick again integrate by paths the second terms and remember each of this Eta i must be 0 at the end point which means when you do the integration by parts the second term and remember each of this Eta i must be 0 at the end point which means when you do integration by part you get a surface term and this surface term will be 0 because you have Eta which is individual Eta i which is 0 at the boundary points.

So this again will simple come out to epsilon times integral from t1 to t2 and then you have summation over i this will be there del yi – d/dt of del yi dot times eta i dt okay. Now this must be 0 for every arbitrary eta i for every eta i and not only that eta 1, eta 2, eta 3 these are all independently or can be independently chosen. So one of the things we can do is in fact choose eta 1 = 1 and all the remaining eta's to be 0 and then the simple calculus tells us that for each i del i over del f over del yi – d/dt yi dot must be equal to 0 may be go.

So in case of Hamilton's principle all that is going to happen is J is nothing but action and the integrant function f is in fact the Lagrangian and we write this as function of q,q dot and t which is the short form for q1, q2 and so on. In which case for each i I get del L over del qi – d/dt qi dot must be 0 and these are nothing but the Lagrangian's equations okay. So what we have shown here is that the Hamilton's principle implies Lagrange's equations.

So the statement of the Hamilton's principle of course we will modify a level instead of saying the principle of least action the principle now modifies that the natural system or a mechanical system would choose a path which is stationary instead of saying minimum stationary with respect to the action okay. So what we have seen here is that Hamilton's principle implies Lagrange's equation we also have to do the inverse proof which is starting from Lagrange's equation we have to prove the Hamilton's principle.

That of course I will leave it for you people to do now what I will do is form this point on wards we want to see whether the Lagrange's equation give us the conservation loss which we obtain from Newton's N mechanics or Newton's equations. But see the Lagrangian's equation in fact have lot more information in it is you can the best thing that can happen with Lagrange's equations is you can do arbitrary coordinate transformations. And the general form of Lagrange's equation remains same and this is what we go going to look at in the next section.