Theoretical Mechanics
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Module No # 02
Lecture No # 06
Hamilton's Principle

Okay welcome to the second week of this course in the first week we learnt about we learnt new ideas like constraints generalize coordinate configuration spaces and so on. And then final we use the D'Alembert's principle to derive Lagrange's equation.

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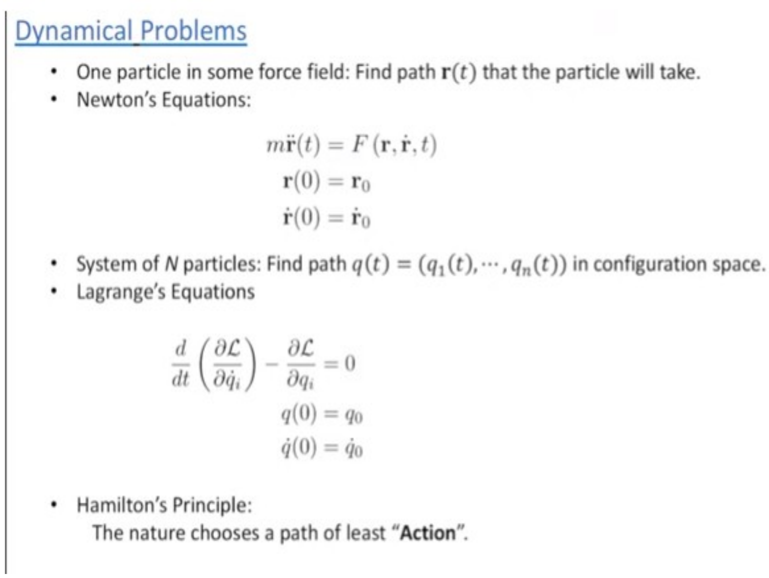

So that basically summarizes the program of classical mechanics what I will do is first post the dynamical problem here. See even if you are considering single particle what is that you wanted to? If we know all the forces on the particle then if we know it is initial position and it is initial velocity then we want to predict the path it is going to take okay. And then of all the path that it is possibly choose which path it will take?

It will take one of those paths which satisfies the Newton's Law's okay which is given as a form of differential equation which is mr double dot = force which may be function of r, r dot and tn so on. Even if you take more complicated system with all the constraint thrown in with potentials given and so on. Even then the problem really reduces to now you are going to describe the system in terms of a path in a configuration space okay.

So put everything together constraints then you find out the accessible configuration space and then your system as a whole is just one point in that configuration space. And if we know the initial condition that is if we know q and q dot at some instant of time then we want to predict the path of the system in the configuration space at some later time t. And which path would it take again you have this Lagrange's equations which are second order equations and you would say that path take by the system is the one which satisfies the Lagrange's equations.

Now you see for us people who believe in nature it is slightly you know uncomfortable feeling that it so technical definition of which path the system is going to take it is very mathematical definition. It is does not seems the set of you know a nice idea remember it is a feel it is a nice idea about what nature is going to do and that is where the Hamilton's principle come in. And the Hamilton's principle this very elegant principle that it says that the nature chooses that path which has a least action.

So of course we will have to make a technical definition of action but there is a elegance in this statement it sounds nice. But of course underneath there is mathematical definition of course but it is such a simple statement and then in the course of this week we will show that the Hamilton's principle is also equivalent to Lagrange's equation or Newtonian formulations that. So this is our program for this week.

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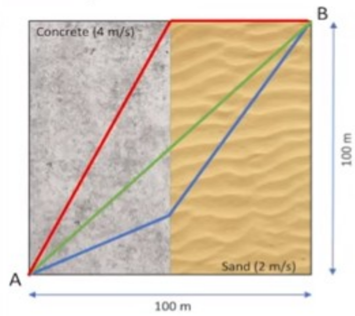
• Example: A Puzzle



Aim: Go from A to B in shortest time.

Path	Time	Distance
Green	53.03 s	141.42 m
Red	52.95 s	161.80 m

Now I will start the introduction with classic puzzle okay look at this diagram here. In this diagram it is like a big field which is 100 meters by 100 meters and on the left side of that field is the concrete and on the right hand side on the right half there is sand and there are these diagonal points which have marked as A and B and the aim is to go from A to B in shortest possible time okay. What is the catch? Catches us.

Concrete is the nice hard surface you can run on it with a speed which is 4 meters per second but on sand you cannot run as fast as that. So for example I have taken that to be 2 meters per second okay now which path is the question? The path that would take the least amount of time now as far as the concrete is concerned we can immediately figure out how to do this? Because in the concrete region the time taken would be the length of the path divided by the velocity which is fixed.

So basically we should be running in a straight lines on concrete and in straight line on the sand so if you do not know the answer you probably want to pass the video here think about the answer and then start again. So here what we must do is run in the concrete in straight line run in the sand in straight line. How about a path like this? A path must be somewhat like this of course is not a pretty good example.

How about the diagonal path? The diagonal path is the shortest path but since we can run faster in concrete it is common sense that we probably should long longer distance or as much distance in concrete as possible to get the shorter time. So probably we might think of a path like this in this path you would run all the way from A to the upper corner of the interface and then run parallel to point B.

Now here is the comparisons of these paths so the green path which is diagonal which is shortest about 141 meters it takes about 53 seconds but wait the red path is about 162 meters but it takes tiny bit less amount of time see we are having much longer distance but still you would end up taking less time okay. So the question is how do I find? Is there is path somewhere in between the green path and red path which probably is the shortest path, shortest time path

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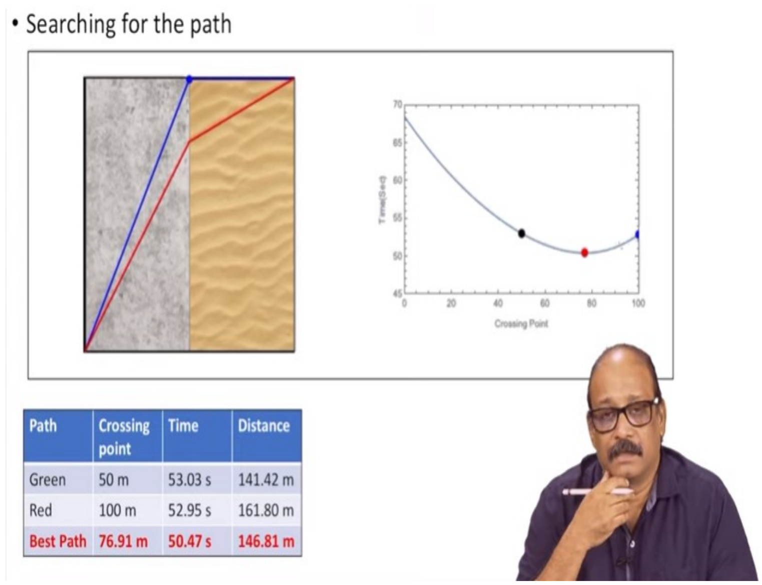

So how do we find this? One of the ways you can do is simply search for the path now we have already figured out you know what is the nature of the path so what I will do is there is a point of crossing from concrete to sand and let us say bottom of the picture to the crossing point that distance actually defines the path completely okay where you can do cross because once you fix the crossing point A to that crossing point is straight line and crossing point to B is another straight line okay.

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So here what I have done in this one is you can actually change the path as it function of crossing point here so on the right hand of the graph you have crossing point and we have just crossed the midway mark and there you go.

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The time taken which is plotted on this graph on the right hand side immediately shows that between that diagonal path and the extreme path on the where you have run all the way to the top of the figure somewhere there at about 76.91 meters you have a best time path and which takes about 50.4 seconds 50.5 seconds and the distance run is of course larger than the diagonal path but this is the one we wanted to find okay.

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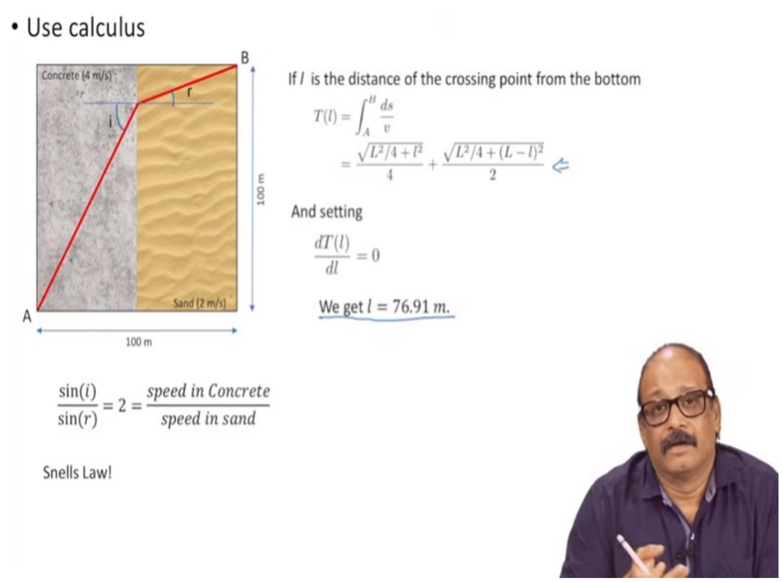

Now there is another way use calculus see since we already have figured out the nature of the path calculating the time for this path is very easy. So the path can easily be given by an integral of ds over v where ds is small length element along the path and then because the velocity is on concrete side is fixed velocity on sand side is fixed the formula can be immediately integrated to something like this.

So here is the time now all we want to do is find a path or a crossing point where this particular time would be minimal so all that we do is take a derivative of this with respect to l that is the distance of the crossing point and set it equal to 0 that immediately gives you value of l as 76.91 meters okay. So this is the but something more to this problem which I have wanted to show you and that is.

If I draw a perpendicular the interface at the crossing point and then I mark the angle here this one as i and on the sand side I will mark it as r and then you calculate sin i / sin r that is equal to 2 right that is just the speed in concrete divided by speed in sand now this is just like the Snell's law what is the connection here? The connection is of course the famous Fermat's principle.

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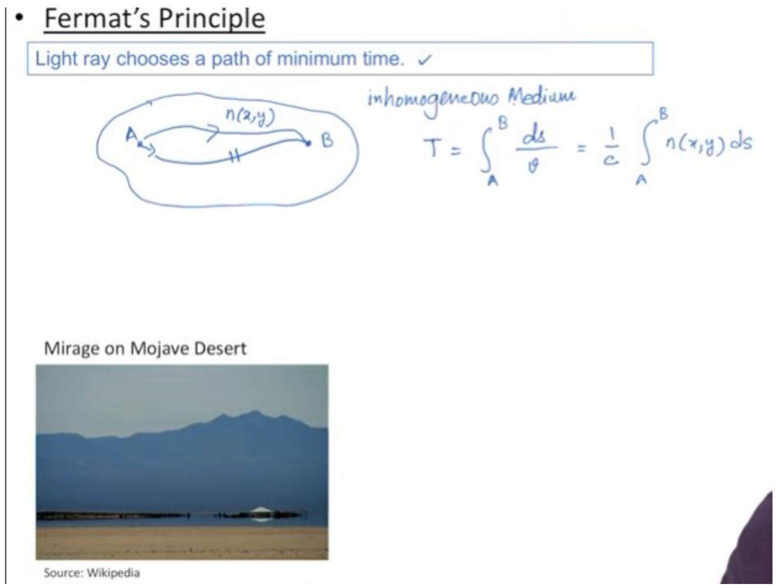

Now Fermat's principle for the light says that this is geometrical optics that the light ray would take such a path between two points which takes the minimum time or which takes the shortest amount of time. So put it more formally what I will do is let us take one say some region. In this region I have these two points okay and I want to find out which path if one of the point emits ray which path this ray would take to reach point B the catch is of course that the refractive index here is probably function of the space it is the inhomogeneous medium.

So in the inhomogeneous medium and then what happens to the velocity at each point? At each point velocity of light would be different and will be given by c / n okay. So what we calculate is first of all you take any arbitrary path between these two points and look at the small length element at some point and your total time taken for this path is integral from point A to point B of the small line segment divided by the velocity of light at that point.

And we can of course readjust this little and write this as 1 over c integral from A to B n of course remember this function of the point ds okay. At what Fermat's principle says is that of all path that connects point A and point B so they could be infinitely many paths which would connects A and B only that path will be taken by the light for which the time taken is minimum okay. So this is what the Fermat principle says.

And this can be of course used to explain the variety of phenomena there and one of the common phenomena that people usually show is that of the Mirage. What happens in Mirage is that near the ground the air is warm so the refractive index is higher and hence the speed is speed of the light is faster near the ground than above the air. So what would happen is that the ray while it passes from the source to the observer it actually goes through slump like this because it tries to travel more and more distance near the ground rather than.

And then of course the observer would interpret as if the light rays are coming from the ground and he sees the reflection of the object even though there is no water or there is no reflecting surface. Okay the question that we want to ask is this now if there is a Fermat's principle for light and remember the light is in principle that is not covered under the classical mechanism.

If there is the Fermat's principle for light why cannot I have such a principle or what is that principle which would apply to the classical systems or classical mechanics.

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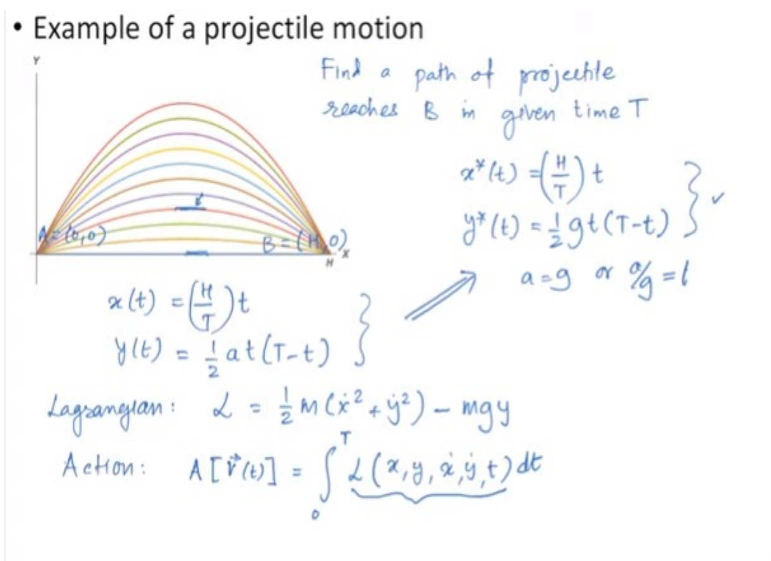

And for that I will take one more example now look at this example of a projectile motion. Now in this projectile motion a particle is thrown from point A whose coordinates are 0, 0 to the point B here whose coordinates of point B are H 0 okay. Now I want to post this problem slightly differently. So your horizontal range is fixed you want to throw a projectile from A to B not just the horizontal range but the question is to find path of projectile such that it reaches B in given time and I will say T okay.

So we want to find out a path which so normally the problems are post slightly differently see normally you would be given the initial point and the initial velocity which means the speed and the angle of inclination or angle of projection and then you are ask to find where is lands and after what time. Here we are posing the problem as boundary value problem I have already given you the end points and I have also said find a path which takes exactly that much amount of time.

Now we of course we already know the answer to this you can immediately put it in we know the answer is parabolic path and so on and the answer we already know and I will call this path as x star t which is the times t so this is and y star t will be equal to half gt times T – t you can immediately verify that this is the correct equation or correct path for the projector and also gives the time dependence.

So at t = 0 the y coordinate will be 0 at t = T also y coordinate will be 0 and the at small t = 0 x = 0 and at t = T x star will be equal to H. So this is the correct path that we know okay now what is special about this path is the question we are trying to ask and that is what is going to be your Hamilton's principle. So what is special what I am going to do is first of all I will look at few other parts which are nearby okay.

So let me define new paths which are x of t which is equal to H / T times t and y of t which is half but instead of g I will put some other number there a okay times t times T – t what is this give you? This of course gives you lots and lots of parabolic path all these paths remember start at 0, 0 end at point B which is H, 0 in time T okay. But they are different parabola so basically the heights at the midpoint will be different depending on the value of a.

And we know only one of those path is correct paths that is correct path is this when a = g or a/g = 1 okay this is the correct path. Now let me define since we know the Lagrangian of this system. So let me write first of all Lagrangian of the system and the Lagrangian L will be equal to square and minus so that would become + mg times y so sorry – mg y. And I am of course we can now apply Lagrange's equation and get that equation.

But I will do I will define a new quantity called as action so definition of action is this so for every path from point A to point B whatever that path may we have chosen only a small family of those path which are only parabolas but any possible between A and B which travers says in time T we will assign action A which of course is function of the path of the path okay. So r of t is nothing but x of t and y of t will define this as integral from 0 to T L.

Remember L is function of x, y, x dot, y dot and t integrate with it so will happen is that this integral is of course easy to do I have already given you x as a function of t y as a function of T so we can calculate x dot y dot also as functions of t. So the whole of this can be expressed as a function of T and then you can integrate okay and I will leave that exercise to you people to do. So after integrating what we get is action A and I want to look at this action A for different paths about the actual path that we have found out which is x star y star.

So if I so these are the various path that you would get if I change the value of a for example the value of a here at the lowest one here a would be 0 and somewhere in between a = g path is there that is the correct path.

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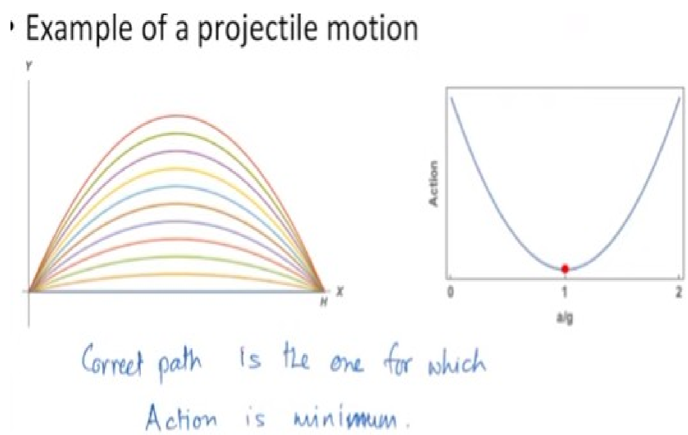

So what I want to do is this for each of this paths I will plot action okay as a function of a / g you can immediately see what happens here at a = g or a / g = 1 the action is minimum see this is what separate this path from the other path and this is in fact the statement of the Hamilton's principle. So the correct path here is the one for which action is minimum there you go okay.

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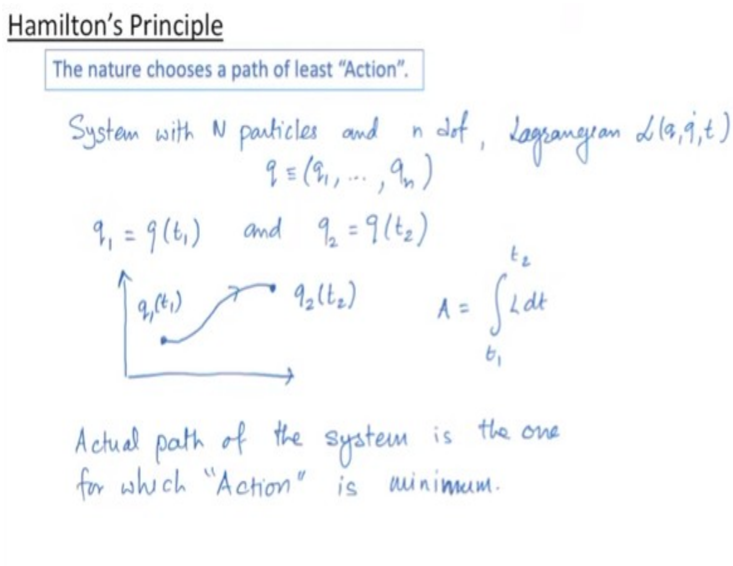

So I will now post the statement of the Hamilton's principle the nature chooses the path of least action but let us make this more accurate okay. So let start by defining consider a system with N particles and N degrees of freedom and you have looked at all the constraints and everything. And then the system is described by Lagrangian which is L and this is of course function of q, q dot and t and remember each of this q is q1 to qn because there are n degrees of freedom.

Now let me post the problem just way I post a projectile problem so let q1 be a point which is q at t1 and let q2 is another point at t2. So in the space okay I cannot draw the configuration space of this schematic diagram in this one I have this two points one point is q1 the other point is q2. The aim is to start from q1 at time t1 and reach q2 at time t2 and then of course there are infinitely many paths which connect the 2 and for each path we assign.

So for each path we assign action which is defined as integral Ldt from t1 to t2 and remember once I give you path I give you the time dependence of all the q variables and from there you can calculate q dot then the Lagrangian is just merely function of t you can do this integral. And then the Hamilton's principle says that the actual path of the system is the one for which action as defined there is minimum.

Now this probable is not the accurate statement in the following sections I will refine this definition it is actually considered as what is called as the stationery point but this is the Hamilton's principle. And what we are going to do in the remaining section is this it is not only this problem so Fermat's problem is 1 which is some sort of least time principle this is a least action principle then I can probably think of you know two points in this space and then ask the question which is the path which as the shortest distance such a least distance path problem.

So this is class of problems which go under the name of variational problems and the mathematics that underlies all this is called as a variational calculus. So in the next few sections that is going is to be our focus to understand the understand how to solve these problems variational problems of this kind and then we will proceed with application to Hamilton's principle.

So in this section we are going to focus on variational calculus now there are set of problem which we are interested in and these problems are called as variational problems they all fall in this category which is basically trying to find paths which have something which is minimized. So in all these problems they would be some fixed points and you would be having paths between them with each path you assign some quantity called as functional and then trying to find a path for which this functional becomes minimum or rather extremum.

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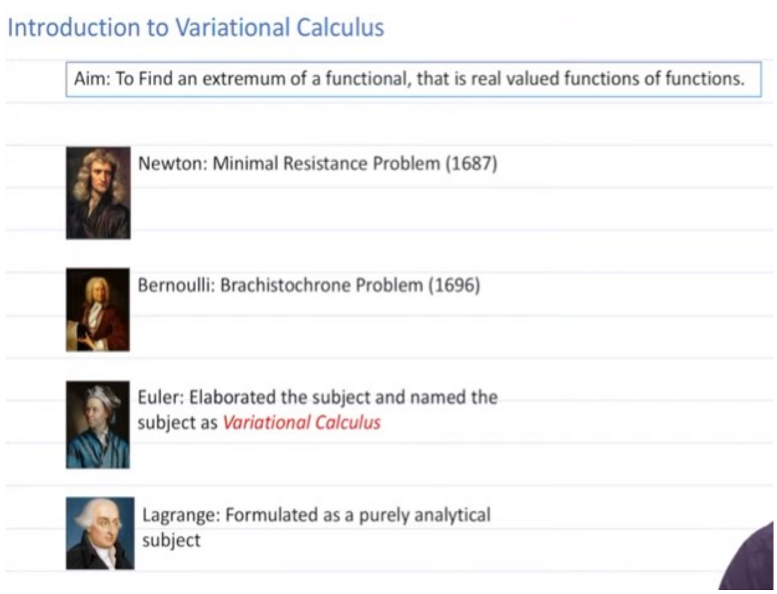

And the history of the variational calculus is also fascinating you should definitely try and read the way it was developed. So it begins with Newton and he was trying work on a problem where you have a solid surface of revolution and the surface of revolution so this particular solid is moving through viscous fluid and what kind of surface would give you minimum resistance.

Something looks like a aerodynamics problem you have a vehicle traveling through or Aeroplane travelling through air and you want to find what should be shape of the nose so that it has the least resistance in the air. The second after that about 10 years later Bernoulli post is famous Brachistochrone problem and he post it as a open challenge and eventually Newton also solved it.

This is very interesting reading and then finally Lagrangian Euler also work extensively on this and Euler is the one who actually coined the name variational calculus.

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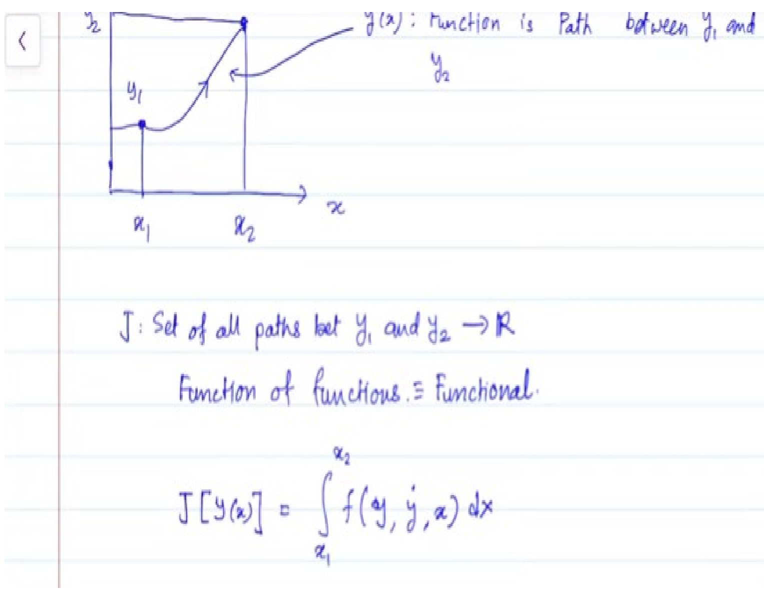

Okay so let us start by defining the notion of functional okay so in this space I am going consider only one dimensional case to begin with. So if you have a space and I will mark this as a y axis and this as x axis and you take these two points here and this one is y1 at x1 and this is y2 sorry for this y2 and x2 okay. Now there will be infinite many paths which join the two points x1 y1 to x2 y2 and all these paths basically are continuous or sufficiently smooth functions of x.

So y the path represented by function which is real valued function it is continuous it is sufficiently smooth this we will call it as this function is called as is path between y1 and y2 remember the x1 x2 is the domain which is fixed and it is given and what we are going to do is with each path we will assign a quantity called as so scalar value J is basically a function from set of all paths between y1 and y2 to real numbers okay.

So we assign scalar real number to each possible path this is in some sense function of functions. So the paths themselves as functions and J is basically assigning to each path some scalar value. So that is why this one as special name so this one is actually called as functional okay and what we are interested in is not any arbitrary functional but very specific kind of functional which appear in all the problems that we describer earlier or which we will discuss later and that is a large class of problems we are interested in.

So I will define this functional which is of very specific kind as a functional which is written as y of x as integral from x1 to x2 of some integrand which I will call it as f and f is of course function of y, y dot here y dot is derivative of y with respect to x commonly we of course write it as y prime and x, dx. So these are the kind of functional that we are interested in. Why does it not depend on y double dot I will leave that question to you that figure that out okay.

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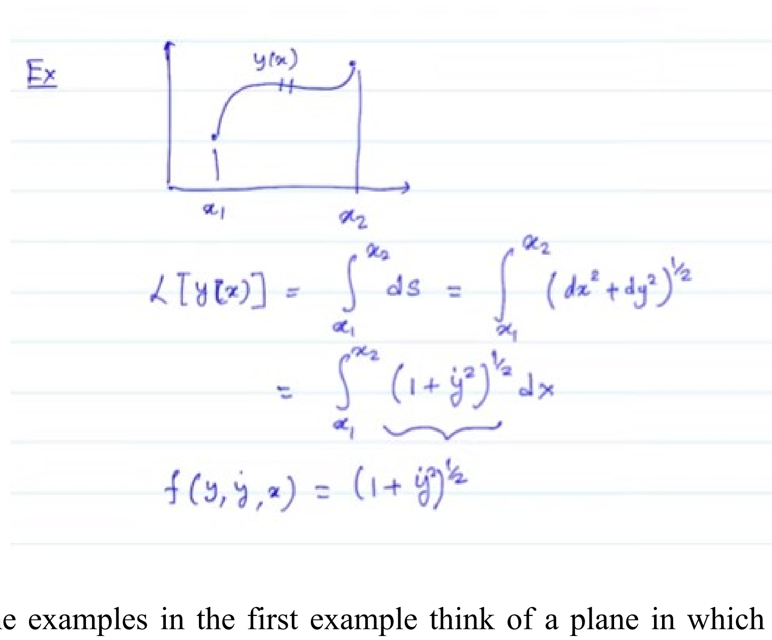

Now here as some examples in the first example think of a plane in which you are given two points and we want to find out which path gives the shortest distance between these two points if it is a Euclidian plane we already know the answer it is a straight path between the two but anyway we will post this problem. So if take this some path between these two points so this is x1 and this is x2 if I take any path suppose y of x I immediately can assign the length of the path as so length of the given path as integral from x1 to x2 and integral over ds.

What is ds? ds is the small element here and assuming that this is a Euclidean plane in which case this would become integral from x1 to x2 of dx square + dy square to the power half and that I will write as integral from x1 to x2 1 + y dot square to the power half dx. And now we can immediately identify the integrand so your length l is the functional of y and that is basically integral of the function. So the integrand function which is function of y, y dot and x here is in fact just 1 + y dot square to the power half okay.

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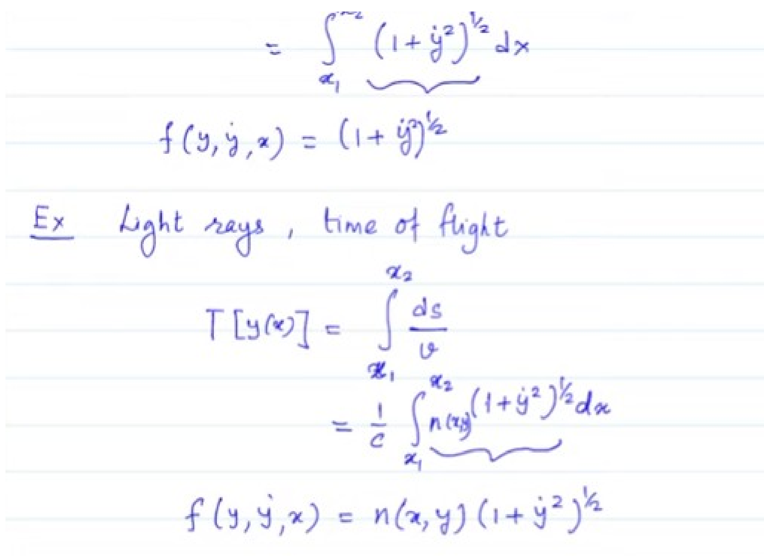

And then the second example us which we have already seen which is if we have light travelling from point x1 y1 to x2 y2 in some inhomogeneous field then we assign with each path for the light rays time of light between x1 y1 to x2 y2 again I will write it as T which is function of path which is given as integral and this is nothing but ds divided by and from some x1 to x2. Now this again I will write it as c so sorry I will write this as 1 over c integral from x1 to x2 of we already know how to write ds that is nothing but 1 + y dot square to the power half dx okay into n which is function of x and y okay.

So here again we can identify the integrand function the integrand function which is function of y, y dot and x is n which may depend on x and y into 1 + y dot square to the power half okay.

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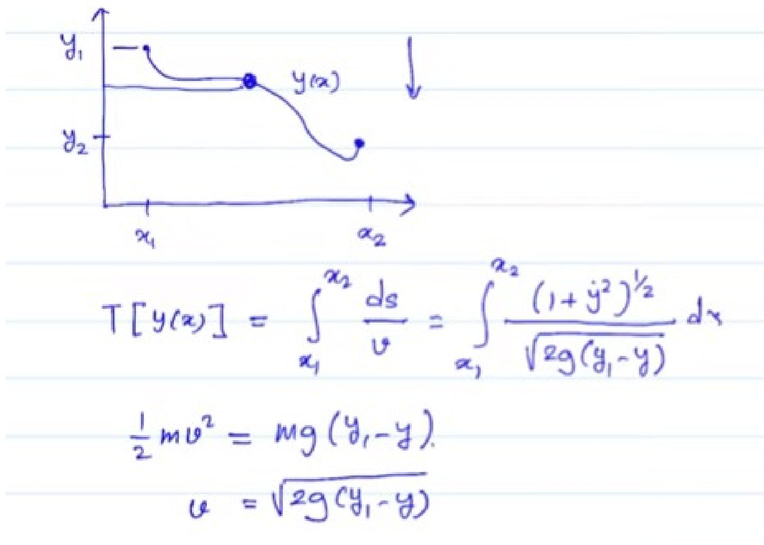

Now one more example this is the famous Brachistochrone problem so think of a situation where you have the vertical direction and horizontal and then you have 1 point here and the other point here and you have some wire which connects two points okay and then there is a bead. So what we are going to do is so there is gravity pointing downwards what we are going to do is this.

From the upper point you drop the bead with 0 velocity and it will slide frictionlessly over the wire the shape of the wire is fixed or is given and then I would it would reach the other point here and question that we want to ask is this. So if this is y1 this one here is y2 this x1 and this is x2 the question we want to ask is this what is that shape of the wire which will give me the minimum time of slide okay.

So let us pose this the time taken so if the shape of the wire is given then the time taken is equal to again similar to the previous problem it is integral from x1 to x2 ds divided by velocity v but we know velocity v at each point. So at each point half mv square must be equal to so if it as drop to this point which is y then clearly this must be equal to mg times y1 – y and that immediately gives you velocity v which is nothing but square root of 2 times g times y1 – y okay.

And then I will put it back into this equation so this becomes we already know how to represent ds so this is x1 to x2 ds is nothing but 1 + y dot square to the power half and divided by the velocity now is square root 2g times y1 – y dx and in this case the integrand or integrand function is 1 + y dot square to the power half y1 – y. And lastly among the class of this problem we already have the Hamilton’s principle.

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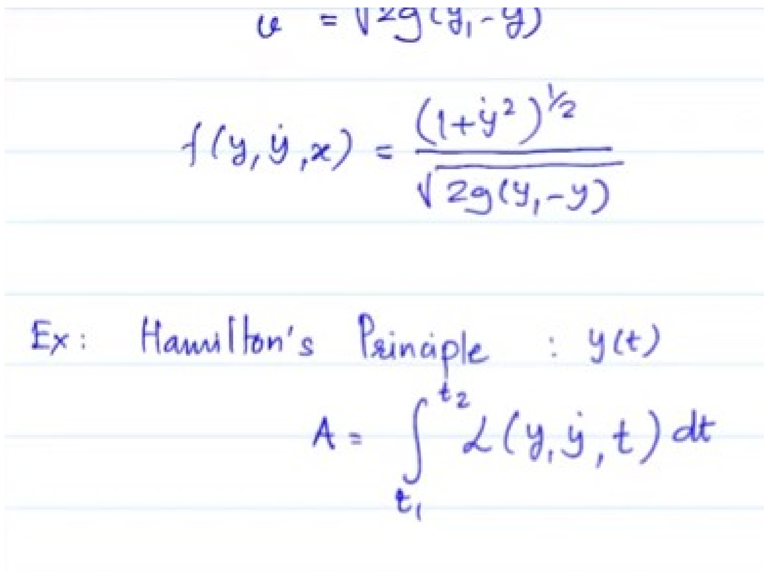

So the Hamiltons principle or Hamilton’s statement is in fact exactly in the form of functional so here the action in fact is already defined as t1 to t2 of which depends on y, y dot and t dt remember here the roll of x will be played by time and y dot here is basically dy/ dt and y of course it is posed as function of time t and this is the path that we are assigning the action to. So here this is how a functional is assigned to paths and all these problems what we want to find out is how to extremize? How to minimize the functional or you know fondly physicist always say action. For every kind of functional that is the focus of our next session.