Theoretical Mechanics

Prof. Charudatt Kadolkar
Department of Physics
Indian Institute of Technology-Guwahati

Lecture - 01
Introduction, Constraints

Hi there. Welcome to this first week of our Theoretical Mechanics course.

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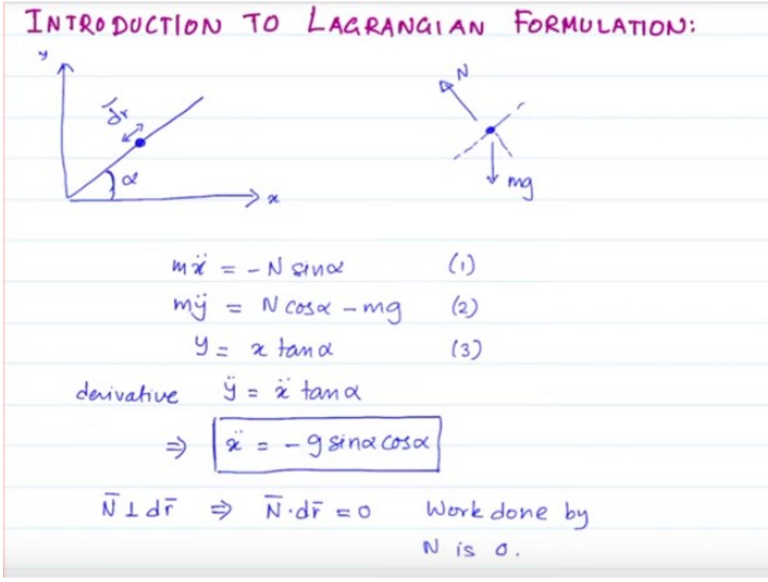

In this week we are going to learn about Lagrangian formulation. Lagrangian formulation is another or alternate formulation of Newtonian mechanics. Now, there are great many reasons to learn Lagrangian formulation. But in this week I am going to introduce the Lagrangian mechanics as a great problem solving tool. This is based on the D'Alembert's principle and in the next week we will derive the same Lagrangian formulation from the Hamilton's principle.

But this week we will follow the treatment which is given in Goldstein's classic book. So here we begin. First I will give you couple of examples to illustrate the need for Lagrangian formulation or the use of Lagrangian formulation as a great problem solving tool. So here let me take the first example. Consider a bead which is moving on a plane wire okay.

So a bead moving on a plane straight wire and this is moving along the wire without any friction. So let me draw a x axis and y axis. Let us say the wire makes an angle alpha with the x axis okay. We want to solve this problem using Newtonian mechanics. Now in Newtonian mechanics the first thing you would do is to draw a free body diagram. So for a free body diagram this is the bead. Then there is a weight.

This is due to gravity and then there is a normal reaction. This is due to the wire okay. And if I draw the normal to the wire there. Now, the next thing we do is write the equations of motion. So if I write this along x axis and along y axis then the x axis equation would be equal to mx double dot is equal to -N sin alpha and my double dot will be equal to N cos alpha – mg.

Let me call these two equations as equation 1 and equation 2. Now we cannot solve this equation because the normal reaction, we do not know this normal reaction a priori. So that means I have 2 equations and I have 3 unknown. So I have to determine x double dot and y double dot and N, all of them are unknowns. Wait, but there is something that happens here.

The wire or the normal reaction, the entire purpose of the normal reaction is to keep the bead on the wire okay. So I have one more equation now. The third equation is y coordinate must always be equal to x times tan alpha. I will call this as equation number 3. And now you see what happens. I have 3 unknowns and 3 equations. And then of course I can eliminate N from these equations.

And then I would get, so you can take a double derivative of equation 3. Take a derivative that is y double dot is also equal to x double dot tan alpha. And then if you eliminate N from here and also y dot from there then I would get finally one single equation which is x double dot is equal to - g sin alpha cos alpha okay. Now, see intuitively we always knew that this is a one dimensional motion.

Which means I really have only one equation of motion and which is what I have gotten here. However, to find this equation of motion I still have to write the two Newton's equations and a constraint equation. Now question that we will be asking is thus. Is it possible to write the same equation without referring to the normal reaction? Is there a prescription which does this and if you notice in this problem the normal reaction is always perpendicular to the motion of the bead.

So N is always perpendicular to the motion of the bead which I will show as dr vector because the dr vector is along the wire and the normal reaction is perpendicular to the wire which automatically means that the work done by normal reaction is zero. So work done by

N is always zero. Now I can use this fact to actually arrive at the same equation of motion by using the energy conservation.

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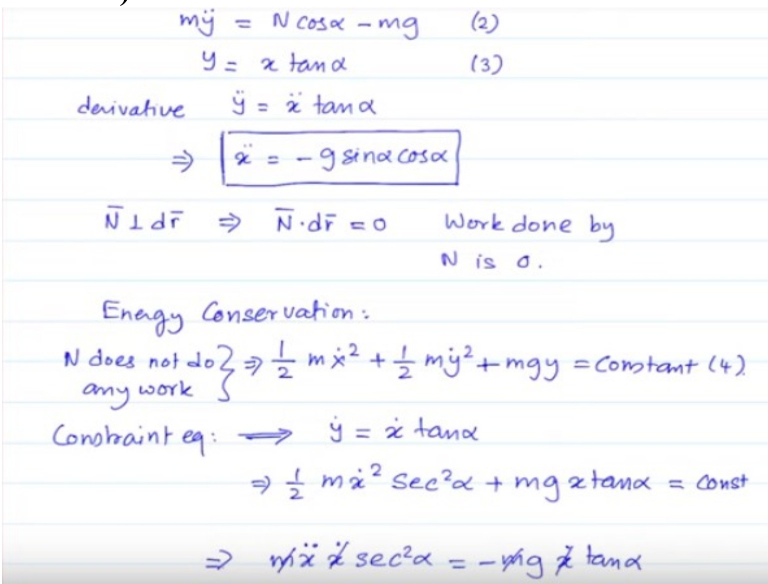

See in this problem, there are only two forces. One is a gravity and the other one is normal reaction. Now normal reaction does not do any work and the gravity is conservative. So I can of course write down the energy conservation formula immediately leads to, so 1/2 mx dot square + half 1/2 my dot square and + mgy, this must be equal to some constant, okay.

We will call this as total energy e and now I have equation number 3. So this is equation number 4. I can use equation number 3 which I also know a priori that y is equal to x tan alpha which means y dot is equal to x dot tan alpha. Now all that I have to do is substitute this in the equation above and that immediately gives me 1/2 m x dot square into sec square alpha + mgy sorry mgx tan alpha must be equal to constant.

And then we will take one derivative of this equation with respect to time and I get mx double dot sec square alpha is equal to – mgx dot tan alpha multiplied by x dot. And now if I cancel x dot from both sides and also m there we go. We get the equation that we had derived earlier. And while doing this what is it that we used. The first fact we used was the normal reaction does not do any work.

So that is the constraint force. I will call normal reaction as a constraint force and n does not do any work. This input goes into energy conservation statement here and the constraint equation which was equation number 3, that input goes here, okay. You used only these two facts and when you do this what happens is in this particular case we do not have to talk about the normal forces at all.

So we can completely bypass the notion of the restrictive forces or the constraint forces. I will take one more example which is similar to this.

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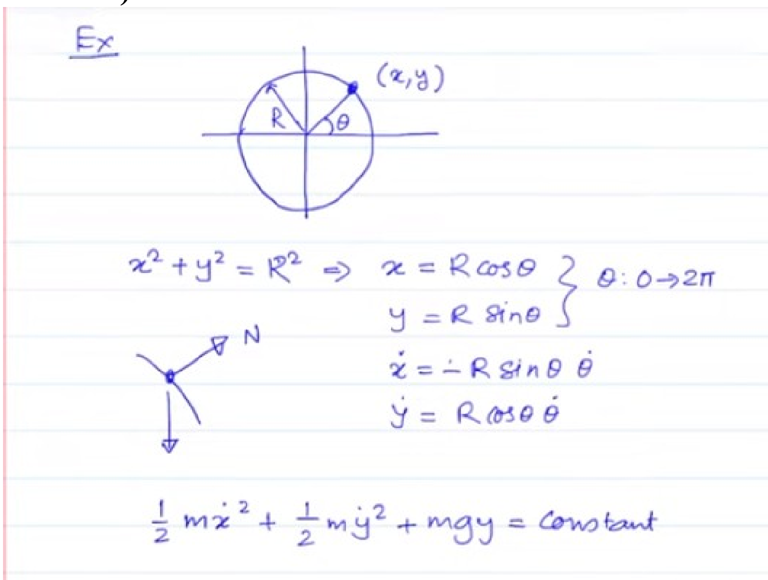

So in second example, consider exactly the same situation except that the wire here is now in a circular shape. So you have a wire here and you have a bead here and the radius of this circular hoop is R and the coordinates of this point are x and y and we will assume that the angle made here is given by theta, okay. Again, the constraint equation here or the restriction on the motion of the bead that it stays on the wire says that your x square plus y square must be equal to R square.

And we know when this happens we can immediately write down x as R cos theta and y as R sin theta and here of course theta changes from 0 to 2 pi. And now I will use the same idea. Notice that the normal reaction here, so the free body diagram is like this and then there is a gravity downwards and in normal reaction is perpendicular to the motion of the bead which means it does not do any work.

So we can use that as one fact. So the problem becomes conservative problem. The total kinetic energy plus the potential energy due to gravity will remain constant and that if I write here then we get 1/2 mx dot square + y dot square + mgy is some constant. And we are going to use the constraint equations which are now written in terms of the angle theta, okay.

So your x dot is nothing but – R sin theta, theta dot and y dot is R cos theta, theta dot.

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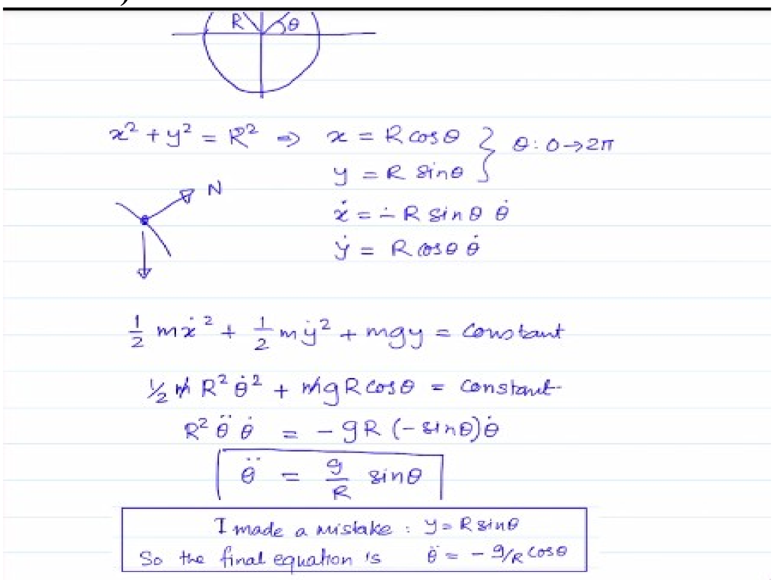

So substituting this back into this equation immediately gives me 1/2 mR square theta dot square + mg and y is nothing but R cos theta must be equal to constant. And now you take one more derivative so you can immediately cancel m from here, take one more derivative with respect to time and that immediately gives you R square theta dot into theta double dot is equal to – gR and – sin theta, theta dot, okay; so the angle theta there.

And this immediately gives us the equation of motion which is theta double dot is equal to g by R into sin theta. So see here too I have managed to get, again intuitively we know that this is a one-dimensional motion and we are going to have only one equation of motion and that equation of motion is written in terms of theta but while deriving this equation of motion I of course did not use the notion or idea of the normal force. We just used the fact that the normal forces do not do any work, okay.

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Ex Plane Polar co-ordinates (r, θ)

Cartesian (x, y)

$$m\ddot{x} = F_x$$
$$m\ddot{y} = F_y$$

Polar: $$m(\ddot{r} - r\dot{\theta}^2) = F_r$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta$$

Here is one more example. Now, we routinely use plane polar coordinates. When we use plane polar coordinates r and theta in place of the usual Cartesian coordinates x and y, so these are Cartesian coordinates. When we use that then in the Newton's equations the equation of motion for Cartesian coordinate is straightforward. It is simply m times x double dot must be equal to x component of the force.

And the second equation is m y double dot is y component of the force. However, when you change your coordinates to plane polar coordinates, the equation of motion do not look as simple as that. In fact, the equation of motion, so in polar coordinates it is m r double dot - r theta dot square is equal to r component of the force and the theta equation is m r theta double dot + twice r dot theta dot is F theta.

Here the equation of motion is not simply m r double dot is equal to r component of the force. And remember how do you get these two equations here. You start with your Cartesian components and then you find out r cap, theta cap. When you are taking derivatives you also have to remember to take the derivatives of r cap and theta cap.

And then with little bit of algebra we are arrive at these equations after taking two derivatives with respect to time. Now, every time we use a new coordinate system, you will have to do this. You will have to find out the unit vectors along those coordinates and then take two derivatives and arrive at this. The question we will be asking is this.

Do I have one single equation or one single procedure prescription which immediately gives me these equations or if I change to some arbitrary coordinate system does it give me very quickly the new equations of motion in those coordinates. And this is exactly what Lagrangian mechanics facilitates. So this is what we will do now but we will of course be deriving this from the D'Alembert's principle.

So before I go to Lagrangian formulation, there is considerable amount of background formulation to be done. So in next two sections we will first do that and then go over to the Lagrangian mechanics. Alright, in this section we are going to talk about constraints. By constraints what I mean is thus. See usually when we talk about mechanical system we are talking about a small collection of particles and not the entire universe.

And if the motion of these particles is restricted in space somehow we say that this mechanical system is constrained mechanical system. Now the constraints come in various forms and some of those are useful constraints, some of those are not useful constraints. When I say the word useful, what I mean is it is useful for solving problems or it is not useful for solving problems.

So in this section we will talk about different types of constraints and their classification. But before that we will of course start with few examples of the constraints.

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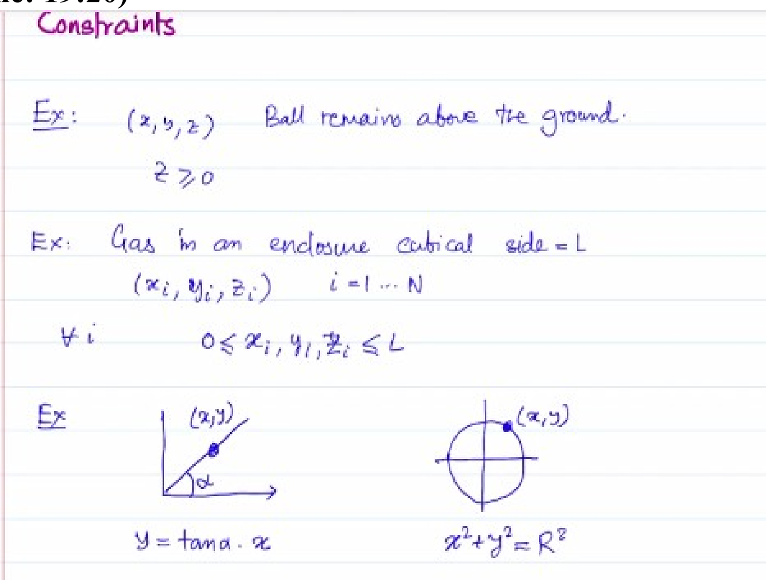

So here is the first example. When you play with a ball on the ground, then ball of course bounces off the ground. It does not vanish inside the ground. Now, if the ball has coordinates which are given by x, y, and z where the xy plane is your ground and z axis is vertically upwards. Then what is the constraint that z coordinate must always be greater than or equal to 0 because the ball remains above the ground.

Here is second example. Similar to this, if I look at a gas in an enclosure. The enclosure say is cubicle with side length equal to L and the coordinates of each of those molecules are given by x i, y i, and z i where i goes from 1 to N. That is the number of molecules in the system. Now what is the constraint on the coordinates now? Clearly for each particle, for each i, 0 must be less than x i or y i or z i and this must be less than or equal to L.

That it remains inside the box okay, each molecule remains inside the box. Here is one more example. This of course refers to the examples which we saw earlier. So in first example I had a bead which was traveling on a wire where the angle made by wire with the horizontal axis is given by alpha. Here the coordinates of the bead are x and y and the constraint and the coordinate is y must be equal to tan alpha times x.

Or the other example in which the bead was constrained to move on the surface of the circle. There the constraint on the coordinates was x square + y square must be equal to R square. That is the radius of the circular wire there. One more example.

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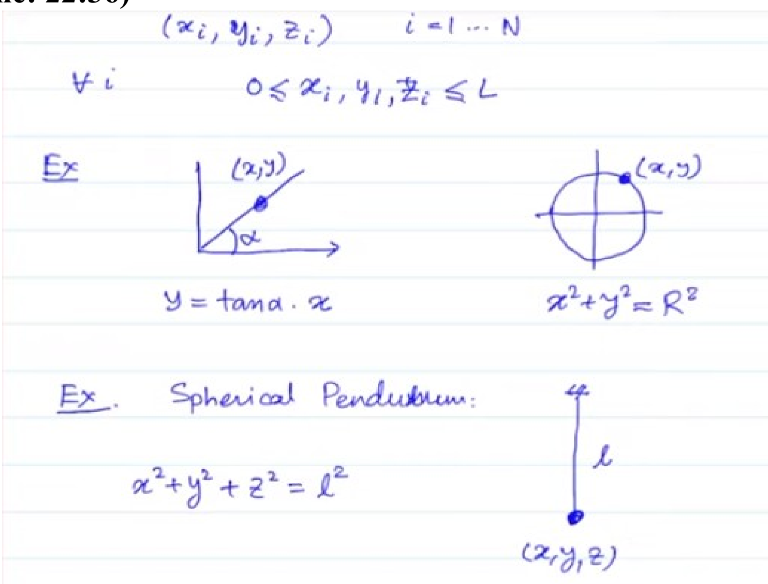

Think of a spherical pendulum. So this is a pendulum, spherical pendulum. This pendulum is suspended by one point and the massive bob is connected to the suspension point by a rigid but lightweight rod, okay. So there you have a support and the bob is hanging and this is the rigid rod there. In that case, the coordinates of the bob are given by x, y, z.

Then the constraint equation is x square + y square + z square must be equal to length square where l is the length of the pendulum.

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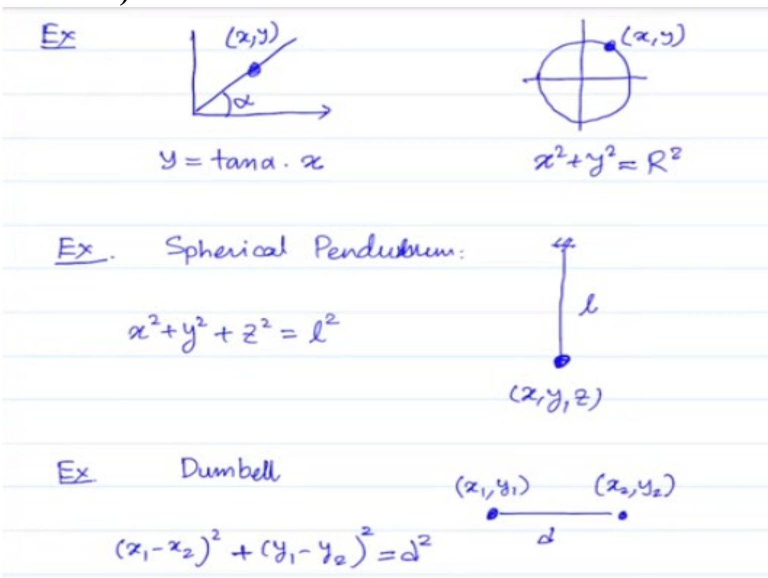

In the next example consider a case of dumbbell. The dumbbell has 2 masses and they are connected by a rigid rod, okay. So that the distance between the 2 dumbbells, that remains fixed.

Let us say that is d. and if the coordinates of the first mass are given by x 1, y 1, the coordinates of the second mass are given by x 2, y 2 then the constraint equation in this case becomes x 1 – x 2 whole square + y 1 – y 2 whole square must be equal to the distance square. This is the constraint on the coordinates of the 2 bobs there, the 2 masses there.

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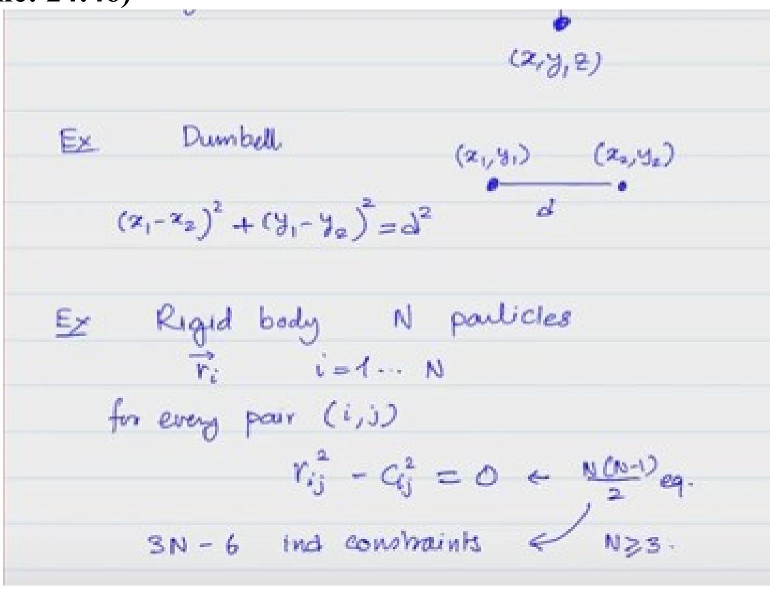

In the next classic example, let us consider a rigid body with N particles in it, okay. And let us assume that the coordinates or position vectors of this N particles are given by r i vector where I can go from 1 to N. So there are 3 N coordinates here because for each particle, there will be 3 coordinates. There are N particles, so there will be total of 3 N coordinates.

But the definition of rigid body says that for every pair, so for every pair of particles the distance between the two that is r i j square – C i j square must be equal to 0. That is the distance between particles for every pair remains same or is constant throughout the motion of the rigid body. Now how many such constraints are there? There are of course large number of constraints.

There will be N into N – 1 by 2 constraints here. Now, not all these constraints are independent of each other. It is very easy to see by a simple consideration of rotation of the rigid body that rigid body actually has only 6 independent degrees of freedom. Now, 3 of those can be assigned to the center of mass and the remaining 3 must be the orientation of the rigid body. So how many of these constraints are actually independent constraints.

Very few are actually independent constraints. Eventually, 3N – 6, these many independent constraints must be there. Remember there are N (N – 1) / 2 equations here. But out of these only these many are independent constraints. And this of course is true if N is greater than or equal to 3. Now, all the examples which we have taken here, the constraints they are independent of time. But that does not have to be the case.

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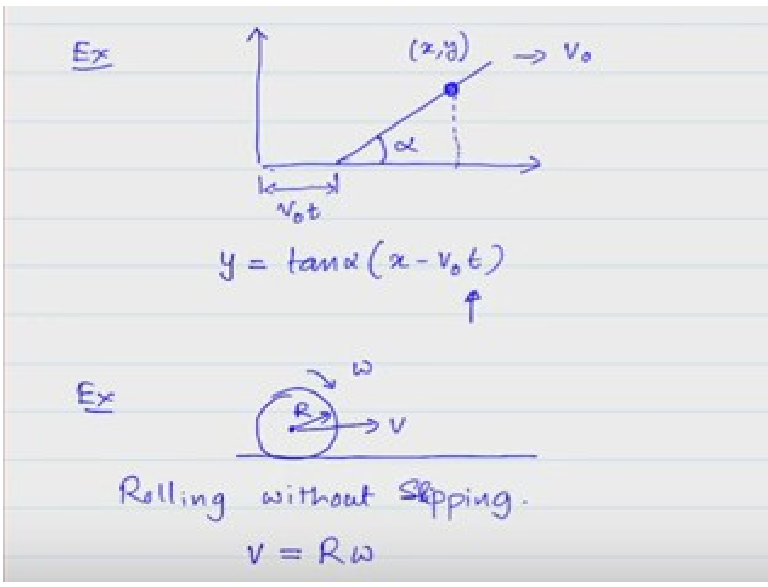

One more example here. In this example I will refer back to the first problem where you have this wire and a straight line wire. But now this wire is moving parallel to x axis with some constant velocity v naught by maintaining the angle that it makes with the x axis. Now what is the constraint? Suppose the bead is here. Its coordinates are given by x and y. Then we can immediately see if at t is equal to 0, the contact point was at the origin.

Then in time t the distance travelled here is v naught times t. And that immediately gives you the constraint between y coordinate and the x coordinate which is y must be equal to tan alpha times x minus v naught t. So if I complete this triangle here then you can immediately see the constraint there. Now, the constraint equation here involves time explicitly. And these are not the only kind of constraints you have.

Many times in mechanical systems you have constraints which are velocity dependent. For example consider a disc rolling on a single line, okay. The disc of radius R and this is rolling without a slip, without slipping. What is the constraint equation here? So if the angular speed is omega and the center of mass velocity is v, then v must be equal to R times omega.

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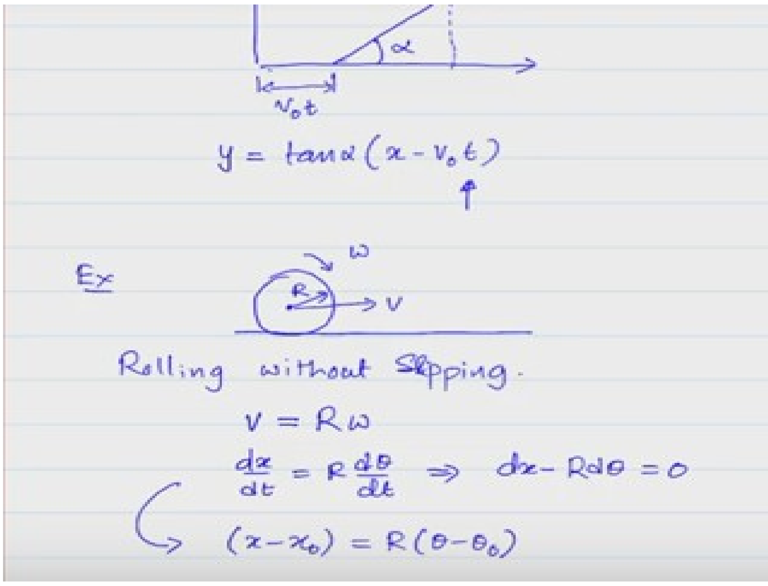

Now if I were using the x coordinate for the motion of center of mass and theta coordinate for the amount of rotation of the disc then this is same as dx/dt = R times d theta/dt and we will write this constraint in a slightly suggestive form. This is dx – R d theta must be equal to 0. But we can immediately see that I can actually integrate this equation very easily.

And if I integrate this equation, I get the equation which is not in terms of differentials but it is in terms of the actual coordinates. So after integrating this we can immediately see x – x naught must be equal to R minus sorry R (theta - theta naught) for some constants x naught and theta naught which we can find out from the initial conditions. So this constraint even though when it was post was in terms of differentials but we can integrate this into an equation between the coordinates themselves.

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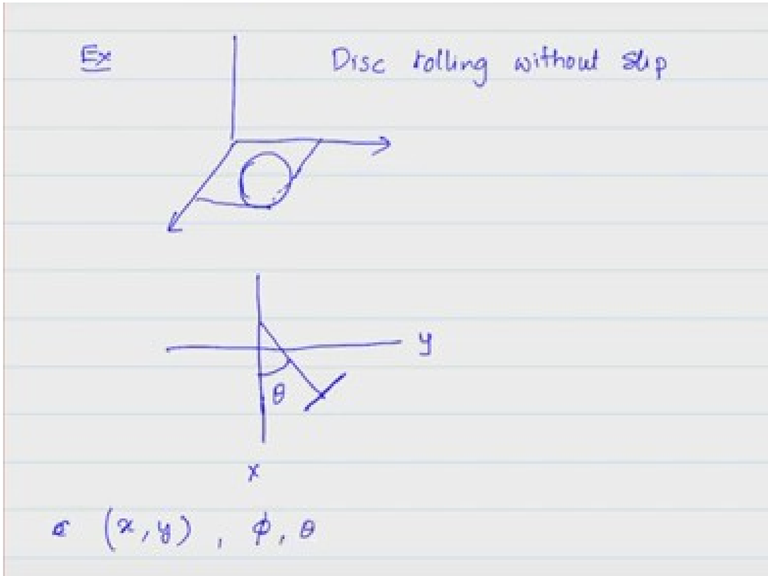

One more example, now in this example what we are going to do is this. We have this xy plane and in the xy plane we have this disc which has a contact point with the xy plane and the disc rolls on the xy plane while remaining vertical but rolls without slipping. So again the same problem as before. Disc rolling without slip, okay. Now, how many coordinates are needed first of all?

If I look at this from the top view then you have x axis, y axis, z axis now is pointing out of the paper or out of the board and say you have the disc here, okay. And the normal of the disc suppose makes an angle of theta with x axis and the rotation of the disc is measured in terms of angle phi. So you have the contact point which are given by x and y. then the amount of rotation which is given by angle phi and the orientation of the disc with respect to xy axis is given by the angle theta there, okay. And what are the constraints here?

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$$\Rightarrow \quad v = R\, d\phi/dt$$

$$v_x = V \sin\theta$$

$$v_y = V \cos\theta$$

$$\Rightarrow \quad dx = R \sin\theta\, d\phi$$

$$dy = R \cos\theta\, d\phi$$

Can we get $f(x, y, \theta, \varphi) = 0 \Leftarrow$ not possible.

The constraint equations would be the velocity must be equal to R that is the radius of the disc times d phi/d t, okay and also what you must have is v x is V. So v x is pointing that way v x will be V sin theta and v y is V cos theta. Now if we look carefully I can substitute this v back into the other 2 equations and you would get the differential, the constraint in terms of differential which will be d x = R sin theta d phi and d y = R cos theta d phi.

Now question we are going to ask is thus. I now have 2 differentials or 2 equations in terms of differentials. Can I integrate these 2 equations to get an equation of this find? Can we get some function f which is function of x, y, theta and phi is equal to 0. Can we find such a function f which also means finding integrating factor in this particular system and it is actually very easy to see that, that cannot be done here.

There is no way if you go back to the diagram there then you can see that you can go to another configuration here, this one, by either going this way or you could actually take circular path this way and in both cases, you would reach the same angle theta, same contact point x y but the value of phi would be different depending on which way you actually travel to that point.

Which automatically tells you that finding such an equation is not possible, okay. This is not possible. So this is constraint in the form of differentials but which cannot be integrated into a nice form there. So let me summarize the what we have done here.

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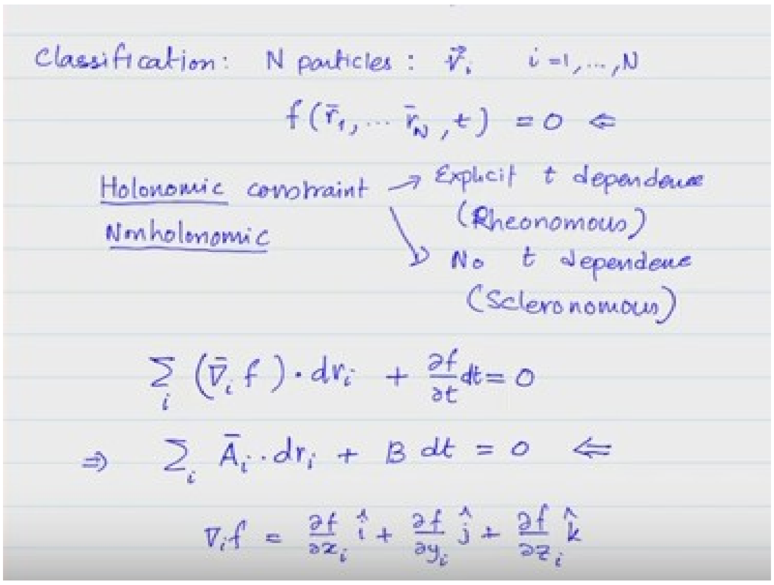

So, the classification. First of all we will define constraints. Some of these constraints are in neat form. These are in the form of so you start with system with N particles with their position vectors given by r i where i goes from 1 to capital N, okay. And these constraint equations are actually some sort of function. So except for the last example in all sorry except for the last example and the first two examples, the first two examples were in the form of inequalities.

The last one is in the form of differentials which cannot be integrated to this nice form. So r 1 to r N and possibly also time t is equal to 0. If our constraint is in this equation form that is some function of r 1 to r N and time t is equal to 0 then we say that this is a holonomic constraint and every constraint which is not of this form is nonholonomic, nonholonomic constraint. Okay, so the first classification separates constraints into 2 parts.

One are holonomic constraints, the other one is nonholonomic constraint. Now among holonomic constraints some constraints explicitly depend on time. So explicit t dependence. These are called as rheonomous. So these constraints are called as rheonomous constraints and the ones which do not, no t dependence, these are called as scleronomous, okay.

Now, look the nonholonomic constraints, the ones which are in the form of inequalities they are of course cannot be, they cannot be used for problem solving in any way. But the ones which are shown in the last example which were in the form of differentials or we can use that. Because you see the holonomic constraint here itself can be put in a differential form. So I can take a derivative of this and write the differentials.

So I can write this as gradient of f doted with d r i and summed over i + del f/del t. This must be equal to 0. This is just taking a differential of this equation. And I will write this as sum over i, now this del i here is nothing but del f/del x with respect to i times i cap plus j cap plus into k cap. So this is just a gradient vector but taken with respect to the position vector or r i instead of general r.

So here I will write this as a vector A i dot d r i plus sorry I forgot to write dt here. So I will write that now, del f/del t dt and I will also replace this del f/del t by another symbol which I will call it as B dt = 0, okay. So now you see we can have nonholonomic constraint which also have similar form along with holonomic constraint. Of course not all nonholonomic constraints. But some nonholonomic constraints can be put in this form.

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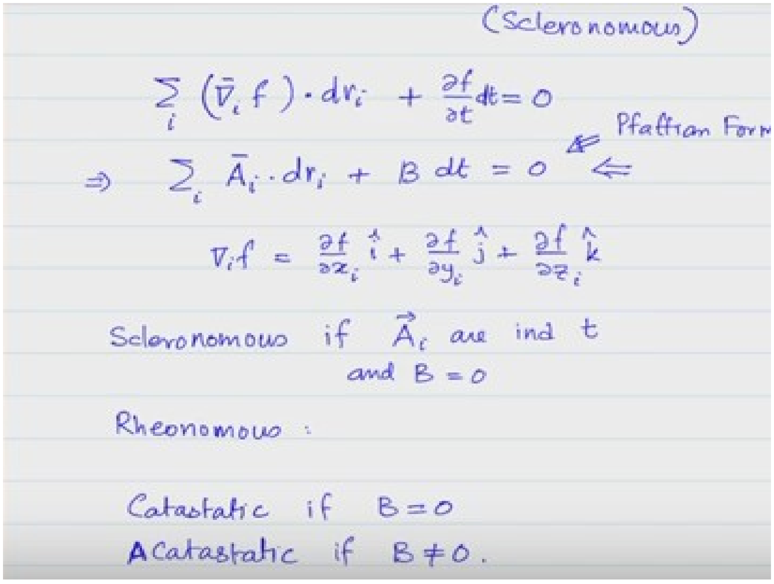

Now, the categorization that the constraint is scleronomous if all A i vectors are independent of time t and B = 0. If that is not the case then you have rheonomous constraint. In addition to this, another categorization is made where you say a system is catastatic if B = 0 and acatastatic otherwise, okay and each of these categorizations or classifications can be used while solving the problems.

So this summarizes the entire or in this particular course what I am going to do is I am going to restrict ourselves to the constraints of this form. This contains both holonomic as well as nonholonomic constraints. Wherever required we will make the distinction between the two. Otherwise, this form in general is called as pfaffian form, okay. Now, how do constraints affect the description of motion is what we are going to see in the next section.