Theoretical Mechanics

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Lecture - 05
Lagrange's Equations

Okay, after having looked at the three examples of D’Alembert’s principle but if you looked at those examples, they seemed rather lengthy and hard. But you see what happens here. I have this D’Alembert’s principle and the D’Alembert’s principle is written in terms of the Cartesian coordinates. Now the Cartesian coordinates because of constraints are not independent of each other.

So we have to write the entire equation, identify the independent variables, then find the coefficients of the virtual displacements of these coordinates and that finally will be able to give or that finally will give us the equations of motion. But then this procedure immediately tells us that I can now combine the D’Alembert’s principle with generalized coordinates.

Because generalized coordinates we had already identified as independent coordinates which will give us the mapping between independent coordinates and the Cartesian coordinates, okay. So if we put these two together then we get what is called as the Lagrangian equations of motion. And Lagrangian equations of motion will make the procedure of finding equations of motions extremely easy.

We will see that in the examples towards the end. Okay, again this is a lengthy derivation and remember in this derivation we have to always remember that we are working in the state space which means all these quantities are treated as functions of not only coordinates but also the velocities and the time.

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- D'Alembert's Principle:

$$\sum_i \left(\underline{F_i^{(a)}} - \dot{\bar{p}}_i \right) \cdot \delta \bar{r}_i = 0$$

- k holonomic constraints $\Rightarrow$ $3N - k = n$ generalized co-ordinates

$$q = (q_1, \dots, q_n)$$

- transformations $\bar{r}_i = \bar{r}_i(q_1, \dots, q_n, t)$

- Velocities $\frac{d\bar{r}_i}{dt} = \bar{v}_i = \sum_j \frac{\partial \bar{r}_i}{\partial q_j} \cdot \dot{q}_j + \frac{\partial \bar{r}_i}{\partial t}$

$$\Rightarrow \boxed{\frac{\partial \bar{v}_i}{\partial \dot{q}_j} = \frac{\partial \bar{r}_i}{\partial q_j}} \quad \text{—} \ (1)$$

Start with $\frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \sum_l \frac{\partial^2 \bar{r}_i}{\partial q_l \, \partial q_j} \dot{q}_l + \frac{\partial}{\partial t} \frac{\partial \bar{r}_i}{\partial q_j}$

$$= \frac{\partial}{\partial q_j}\left(\sum_l \frac{\partial \bar{r}_i}{\partial q_l} \dot{q}_l + \frac{\partial \bar{r}_i}{\partial t} \right)$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \frac{\partial v_i}{\partial q_j}} \quad \text{—} \ (2)$$

Okay, let me begin by writing down the D'Alembert's principle here which we have already seen and this D'Alembert's principle only talks about the applied forces. And if the system has a holonomic constraints then we know that for the capital N particles we had 3N Cartesian coordinates minus the number of holonomic constraints that is k in this case, 3N – k are the total degrees of freedom for this system.

And I will denote it by small n and what we need is small n generalized coordinates and I will denote those by q 1, q 2 and so on up to q n. And in the successive derivation, many places we may have to write function of q 1, q 2 and q n, up to q n. I will use a short form for it and just write q and if write q that means I am actually talking about the set of generalized coordinates which is q 1 to q n, okay.

Now, we already have the transformations from the generalized coordinates to the Cartesian coordinates. So here are the Cartesian coordinates and they are expressed in terms of the generalized coordinates. So I will call this as transformations. Now from these transformations I can immediately calculate velocity. So velocity of ith particle, remember this is nothing but dr i/dt. So this is velocity of ith particle.

And that we can use the chain rule and then I get velocity equal to partial derivative r i with respect to q j into q j dot and plus del r i by del t. And now you see the positions or Cartesian coordinates were functions of generalized coordinates. But what about the velocity? Velocity is not only the function of coordinates, generalized coordinates but also functions of generalized velocities.

Because in the expression of velocity, generalized velocity appears here. Only thing that is nice about it is it is linear in generalized coordinate. So the velocity of the particle when expressed in terms of generalized velocities it is linear function of generalized velocities. So if I take the partial derivative of V i in this equation with respect to q j dot. But then we can immediately refer back to the previous equation.

And from the previous equation I know that this would be equal to del r i/del q j, okay. Once you take the derivative with respect to q j dot, the coefficient of that term will remain. This identity we will use in future. So I am going to call this as identity number 1.

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$$q = (q_1, \dots, q_n)$$

– transformations $\quad \bar{r}_i = \bar{r}_i(q_1, \dots, q_n, t)$

– Velocities $\quad \frac{d\bar{r}_i}{dt} = \bar{v}_i = \sum_j \frac{\partial \bar{r}_i}{\partial q_j} \cdot \dot{q}_j + \frac{\partial \bar{r}_i}{\partial t}$

$$\Rightarrow \boxed{\frac{\partial \bar{v}_i}{\partial \dot{q}_j} = \frac{\partial \bar{r}_i}{\partial q_j}} \quad \text{— (1)}$$

Start with $\quad \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \sum_l \frac{\partial^2 \bar{r}_i}{\partial q_l \partial q_j} \dot{q}_l + \frac{\partial}{\partial t}\frac{\partial \bar{r}_i}{\partial q_j}$

$$= \frac{\partial}{\partial q_j}\left(\sum_l \frac{\partial \bar{r}_i}{\partial q_l} \dot{q}_l + \frac{\partial \bar{r}_i}{\partial t}\right)$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \frac{\partial V_i}{\partial q_j}} \quad \text{— (2)}$$

Now look at his expression. If I calculate this d/dt of del r i/del q j then what I get is this. First, again you see this is the entire derivation is all about use of chain rule and partial derivatives. So when I take this derivative, what I get is this. I take the derivative of del r i/del q j with respect to q l and into dq l by dt which is just q l dot here, okay.

And one more derivative with respect to partial derivative with respect to time because time appears explicitly. And as long as these transformations are nice in some sense, nice means they are smooth. The easiest thing that we can do is if they are smooth then I can interchange the order of derivatives. That means if I take these two derivatives here, I can take del q j derivative first and then the derivative with respect to time. Same thing about this also, this term too.

So now what I am going to do is I will write everything inside the bracket here which is derivative with respect to q l and derivative with respect to t and derivative with respect to q j I am going to do last. So that comes out of the bracket here. But identify this bracket. What is this bracket here?

This bracket here, oh that is nothing but V i. Just go back one step and there it is. This is exactly same as V i. It is a summation over q l summation over l del r i/del q l into q l dot + del r i/del t and compare it with this equation here. So this is just equal to del V i/del q j, okay. This actually tells you what. If the coordinate transformations are nice, then I can actually interchange the partial derivative with respect to q j with the total derivative with respect to time.

That is exactly what has happened on the right hand side there okay. So that is the identity number 2. Now look at the virtual displacements.

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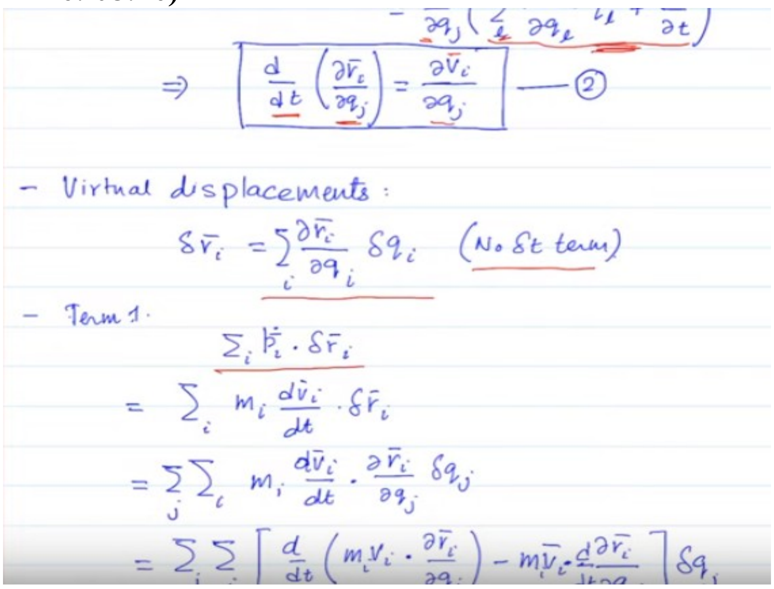

The virtual displacements remember are delta r i is equal again by chain rule it is del r i/del q j into delta q j. And what happened to the time term? Remember, these are virtual displacements. That means these displacements are taken at an instant t so no delta t term for this one, okay. Now go back to the D'Alembert's principle and look at the first term. One of the terms is p i dot into delta r i and then summed over i.

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$$= \sum_j \sum_i m_i \frac{d v_i}{dt} \cdot \frac{\partial r_i}{\partial q_j} \delta q_j$$

$$= \sum_j \sum_i \left[\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} \right) - m_i \bar{v}_i \cdot \frac{d}{dt}\frac{\partial \bar{r}_i}{\partial q_j} \right] \delta q_j$$

$$= \sum_j \sum_i \left[\frac{d}{dt}\left(m_i \bar{v}_i \cdot \frac{\partial \bar{v}_i}{\partial \dot{q}_j} \right) - m_i \bar{v}_i \cdot \frac{\partial \bar{v}_i}{\partial q_j} \right] \delta q_j$$

$$= \sum_j \sum_i \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} m_i v_i^2 \right) \right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2} m_i v_i^2 \right) \right] \delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j \qquad T = \sum_i \frac{1}{2} m_i \bar{v}_i^2$$

- Second term:

$$\sum_i \bar{F}_i \cdot \delta \bar{r}_i$$

$$= \sum_j \left(\sum_i \bar{F}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} \right) \delta q_j$$

Now let us start analyzing this, simplifying this. The first thing I will do is write p i as m times dv/dt okay. Then second step I will take this delta r i and write it as, this is given by the equation here okay. Now what I have here is this. I have m i into derivative of one function with respect to t into second function. So think of this as your second function and v i as your first function and write it as time derivative of m i v i into the second function.

But if I use, if I take a derivative of this, then it would be derivative of v i into the second term plus v i into derivative of the second term. So we can immediately see that you need to subtract the second term too. And if I subtract that second term that is what we get here, okay. So it is m i v i into time derivative of the second term. Now we simplify this. Now identity number 1 goes here.

This part here gets replaced by derivative of v i with respect to q j dot okay. And secondly on this side I have d/dt. This term here gets replaced by the term here. And now you see what happens, something nice. So this one here is m i v i into del v i by del q j dot but this is nothing but del/del q j dot of half m i v i square. And similarly on this side, this term becomes del/del q j of half m i v i square.

But we identify half m i v i square. What is that? That is the kinetic energy of the ith particle. Now if I take this summation inside, I can do that because d/dt and the partial derivatives are linear operators. So I can take this summation inside and then substitute t is equal to sum over i half m i v i square okay. So this term from the D'Alembert's principle simplifies to this. All along there is a delta q j sitting outside the bracket.

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- Second term :

$$\sum_i \bar{F}_i \cdot \delta \bar{r}_i$$

$$= \sum_j \underbrace{\left[\sum_i \bar{F}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} \right]}_{Q_j} \delta q_j$$

$$= \sum_j Q_j \, \delta q_j$$

Generalized Force.

- Finally:

$$\boxed{\sum_j \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0}$$

Hence

$$\boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \qquad j = 1 \dots n.$$

2^{nd} ordered n DE

Now the second term has sum over i F i vector dot delta r i. Now this I will write here again and write down the expression for delta r i. That is nothing but partial derivative of r i with respect to q j into delta q j summed over j, okay. And now if you rearrange then what you get inside this bracket, this bracket here, I will call this as q j. And this q j is called as generalized force, okay.

Now finally what do I get? Put everything together, the first term and the second term and the D'Alembert's principle looks like this. In this case, I have summation over j. Remember the summation over i in the kinetic energy term is over the particles. So that goes from 1 to capital M. This summation here is over generalized coordinates. This goes from 1 to small n.

Remember small n was 3N – k and k was the number of holonomic constraints. So delta q j's are independent of each other. What does that mean? This identity must be true for any arbitrary delta q j immediately implies that each one of these coefficient brackets themselves must be 0. This is how you get these n. So there are n equations here, these n equations, okay.

What is the nature of these equations? These are second order differential equations.

You can immediately see that t will be treated as function of q, q dot and then when you take the derivative of t with respect to q dot I get first of all terms in q dot. And then I have one more time derivative with this. So basically the final equations will involve q double dots. So these are second order n differential equations, okay.

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– If $f_i = -\nabla_i V(r_i, t)$ then

$$Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} = -\sum_i \nabla_i V \frac{\partial r_i}{\partial q_j}$$

$$= -\frac{\partial V}{\partial q_j}$$

and if V is ind of $\dot{q}_j \Rightarrow \partial V / \partial \dot{q}_j = 0$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}[T - V]\right) - \frac{\partial}{\partial q_j}(T - V) = 0$$

Define $\mathcal{L}(q, \dot{q}, t) = T - V$

and Lagrange's Equations, then are

I will take this a little further and look at only a special case. Remember this special case includes large number of problems. So if I can express the force on the ith particle which is f i as minus gradient of, gradient with respect to ith coordinates or coordinates of ith particle of some potential function. So here V is not velocity, but this is capital V. This stands for the potential.

So if our forces can be derived from some scalar potential by using this process, then I can calculate q j immediately; q j will be equal to minus of grad of v i. Remember grad is with respect to i or with respect to x i, y i, z i into del r I by del q j. But by chain rule we immediately know that this is exactly equal to, so this is exactly equal to minus of del v by del q j. So make a second stipulation.

See we made a first stipulation that the forces are derived from some scalar potential. I of course did not say that this is conservative force. This could be function of r i and also time t. Remember if it is independent of time t then this will be conservative force. If it depends on time t then it will not be a conservative force. However, you should be able to define some energy term there, okay.

So first thing we do is that the forces are derived from some scalar potential and second stipulation I will make is that if velocity is independent of q j dot then del v by del q j dot is 0. And then I can now put everything together in my equation.

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Define $\mathcal{L}(q,\dot{q},t) = T - V \Leftarrow$ Lagrangian

and Lagrange's Equations, then are

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0} \quad \underline{n \ 2^{nd} \text{ ordered DE}}$$

So because del over del q j dot, there is a dot here, del over del q j dot of T – V. Remember here the derivative of V with respect to q j dot anyway is going to be zero and minus del over del q j T – V. This one has come from your generalized force term, okay. So I will now define a new quantity called as Lagrangian and this Lagrangian is defined as T – V.

And this of course will be function of all the generalized coordinates q, all the generalized velocities q dot and also may depend on time t. And then for given this Lagrangian I have Lagrangian's equations which are n second ordered differential equations. These are our equations of motion. Rather a length derivation but you see when we looked at the examples of the D'Alembert's principle we anyway manually did this. We started by writing everything in terms of Cartesian coordinates.

Then identified the independent coordinates. Then looked at the coefficients of the independent coordinates and set them to zero. This derivation already has done that. So now equations of motion can be directly obtained by applying this. We do not have to do all that lengthy procedure that we did in the examples of D'Alembert's principle. Okay, immediately we will try this for some examples.

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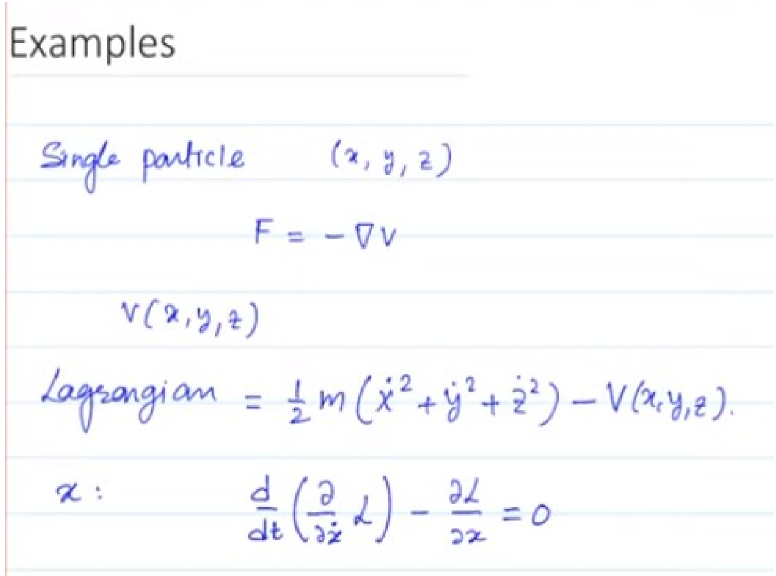

The first thing that I want to try is this must work if my system is free of any constraints. So take for example a single particle whose coordinates are given by x, y, and z okay. And there is a force on this particle. So remember because there are no constraints x, y, z are independent of each other. Why would I not use x, y, z as generalized coordinates?

Of course yes, I will use x, y, and z as generalized coordinates. Now the force say is given by minus of grad V where V is some function of x, y, and z okay. So this is a single particle moving in a conservative force field. And let us try to find the equation of motion. First of all the Lagrangian would be, what is the kinetic energy? That is simple. It is 1/2 m x dot square + y dot square + z dot square and this minus the potential. Remember potential is function of x, y, and z.

So let us write down. There are 3 generalized coordinates, there will be 3 Lagrange's equations. The first equation for x. So this equation would be d/dt of del over del x dot of Lagrangian minus del l over del x is equal to 0. So let us calculate this.

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$$\text{Lagrangian} = \underline{\tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)} - V(x,y,z).$$

$x:$ $$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}L\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(m\dot{x}\right) + \frac{\partial V}{\partial x} = 0$$

$$\Rightarrow \quad m\ddot{x} = -\frac{\partial V}{\partial x} = F_x \Big\} \quad y,\, z$$

$\Rightarrow$ Cylindrical Co-ordinates
(r, θ, z)

What is the partial derivative of Lagrangian with respect to x dot? So that is nothing but d/dt. This is just mx dot - del l/del x. But the first term of Lagrangian does not have any x. So this immediately gives us del v/del x = 0. And this gives us the equation mx double dot = - del v/del x but we know that is just exactly equal to F x. there you go. We recover Newton’s laws which were written in Cartesian coordinates which do not have any constraints written there.

So this is how the Lagrangian will give you the Newtonian equation. Here is the second example. The same single particle, see this is what I had said in our first lecture that when we change the coordinate system and go to a new coordinate system then we have to do lot of work to find the equations of motion. And I gave you the example of plane polar coordinates.

So in plane polar coordinates the equations of motion look very different. They are not simply equal to mr double dot is equal to f i okay. So here I will apply the same situation but I am going to use cylindrical coordinates as r, theta, and z. By the way, in the previous case of course there are two more equations, one for y and one for z too, okay.

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⇒ Cylindrical Co-ordinates

(r, θ, z)

$$\mathcal{L}(q, \dot{q}, t) = \frac{1}{2} m \left(\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2 \right) - V(r, \theta, z)$$

Eq r: $$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

Now in case of cylindrical coordinates, can I use them as generalized coordinates? Of course yes. So the Lagrangian which is function of now the coordinates q, q dot, and t; remember here q represents r, theta, and z; q dot of course represents r dot, theta dot, and z dot. This is equal to kinetic energy. How would I write kinetic energy, oh we already know. The radial velocity is just r dot.

So r dot square plus, what is the tangential velocity? That is r theta dot. So that is plus r theta dot square and plus z dot square and minus v. Now apply, what would the equation for r look like? Equation for r. that would be d/dt of del l/del r dot – del l/del r = 0. This is our first equation and this will be equal to or this simplifies to d/dt and what is the derivative of Lagrangian with respect to r dot?

Oh, that immediately just gives me m r dot and everything else of course, all other terms are independent of r dot. This minus now when you are taking the derivative of Lagrangian with respect to r remember oh, there is a r in this term too. So you should not forget that one. So this one will just become mr theta dot square. And then there is a r here too. There is a r here too.

So that would be equal to minus, so that will be del v/del r and this must be equal to 0.

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$$\text{Eq } r: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot r) - m r\dot\theta^2 + \frac{\partial V}{\partial r} = 0$$

$$\Rightarrow m(\ddot r - r\dot\theta^2) = -\partial V/\partial r = F_r$$

$$\text{Eq}\cdot \theta: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(m r^2\dot\theta\right) + \frac{\partial V}{\partial \theta} = 0$$

And this gives us m r double dot - r theta dot square = - del v/del r. And what is that? That is nothing but r component of the force. So that is del v/del r is F r. What about theta equation? Okay, so let us derive this theta equation too. Equation for theta is d/dt (del l/del theta dot) – del l/del theta and this must be equal to 0 and calculate this. So theta dot only appears in the second term here.

So this will be equal to d/dt of mr square theta dot and plus now remember theta does not appear in the kinetic energy part, theta only appears in the potential part. So this will be equal to del v/del theta = 0 fine. And what is this del v/del theta? Can you make a guess?

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$$\text{Eq}\cdot \theta: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\underbrace{m r^2\dot\theta}\right) + \frac{\partial V}{\partial \theta} = 0$$

$$\frac{d}{dt}\underset{\substack{\uparrow \\ \text{Ang. Mom.}}}{L} = -\frac{\partial V}{\partial \theta} = -r\,\frac{\partial V}{r\,\partial \theta} = r F_\theta$$

$$\boxed{\frac{dL}{dt} = \tau}$$

$$m\ddot z = F_z$$

This is of course the angular momentum of the particle about the given origin. So this is nothing but d/dt of angular momentum. Remember this is capital L, not the curly l that we are using for Lagrangian. So this is angular momentum and this will be equal to what would be del v/del theta. This is nothing but 1/r. So I will write it as r into del v by r times del theta. This is nothing but r F theta sorry with a minus sign.

And what is r into the tangential force? This is in fact torque. So the second equation we get is dl/dt is equal to torque and the force. And what about the z equation? That is straightforward. That we already know. That would look exactly same as the Cartesian coordinates that we used earlier. So that equation simply comes to mz double dot is equal to F z.

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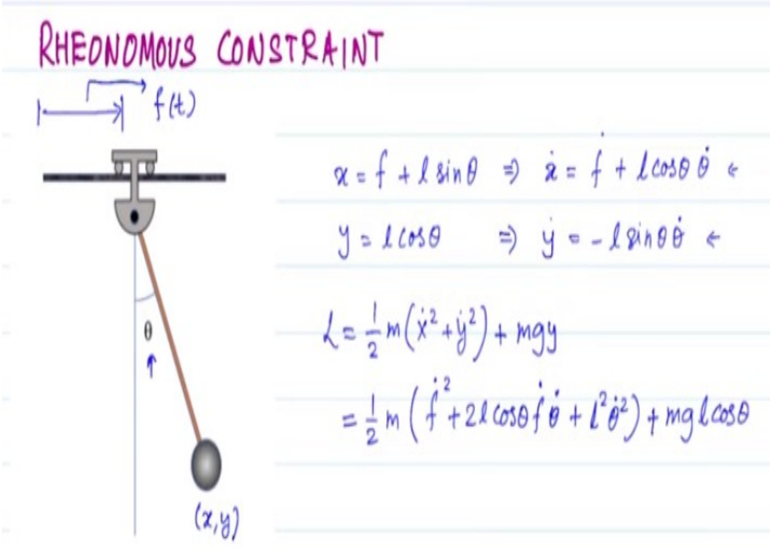

One more application of Lagrange's equations. This is one of the problems we have looked at earlier and remember the amount of work we have to do to solve this problem. This is the one with the rheonomous constraint where the trolley from which the pendulum is suspended, this trolley is moving with some predetermined function of t. So this distance is given by f (t) okay.

Now since you remember this example, I will quickly write down here, the generalized coordinate that I am going to use is angle theta and the Cartesian coordinates of this bob are x and y. So I will immediately write down the transformation equations. Transformation equations would be x = f + l sin theta and y = l cos theta. I can immediately calculate the derivatives of this.

So your x dot is f dot + l cos theta times theta dot and your y dot is – l sin theta times theta dot. Now write down the Lagrangian. Oh what is the kinetic energy term? It is 1/2 m x dot square + y dot square and minus the potential energy. What is the potential energy here? That is minus of mgy. Remember y points down. So the potential energy is –mgy. Now write this in terms of the generalized coordinate theta.

So that would be 1/2 m and this would give you f dot square plus 2 times l times cos theta f dot theta dot plus I would get l square cos square term as a coefficient of theta dot square which is coming from here and l square sin square theta coming from the second equation. We can immediately see that this adds to l square theta dot square. And plus mg times l cos theta, okay. Now let me do steps one by one.

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$$\frac{\partial L}{\partial \dot\theta} = ml\cos\theta\,\dot f + ml^2\dot\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = ml^2\ddot\theta + ml\cos\theta\,\ddot f + ml(-\sin\theta)\,\dot f\,\dot\theta$$

$$\frac{\partial L}{\partial \theta} = ml\dot f\dot\theta(-\sin\theta) - mgl\sin\theta$$

$$ml^2\ddot\theta + mgl\sin\theta = -ml\cos\theta\,\ddot f$$

$$\boxed{\ddot\theta + \frac{g}{l}\sin\theta = -\frac{\ddot f}{l}\cos\theta}$$

First thing we will calculate these del l/del theta dot. Del l/del theta dot is nothing but remember f is independent of that dot. So the first term will go away. Then we get m, so that would be ml cos theta into f dot. That is coming from here. You take a derivative with respect to theta dot and you would get the coefficient and plus m l square theta dot okay.

This is one term and when you take a time derivative of this, so d/dt (del l/del theta dot) this will give us d/dt of ml square theta dot plus so I will immediately simplify it here itself. It will be ml square theta double dot plus now I will get two terms there. Remember you have, this is the total time derivative. That means I must do ml cos theta times f double dot and plus it will be ml minus sin theta into f dot and theta dot okay. And also look at del l/del theta.

Remember now theta appears in several places. So in the Lagrangian theta appears here and theta appears here. So you get two terms there. So I will get the derivative of this will be mlf dot theta dot into –sin theta – mgl sin theta. And when you equate these two, this term here cancels with this term. That leaves us with ml square theta double dot plus, so I will take this to the other side.

So mgl sin theta and this will be equal to –ml cos theta into f double dot. And then you will see we get the final equation which is theta double dot + g/l sin theta = - f double dot by l cos theta. This is the equation we had derived earlier and remember the amount of work that we had to do but with the Lagrangian this becomes very easy. So you see we obtained these equations very quickly from the Lagrangian.

And this is a great utility of the Lagrangian mechanics. I am going to look at the detailed examples of the Lagrangian mechanics and how we extend it to include dissipation or even velocity dependent forces. Remember velocity dependent forces are important. Magnetic force is velocity dependent force. So those forces are also important. So all that extension including applications I am going to do in third week.

In the second week what we will do is we will derive the Lagrangian equations but from a more aesthetically pleasing, more elegant principle called as Hamilton's principle.