Theoretical Mechanics

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Lecture - 04
D'Alembert's Principle

The principle of virtual work applies to only static equilibrium system. In the last section we saw this principle and some of its simple applications. Now what we want to do is extend this to dynamic systems. Now that is very easy to do and the extensions of this is called as D'Alembert's principle. So let me write down the principle.

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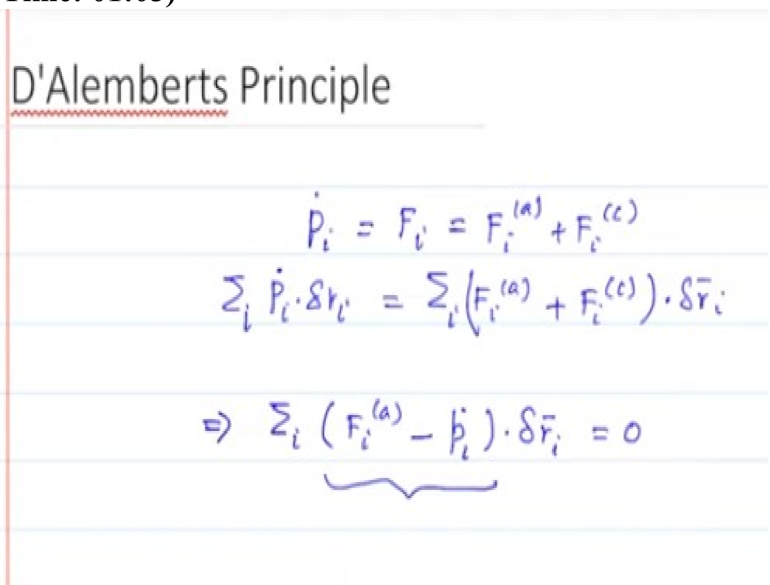

So if you have a dynamical system then for each particle the rate of change of momentum is equal to the net force on the particle and we already have seen that this can be split into two parts, one we will call it as applied force and the other one is the constraint force and now I will multiply both sides with the virtual displacement into r i and then take sum over all particles.

Now we already know from the principle of virtual work that the net work done by the constraint forces is zero so this immediately becomes sum over i F i applied - P i dot times delta r i this must be equal to 0. Now how can we use this principle? Remember again that all these virtual displacements for each particle which is delta r i, they are not independent of each other if the system is constrained.

That means I cannot simply set this bracket to 0 for each i, okay. Now we will immediately try to apply this to an example. Let me start with a very simple example.

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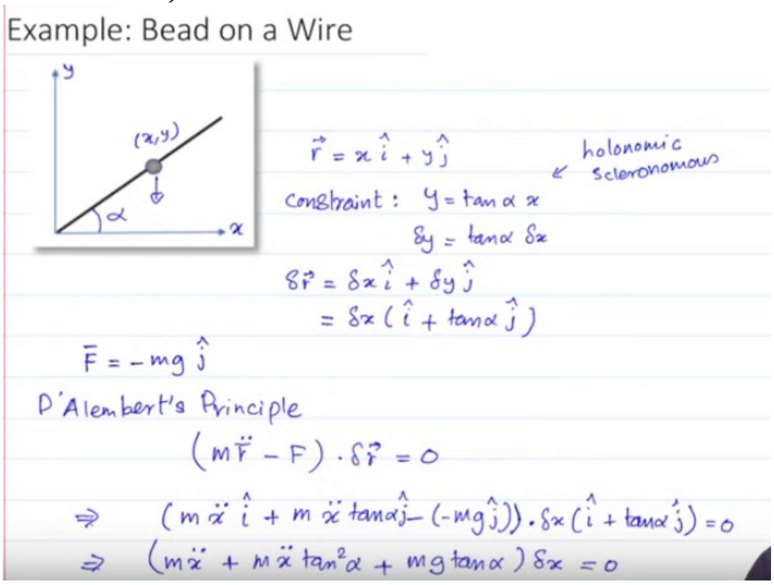

This example we have been looking at again and again. So here this is a bead which is traveling on a wire and there is no friction in the problem. So the coordinate system here is x and y. So the coordinates of this point the coordinates of the bead are x and y. So the position vector here r i . So in this case because there is only one particle I will in fact not write the index i but just the r vector.

r vector here is x i cap + y j cap. But then we have a constraint in the problem. So the constraint is constraint equation is y must be equal to tan alpha times x if this angle is alpha. Now, the virtual displacements delta x and delta y are related because of this constraint. So virtual displacements delta y must be also equal to tan alpha times delta x. This is remember a constraint is holonomic and it's also scleronomous.

That is the time variable does not explicitly appear in this constraint. So the net displacement vector, virtual displacement vector will be equal to delta x i cap + delta y j cap and I will simply this to delta x times i cap + tan alpha times j cap. What are the forces in the problem? We will not look at the constraint forces. The only applied force is the gravity acting on the bead.

So the only applied force F I will again not write a for applied but vector F is – mg j cap. It is in the downward direction. So let us now write down the D'Alembert's principle. So according to this m times r double dot vector minus F multiplied by the virtual displacement, this must be equal to 0. So let us put all the entities there.

So we get m x double dot times i cap plus m, you will have y double dot here but remember y double dot is nothing but x double dot times tan alpha. So this would be x double dot times tan alpha into j cap minus the force which is minus mg times j cap. And then dot product with delta x times i cap plus tan alpha times j cap. And this must be equal to 0, okay. So we can immediately simplify this.

This would be so mx double dot then the dot product of j cap with these two terms. This would be plus mx double dot tan square alpha plus mg tan alpha into delta x must be equal to 0.

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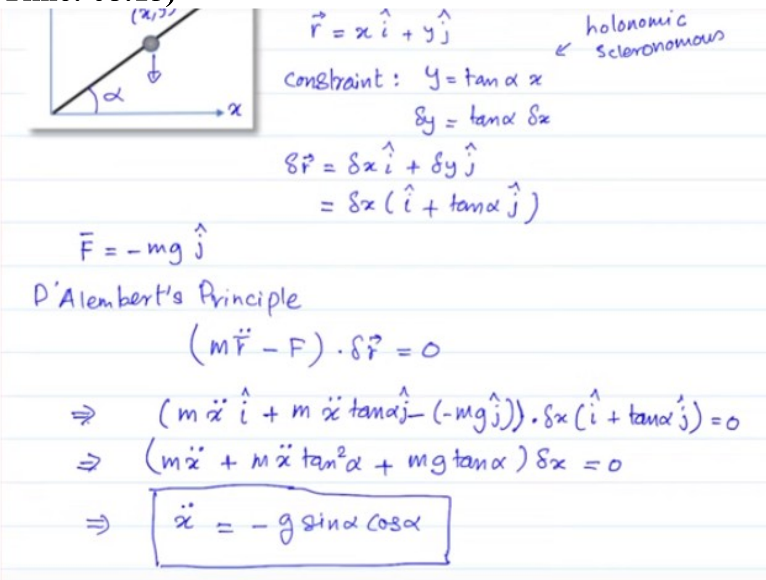

And now we can immediately simplify this to x double dot, x double dot = - g times sin alpha cos alpha. Remember this equation, we have derived this equation many times over earlier. So this is how you apply D'Alembert's principle to dynamical system. Now we actually got the acceleration of the particle in terms of the forces.

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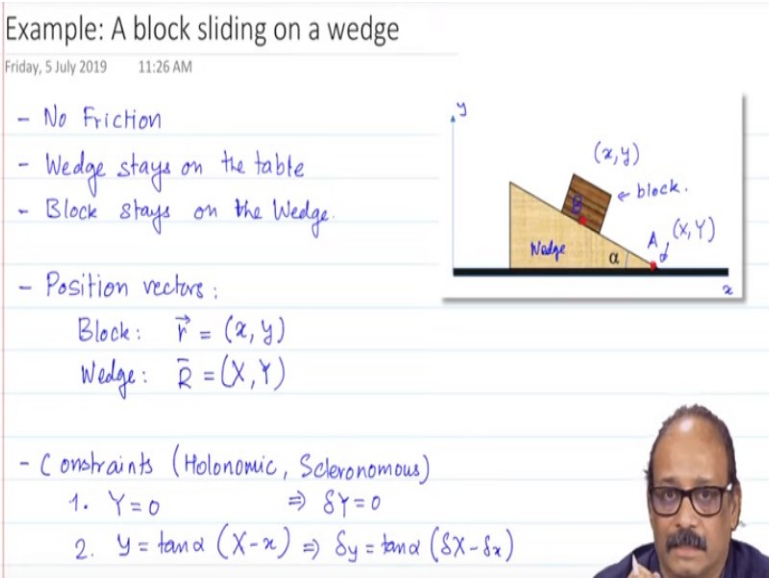

Okay, in this second example we have 2 masses and 2 degrees of freedom. And this example is somewhat longish. So that is why I have already written many steps here. So I will not be writing steps now. But I will just explain the steps as they go. So here is the wedge block system. So this is the wedge here. This is the wedge and this is the block. In the problem there is no friction anywhere.

That means the wedge slides on the table without friction and the block slides on the wedge without friction again. And then the constraints in the problem are that the wedge remains on the table and the block does not leave the surface of the wedge. These are the 2 constraints we have. Now I will start by describing the position vectors first.

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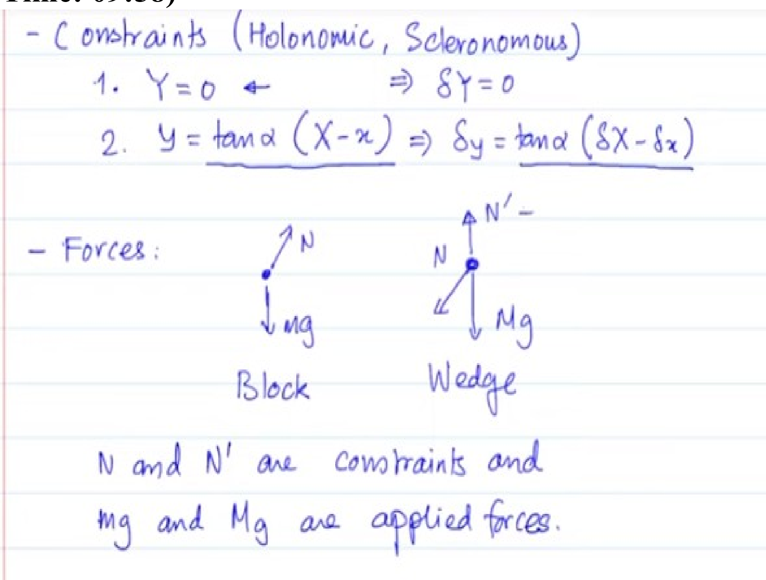

So the position of the wedge, remember neither the wedge nor the block are changing its orientation. So the orientation of the wedge and the block will remain the same throughout the motion. So all that I really need to do is describe one point of the wedge and one point of the block as the reference points. So here point A, I will choose point A which is the vertex of the wedge as the reference point for the wedge.

And the point B here, the point B here as the reference point for the block. And let the coordinates of the point B be xy and coordinates of the point A are capital X and capital Y. So your position vectors are xy and for the wedge they are capital X and capital Y. Now the constraints in the problem these are that the wedge remains on the table which means the y coordinate of point A does not change.

So what I will do is I will choose that y coordinate to be some constant in specifically I will choose it as 0 here okay. And the second constraint is that the block does not leave the surface of the wedge. That means small y, the coordinates of point B that is small y must be equal to tan alpha times capital X minus small x.

If you go back to the diagram, then this is your capital X and the projection of this would be somewhere here. So this distance here is capital X minus small x. And this height here is y. So your y is tan alpha times capital X minus small x. and then because these constraints are holonomic and scleronomous, that means there is no time explicitly occurring in the constraint equations, the actual displacements and virtual displacements are same.

So the virtual displacements would be delta y would be equal to 0 for capital Y and delta y for small y is tan alpha times delta x minus delta of small x. Now out of these since we have 3 coordinates, capital y we have set to 0 anyway. So I have capital X, small x, small y and there is one constraint equation so I actually have only 2 independent variables.

So I will choose 2 independent variables to be small x and capital X, okay. Now what are the forces in the system? The forces in the system, so we will draw this free body diagram and in the free body diagram you can see on the left side, it is the free body diagram of the block. On the block, there is only weight and the normal reaction due to the wedge.

Whereas on the wedge, you have normal reaction because of the block on the wedge, mass of the wedge that gives you the weight and the table also exerts a normal reaction which I am going to call as N prime here. So in this case, your N and N prime are the constrained forces and the weight of each of these blocks are the applied forces, okay. Now let us apply the D'Alembert's principle.

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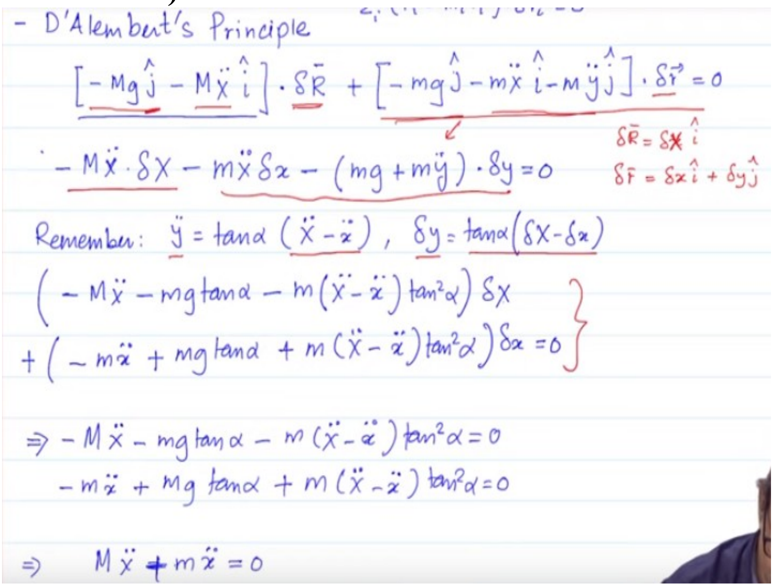

If I apply D'Alembert's principle, for the wedge we have this. Remember the D'Alembert's principle is sum over i F i – m i r i double dot times delta r i. This must be equal to 0. Here there are 2 particles, so your i goes from 1 to 2. So first for the wedge the applied force is –mg times j cap. Then –m r double dot.

So mass of the wedge is capital M times x double dot. And that is of course in the direction of i cap, multiplied by the virtual displacement of the wedge. And then the second bracket for i = 2, that is for the block is the applied force is –mg j cap and then we have r double dot, the acceleration would be equal to mass times x double dot i cap and m times y double dot j cap, okay.

And multiply this by the virtual displacement of the block and the net sum of this must be equal to 0. See the constraint forces we have dropped, there are no constrained forces appearing in this. Let me simplify this. If I write delta r as so delta r is nothing but just delta X times i cap. This is capital X, okay. So I will substitute that here and because it is in the direction of i cap but force mg is in the direction of j cap.

So that term will vanish and you only get Mx double dot into delta X and similarly the virtual displacement for the block is delta x i cap + delta y j cap. So substitute that here and this entire bracket here simplifies to this one here, okay. And at this point we will use the information from the constraints that y double dot must be equal to tan alpha times X double dot minus small x double dot and similarly delta y must be equal to tan alpha times delta X minus delta of small x.

So once you do that now I have quite a bit of work there, but straightforward.

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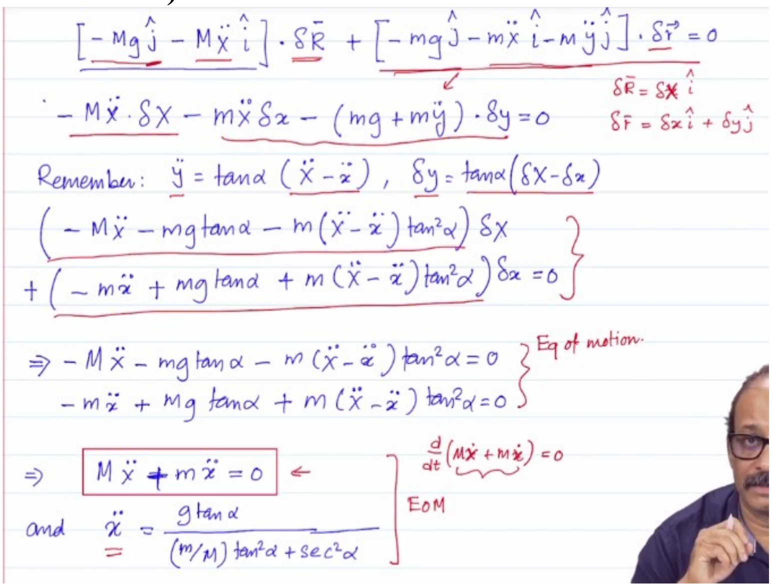

So then you get this one long equation and in this equation remember we have gotten rid of y, we have gotten rid of delta y. So now my one single equation is in terms of capital X and capital delta X or delta of capital X, small x and delta of small x. What we will do is we will collect all the terms coefficients of delta x for both small x and capital X. So this is what we have done.

This is the coefficient of delta X and this is the coefficient of delta of small x. And we already had said that the 2 independent variables in the problem are capital X and small x. That means the variations in capital X and small x are independent of each other. So this equation must be true for any arbitrary delta capital X delta small x. In one case I can of course choose capital delta X to be 0 in which case I will get the first one of the brackets to be 0 and in the other case I get second bracket to be 0.

And that is how I extract the equations of motion. Remember if you look at these equations carefully, they are actually just simultaneous equations, linear simultaneous equations in small x double dot and capital X double dot. Now you can easily separate them and I am going to ask you people to do this algebra now and finally show that the correct equations of motion turn out to be thus. So this is equation of motion.

Out of which the first one, second one of course gives you x double dot directly and from the first equation you can get capital X double dot in terms of small x double dot and small x double dot we already have it. But look at this first statement. Is the statement obvious to you from the problem? Yes. If you look at this problem here or the free body diagram here, it is immediately clear that there is no net external force in the horizontal direction.

So what happens to the net momentum in the horizontal direction? Now that must be conserved and that exactly is what this statement is. If you look at this, this is d/dt of mX dot that is capital X plus mx dot and this is equal to 0 and this is basically the net momentum there, okay. So we will go to the one more example where we will consider the time dependent constraints, so it is a rheonomous constraint.

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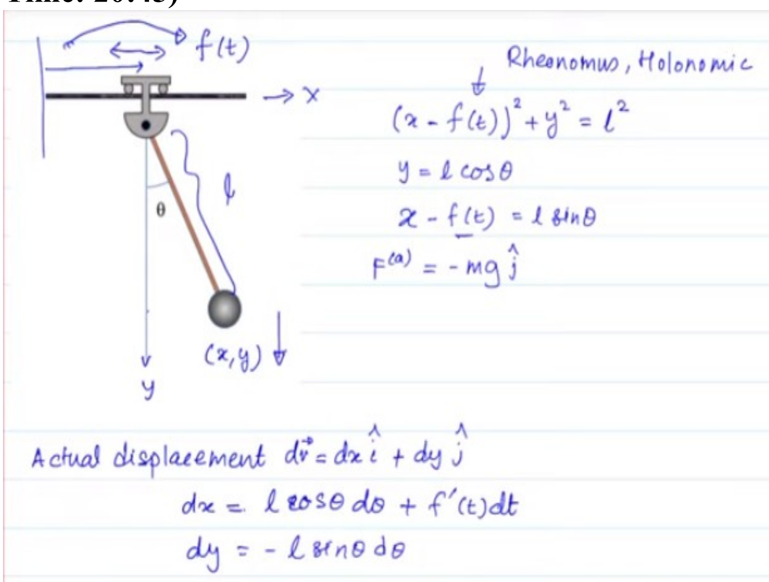

This is a example with time dependent constraints. So here we have a pendulum and this pendulum is suspended from a trolley and the trolley can move horizontally, okay. And here what we are going to assume as the motion of this trolley is already known. And that is given by some function of time. So this point here if I measure it from some arbitrary x axis, so this distance here is given by the function f and it is predetermined.

So it is not, the trolley does not move under the action of forces but there is some external force which is a constrained force which moves the trolley according to some known time dependence, okay. So we will go around setting the other constraints. Coordinates of this point are say x and y. Then the constraint equation can be immediately written. So this is your x axis, this is your y axis, okay.

So the constraint is that the length of this rod remains fixed and is equal to l, okay. So the constraint equation becomes x – f(t) whole square that is the distance between the vertical axis here to the bob plus y square must be equal to l square. And as usual whenever there is a pendulum problem or the problem with the rotation involved in this, we of course immediately make one switch of variable.

So I will write y as l cos theta and x – f(t) as l sin theta. Now what kind of constraint is this? Because in this constraint the time appears explicitly. This constraint is called as rheonomous and is of course holonomic because it comes in the form of an equation involving coordinates and the time, okay. Now, the only force in this, so everything else is a constrained force except the weight of the bob.

So again the force here, applied force is minus mass times g times j cap, okay. And then the remaining part can be immediately done as usual. So what are the actual displacements? The only difference comes when we are considering the actual displacement. The actual displacement here is given by dx i cap + dy j cap. So this would be your dr vector.

And remember here y of course can be used as dependent variable and or I will write this in terms of the coordinate theta. So your dx is equal to l cos theta d theta. But remember there is a time dependent term here. So that would be equal to f prime t dt. And similarly dy would be equal to – l sin theta d theta.

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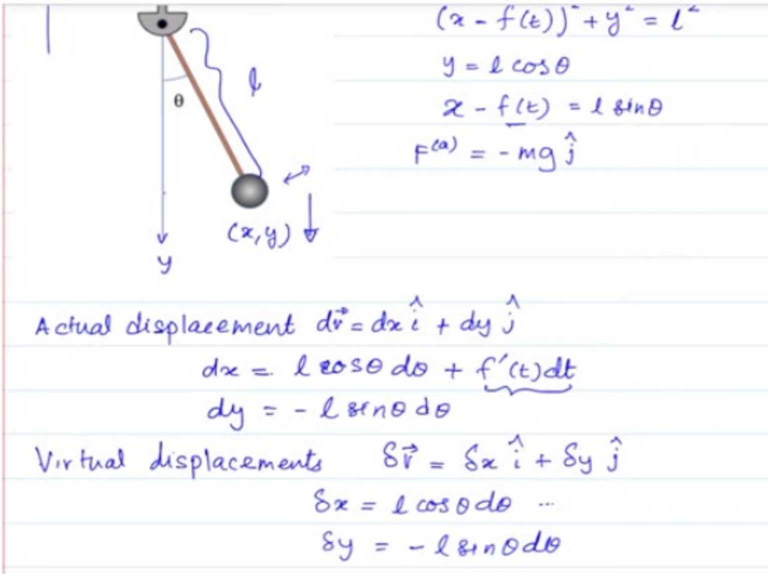

Now these are the actual displacements. What about the virtual displacement? Virtual displacement remember occurs at a particular instant which means I am going to hold this trolley steady and only move this bob. In that case, the virtual displacements are given by delta r vector which is delta x times i cap + delta y times j cap. And this would be equal to because delta x now is simply l cos theta d theta.

And delta y is – l sin theta d theta. Remember in this case that additional term involving f will not be there because since we are considering the displacement at an instant, the dt is 0 and hence this term which appears in actual displacement will not appear in the virtual displacement. Now let us write the D'Alembert's principle.

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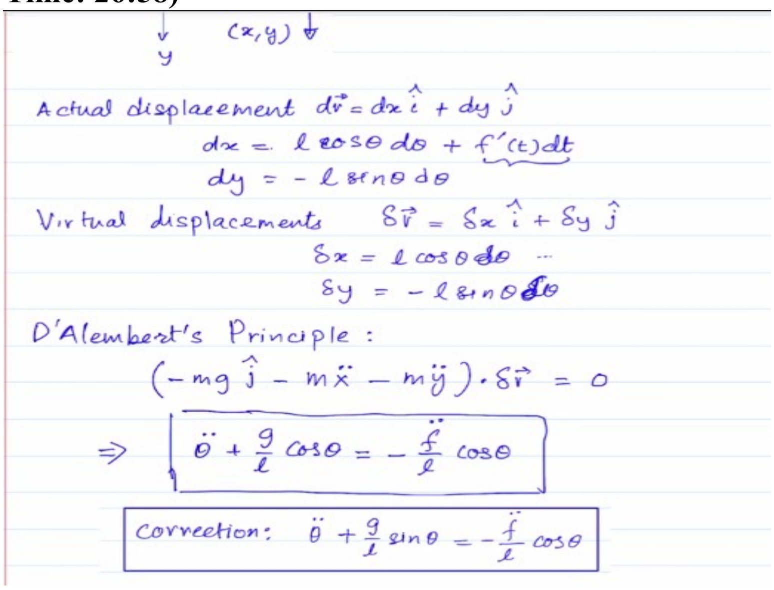

So according to that the only force is (–mg j cap – mx double dot – my double dot) into delta r = 0, okay. So we of course need to calculate mx double dot and my double dot.

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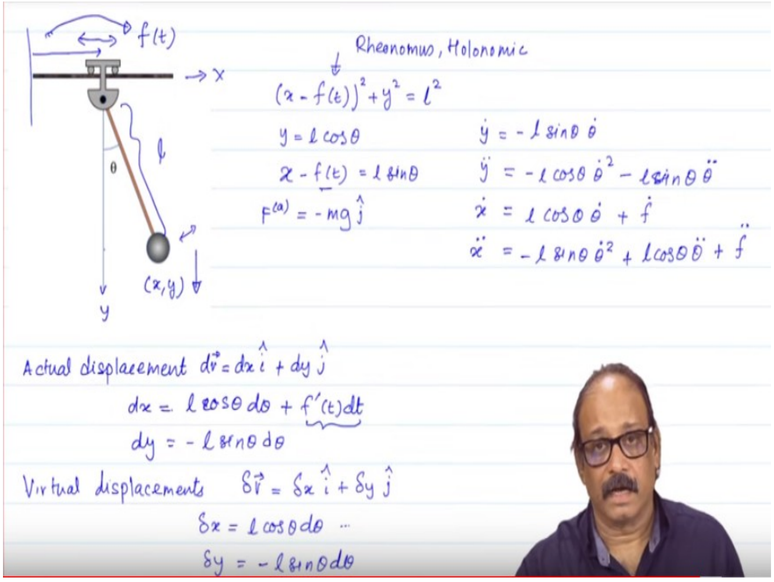

So that we can do immediately here your since x is this y dot is equal to – l sin theta theta dot and y double dot will be equal to – l cos theta theta dot square – l sin theta theta double dot. And similarly x dot will be equal to l cos theta times theta dot.

But remember there is a term f. So this will be f dot and x double dot will be equal to – l sin theta theta dot square + l cos theta theta double dot + f double dot. Now you see the difference between this and the pendulum that we have seen earlier. In the x double dot term we have this x dot term there. So I will substitute this back into this equation here and again I am going to ask you people to work with the algebra.

Here there is only one degree of freedom. So your delta r has only delta theta in it. So the coefficient of delta theta must become 0 and using that I will ask you to prove that theta double dot + g/l cos theta must be equal to f double dot – f double dot/l cos theta. And this we can immediately see that if the suspension trolley is actually not moving which means f of t is constant then its second derivative would be 0 and this would reduce to a normal pendulum equation.

Note that even if this f of t is a linear function of t, the equation of motion is exactly same. So the pendulum in fact would seem to just do a usual oscillatory motion as if the trolley was not moving. But that we already knew because if you go to the trolley's frame, but trolley's frame is also inertia. That means there would be any, no pseudo forces in the problem and you would actually get the same equation of motion for the pendulum in the trolley frame same as the stationary frame, okay.

So after these three examples, we see that the calculation of equation of motion is not as easy. There is already a quite bit of work that we have to do and I have already made few mistakes here. So these derivations are usually norm. But what we are going to do is this. Now we can actually in the formal derivation from D'Alembert's principle and combining it with generalized coordinates we can now write down the final Lagrange equations which make the calculation easier.