Theoretical Mechanics

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Lecture - 03
Principle of Virtual Work

Hello in the last section we saw some examples of constraints and we also looked at the classification of constraints. Now in this section I am going to introduce an idea of virtual displacements and virtual work. Now do you remember in the first lecture we said that we were looking for a prescription to solve problems or get equations of motion in a single step okay without talking about the constraint forces.

Now that prescription or the way we solved it there crucially depended on the fact that the constraint forces do not do any work. So in that problem gravity was the only conservative force and then we could write the energy conservation equation to get the equation of motion, okay. But this is not always true. The constraint forces, there are several examples which we will see where constraint forces do not do any work, but that is not always true. The constraint forces do work. But wait.

There is a very simple work around this. If a force is doing work then you simply classify that as a non-constraint force. But that is not a good idea because if we do that then in that case we have to solve extra equations of motions, okay. So for this reason I am going to introduce the notion of virtual displacement and virtual work. So it may be possible to solve problems where the constraint forces which do not do any virtual work also will be useful, okay. So let us start with examples straightaway.

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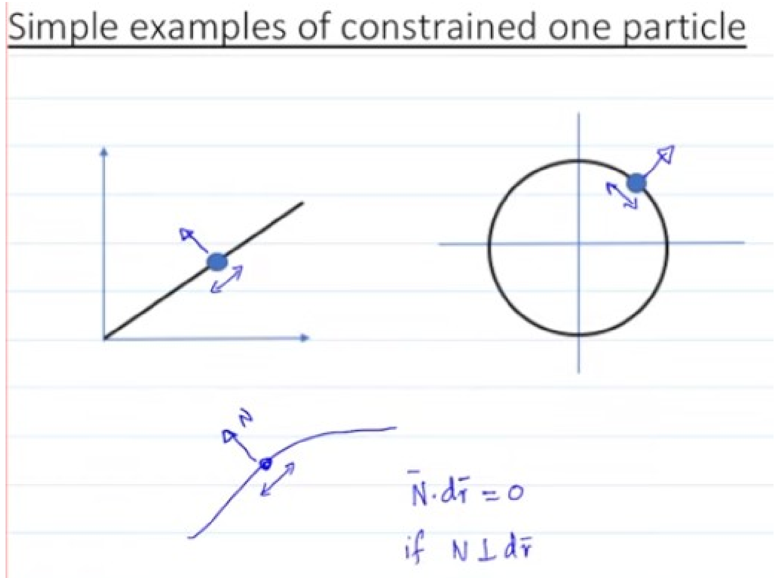

So look at this some simple examples of single particle constraint in the space. The examples which we had seen earlier was, one was the bead traveling on a straight wire or a circular wire. In both cases we see that the normal force which is a constraint force is actually perpendicular to the possible motion of the particle.

So in this case the particle can move along the wire here or in this case the particle can move along the wire here. And it works for every possible configuration like this. If you for example have a wire which has some arbitrary shape and there is a bead there, then the motion of the bead infinitesimal motion is tangential to the curve.

And the normal force is always perpendicular to the force oh sorry perpendicular to the wire in which case the work done which is N.d r will be zero because N is always perpendicular to d r. Okay, let me take one more example.

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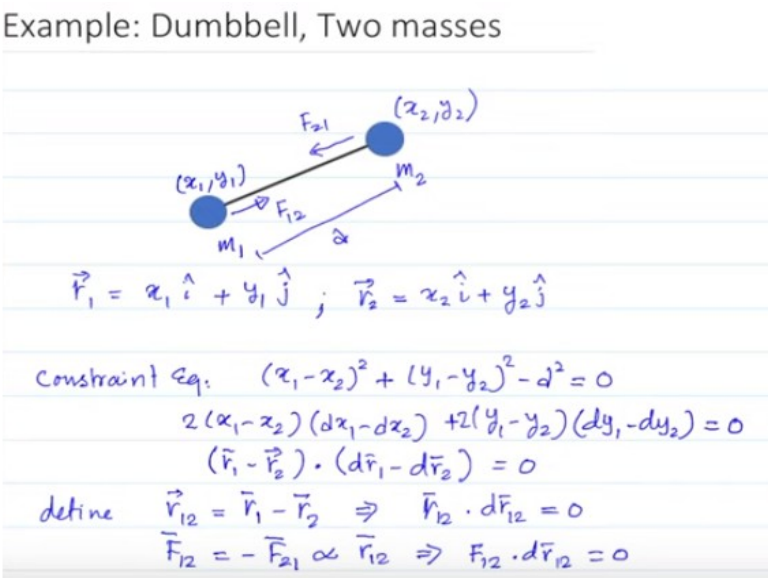

Now in this example I have a dumbbell. Dumbbell has two masses. Let me say this is mass m 1 and this is mass m 2 and the coordinates of this mass are x 1, y 1 and for this mass it is x 2, y 2. So let me write the position vectors. So here the position vector is r 1 vector is x 1 i cap + y 1 j cap and the second vector r 2 vector is x 2 i cap + y 2 j cap okay.

And in this dumbbell the constraint is that the distance between the two masses, the distance between the two masses remains fixed. Now that constant I will express as, so the constraint equation is x 1 – x 2 whole square + y 1 - y 2 whole square - b square, this must be equal to zero, okay. So the distance between the two masses remains fixed. Now what I will do is take differential of this equation.

If I take differential of this equation then I get two times x 1 - x 2 times dx 1 - dx 2 plus into dy 1 – dy 2 and this must be equal to zero, okay. Now I can express this identity in the form of vectors. So if you immediately notice that this one can be written as r 1 vector – r 2 vector dot product with dr 1 vector – dr 2 vector and this must be equal to zero.

So if I define my r 12 vector, so if we define the separation vector and denoted it by r 12, define it as r 1 – r 2 then what we have is this identity becomes r 12 dot product with d of r 12 vector and this must be equal to 0. And now you look at the constraint forces. The constraint forces are there along the rigid support that connects the two and it is a Newtonian pair of equal and opposite forces.

So this is F 21 and this is F 12 and you can immediately see that because it is a Newtonian pair that F 12 vector is minus of F 21 vector and also it is directed along the line joining the two masses. So this is proportional to r 12 vector. And this in fact gives us the identity that F 12 times d r 12 and this must be equal to 0. So what will happen is thus.

In this case 2 the constraint forces will do absolutely no work even though the dumbbell could be moving in space, in whatever possible configuration that it can take, okay. Alright, one more example.

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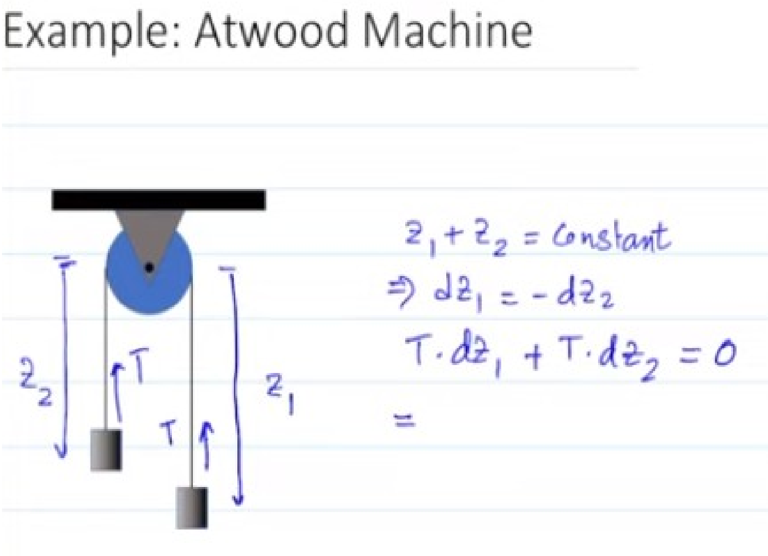

This is a simple Atwood machine. In this Atwood machine, two masses are hung from a pulley and if I use a reference here, then I have this distance as z 1 and this distance as z 2. And what is the constraint here? We can immediately see the constraint is z 1 + z 2 must be equal to a constant and the constraint forces they are of course along the string joining the two.

So the constraint force which we typically call as tension, in both cases the tension is upwards. And if we write down the amount of work done by the constraint forces so the net work will be T. dz 1 + T. dz 2. But we immediately know from the constraint that dz 1 is in fact minus of dz 2. So this immediately becomes 0. So even in this case, here remember the forces are actually parallel to each other as opposed to the previous example where the forces were equal and opposite.

Even then the nature of the constraint is such that the net work done by the constraint forces will still be 0 in Atwood's machine.

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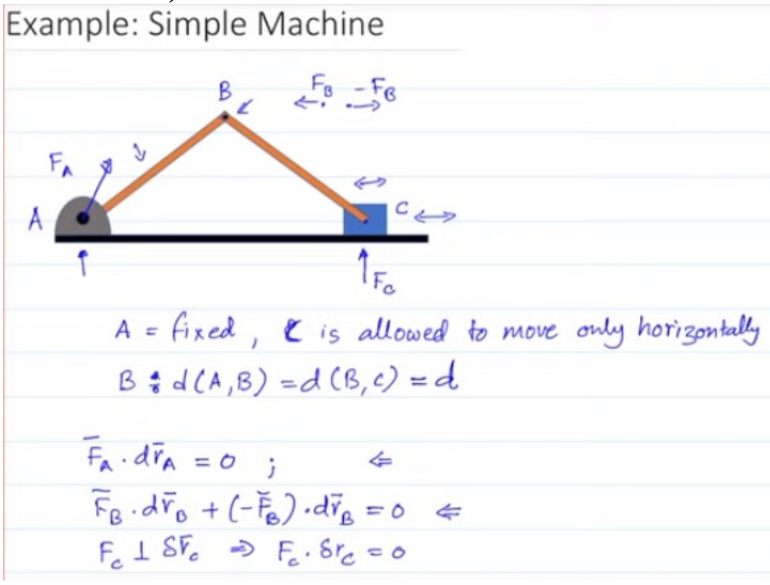

I will take one more example before I will show you one example where the constraint do work. Now look at this simple machine. This machine has two arms and the job of these two arms is to push this block back and forth here. And the constraints are that the first rod, this rod here, is fixed at this point. Then the two rods are joined at this point and the third point here is also only allowed to move along the horizontal direction.

The rod is not allowed to move in the vertical direction, okay. To analyze this I will mark the three points as A, B, and C here, okay. What are the constraint forces in this case. The constraints themselves are that point A is fixed. The second constraint is point C is allowed to move only horizontally, okay. And the point B is constrained by the fact that the two rods are joined at this point.

And the distance between, so the constraint here would be distance between A and B and distance between B and C. These both are equal to some given distance d and that distance does not change because these are rigid rods, okay. Now, you could have of course considered this as a system of a rigid body system in which case each rod in fact has large number of particles and the forces there constraining the positions of individual particles in the rigid body.

We have already seen this in the previous example. If the distance between 2 masses is constant then the corresponding constraint forces do not do any work. So I will not worry about this rod being rigid or the internal forces inside the rod, okay. Now let us look at what are the constraint forces. Because the point A does not move, there must be some force here which I will call it as F A.

Then there is force here F C which keeps the block on the horizontal surface, okay. So the other end of the rod, this end of the rod remains attached to the block and the block only moves on the horizontal surface. And what is there at B? So at B there is again a Newtonian pair of equal and opposite forces. So because the rods are joined to each other they will be on one rod F B and same time on the other rod there will be – F B.

This is what you will have at point B. Remember F B and – F B are working on two different rods there. And how much work is done by all these forces. First of all F A into d r A where d r A is a infinite decimal motion of point A but d r A is 0 because r A is fixed. So this is 0, okay. Then F B and – F B, both these work on the point B or at point B. So you have net work done which is work done by F B and work done by – F B.

But see the displacement of point B is same in both cases and hence this also will be equal to 0. And the third case F C is in fact perpendicular to delta r C. So F C is the motion infinite decimal motion of the block is along the horizontal direction whereas the constraint force is in the vertical direction. And this also immediately gives us F C dot delta r C is equal to 0.

Remember the statement that we want to make here is that the net work done by the constraint forces is 0. It is not individual statement that individual forces are doing 0 work. The example being the second line here. In the second line individual F B does work on individual rods but the net sum is 0, okay. Just as in case of the previous example in case of dumbbell there are individual forces which is F 12 and F 21.

Both these forces do work but the net sum of these two forces is 0, okay. Now I will go to one example where constraint forces actually do some work.

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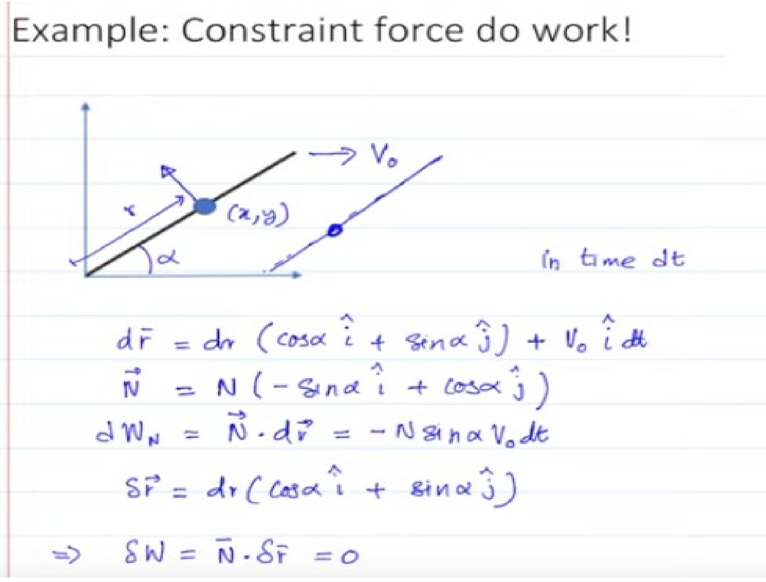

Now here this is a similar situation to the one we had in the first example except now I am going to say that this rod here is moving with uniform speed V naught over the right along the horizontal direction, the V naught is horizontal. Now what will happen? The constraint force, so after some time of course the rod will move to a new position and the bead also would be at some other position there, okay.

Now what is the constraint force here? Constraint force is a normal reaction and remember if the motion of the bead is frictionless on the wire then of course the normal reaction will be perpendicular to the wire. Now what is the displacement of the bead? So the displacement of the bead, so if I call this angle as alpha, this distance here as r and the coordinates of the bead as x and y.

Then the infinite decimal displacement dr will be equal to dr into cos alpha times i cap plus sin alpha times j cap. But wait, this is a small displacement happening in time dt. So in the small time dt remember the rod also has moved a little and hence the bead also moves which is by v naught times i cap, okay. Now what is the normal reaction? So the normal reaction, the vector is the modulus N, the magnitude of the forces N.

This is – sin alpha times i cap + cos alpha times j cap. And now you immediately see that if I take a dot product of the normal reaction, so if I am calculating work done by the normal reaction which is, so infinite decimal work into dr vector but this now is not 0. This is in fact –N sin alpha times V naught, okay. Sorry I forgot to write dt here. So that will be dt, okay. So here is the example where the constraint forces will do some actual work.

Now here what we will notice is this. Look at this displacement which is along the wire which I will write it as delta r and write it as dr times cos alpha i cap + sin alpha j cap. This displacement, I will call it as a virtual displacement and now you see that the virtual work which I will define as dot product of N with delta r. This is still 0.

So every constraint which actually depends on time t explicitly in that case it is very likely that the normal forces or the constraint forces will do some work. So finally let me put all these ideas together in a formal way and define virtual displacements.

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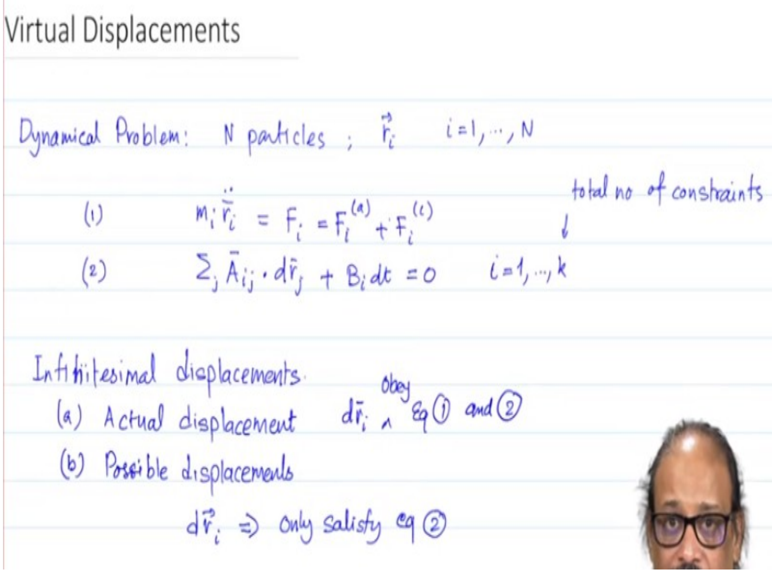

Now let me pose it as a complete dynamical problem. So you have N particles and their corresponding position vectors are given by r i where i is going from 1 to N, okay. Now the Newton's equations, so we will use the Cartesian coordinates in this case. So the Newton's equations will become m i r i vector double dot must be equal to the force on i th particle, okay. And this force I can divide this into 2 parts plus.

So by a I either mean active forces or applied forces. By c here I mean constraint forces. So this is our equation 1. These are the Newtonian equations and I will also write down the constrained equations. Now remember the constrained equations in a most general form was the Pfaffian form. So I will write it as sum over j A i j . dr j +

B i dt = 0.

So this is the constraint equation and here the summation j goes from 1 to n and index i goes from 1 to k where k is the total number of constraints, okay. And based on this, I will make a categorization of the displacements. So define infinite decimal displacements in 3 categories. One I will call it as actual displacement. Or this is the displacement that actually system undertakes when all the forces are acting on it.

So what must happen is these displacements, dr i they must satisfy equation 1 and equation 2, okay. So they the displacements obey equation 1 and equation 2. And the second category I will write it as all possible displacements. Now I will use the same symbol here dr i and what do they do? These only satisfy equation 2. Those are the constraint equation.

So basically all possible paths which satisfy constraint equation will give you infinite decimal possible displacements there and remember only one of these paths will be the actual path or will give you actual displacement.

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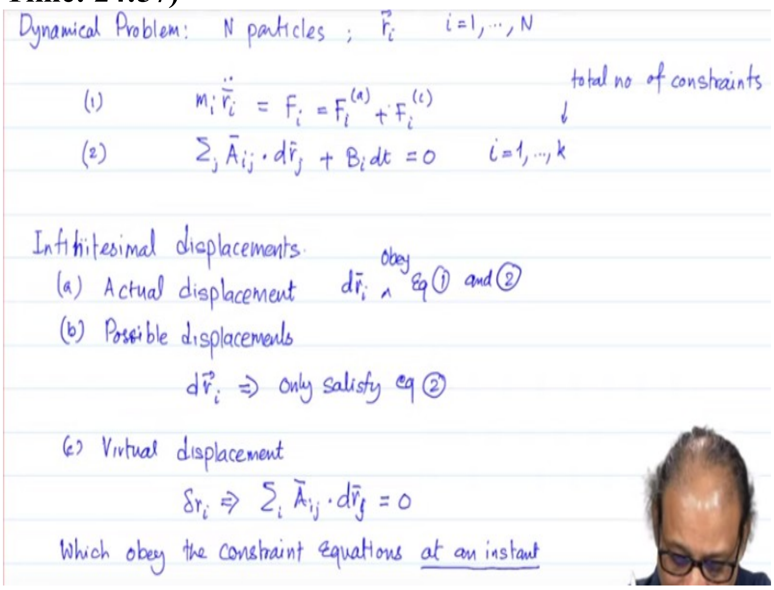

And in addition I will define a third category here as virtual displacement and denote it by symbol delta r i and these will satisfy summation over i A i j vector . sorry dr j = 0. See this is a virtual displacement which is in accordance with constraints. That means they satisfy constraint equations but only at one instant. If you consider infinite decimal amount of sorry time interval then of course the actual displacements will take place.

So the virtual displacements we will make a statement about it that virtual displacements are the displacements which obey the constraint equations at an instant. This is the key remember, at an instant. This is the difference between virtual displacements and the possible displacements that virtual displacements are taken only at one instance. So in our last example the rod was moving horizontally.

If I consider the interval dt, then the actual displacement also has an extra V naught dt component along the horizontal direction. But if I consider the rod to be stationary that is at an instant and look at what possible displacements can take place, those are only along the wire. And those are the ones called as the virtual displacements, okay.

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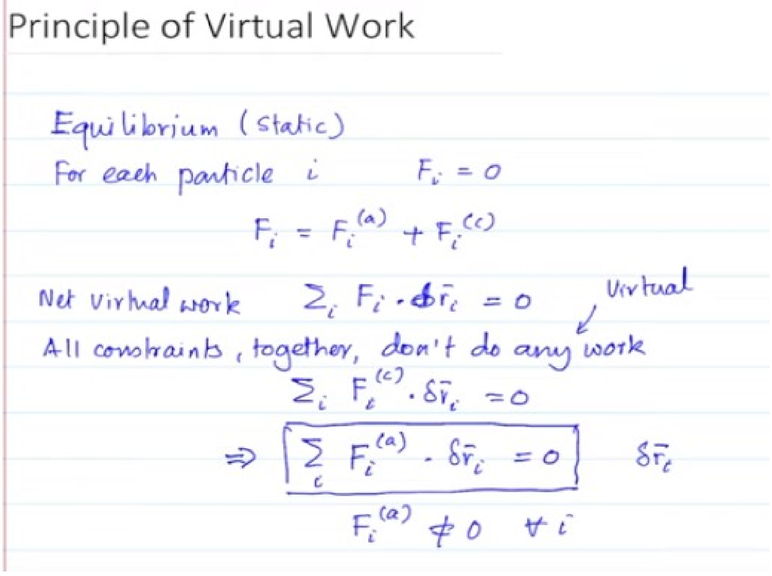

Now, let us put this to use. This can be used in a principle of virtual work. Remember this principle of virtual work is used commonly by engineers, mechanical engineers to inspect the stability or construction of static structures. So the principle of virtual work is thus. If a system is in equilibrium okay or static equilibrium then the forces on all components of the system or all particles in the system are 0.

So that is for each particle i F i is 0 okay. Now, I will write this as F i is equal to F i applied plus F i constrained. And now we look at the virtual work. So the net virtual work done by all forces is sum over i F i dot delta r i but this must be 0 because all the individual forces are also 0 okay. And in addition now we have all the constraints together don't do any work.

Which also means sum over i all the constrained forces delta r i must be equal to 0. Remember do not do any, I will say virtual work, okay. And this gives me an immediate equation which is sum over i F i applied times delta r i must be equal to 0, okay. Now there is some physics that we can extract from here.

Remember in this case the virtual displacements, these are the displacements of individual particles under constraint which means the individual delta r i's are not independent of each other. All these 3N coordinates or 3N infinite decimal displacements will of course depend on each other through constraint equations. Which also means that I cannot immediately put F i a is equal to 0.

The individual F i a is not 0 for any i, okay. This is crucial to remember that we have one single equation here where this is true for any arbitrary virtual displacement but still we cannot put F i a equal to 0 because those are not independent of each other. So how do I use this principle?

To use this principle what I must do is from this equation I will write down the generalized coordinates or delta r i's in terms of generalized coordinates for which the individual virtual displacements will be independent of each other. In that case I will be able to extract equations from it. Let me take one example to make this idea clear.

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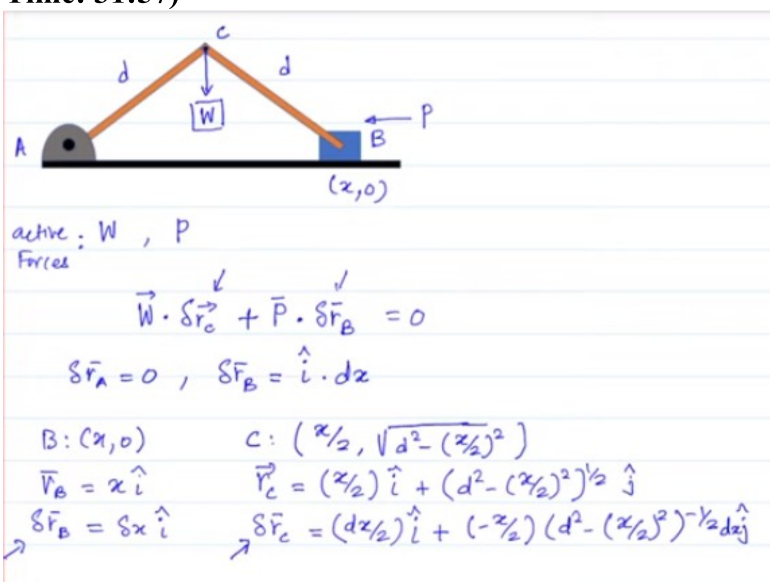

Take this simple machine, okay. Now this is your point A, this is your point B and this is your point C okay. And let us also assume that there is some weight W attached here, okay. And then I have to apply some force here which I will call it as P to make sure that this structure remains rigid or this point B does not slide out. So you have system in static equilibrium.

Now we have already seen what are the constraint forces here. And what are the applied forces now? So applied forces are one is W which is the weight that we have attached and P. So these are active forces, okay W and P. And when I write the equation remember I really am not going to use any of the constraint forces.

So the principle of virtual work tells us that the W dot product with delta r c, the motion of point c because W acts at point c plus P vector dotted with the motion of point B, okay virtual motion of point B. And this must be equal to 0. This is the equation. And how do I extract now the equation of motion from here or find the value of P in terms of W, we will see that now.

So first of all delta A or delta r A is 0 okay. Now B only moves in the horizontal direction. So I will write delta r B as i cap times times dx okay. And the coordinate of this point, B point let it be x 0. If the coordinate of point B is x 0 so point B is x, 0, what must be point C?

The coordinates of point C you can immediately see from the geometry of the figure that if this distance is d and this distance is also d then ABC forms a isosceles triangle and in that case the coordinates of point C will be x/2 and it will be square root of d square. So the vertical distance there would be d square – x/2 whole square. So we got delta B which is i cap times dx.

So the parenthesis notation here is the coordinates of point B or the r B vector is x i cap and here r C vector, so r B vector is x i cap and r C vector is x/2 times i cap plus to the power half j cap okay. And this gives you delta r B. The virtual displacement of point B is nothing but just delta x times i cap.

And delta r C it will be equal to dx/2 times i cap plus you would have –x/2 into d square minus whole square to the power -1/2 times j cap into dx. And now we will write down, put this input r B and r C back into this equation here.

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$$\vec{W}\cdot\delta\vec{r}_C + \bar{P}\cdot\delta\bar{r}_B = 0$$

$$\delta\bar{r}_A = 0, \quad \delta\bar{r}_B = \hat{i}\cdot dx$$

$$B: (x, 0) \qquad C: \left(x/2, \sqrt{d^2-(x/2)^2}\right)$$

$$\bar{r}_B = x\hat{i} \qquad \vec{r}_C = (x/2)\hat{i} + \left(d^2-(x/2)^2\right)^{1/2}\hat{j}$$

$$\delta\bar{r}_B = \delta x\,\hat{i} \qquad \delta\bar{r}_C = (dx/2)\hat{i} + (-x/2)\left(d^2-(x/2)^2\right)^{-1/2}dx\,\hat{j}$$

$$\bar{W} = W(-\hat{j}) \qquad \bar{P} = P(-\hat{i})$$

$$W\frac{x}{2}\left(d^2-\left(\frac{x}{2}\right)^2\right)^{1/2}dx + (-P\,dx) = 0$$

$$\boxed{P = \frac{1}{2}\cot\theta\, W}$$

And this gives us, so W times remember W is only in y direction. So I will put this as W vector is magnitude of W times – j cap vector and P vector is magnitude of P into – i cap vector. And then let us apply this. So this will become x/2 times W and then you have d square minus whole square plus say you have minus P times dx. And this must be equal to 0. And this of course gives you P in terms of W.

So if we use the angle here, call this angle as theta then P, magnitude of P will be equal to 1/2 cot theta times W, okay. There we go. So we have solved the problem of the static structure and calculated, this would be of course the standard aim of this problem that in this structure if I put certain amount of weight W what should be the force that I would apply on the block B, okay.

And that we have calculated here without referring to any of the constraint forces there, okay. So this is how in fact mechanical and civil engineers use the principle of virtual work. In the next section, I am going to extend this principle to dynamic systems where things are moving and can we find out the dynamical equations from such a extended principle of virtual work.