Numerical Methods and Simulation Techniques for Scientists And Engineers

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Lecture 16
Runge Kutta Method

So, we are going to study the Runge-Kutta methods it is popularly called as a RK method. And this is one of the most popular methods of solving a differential equation.

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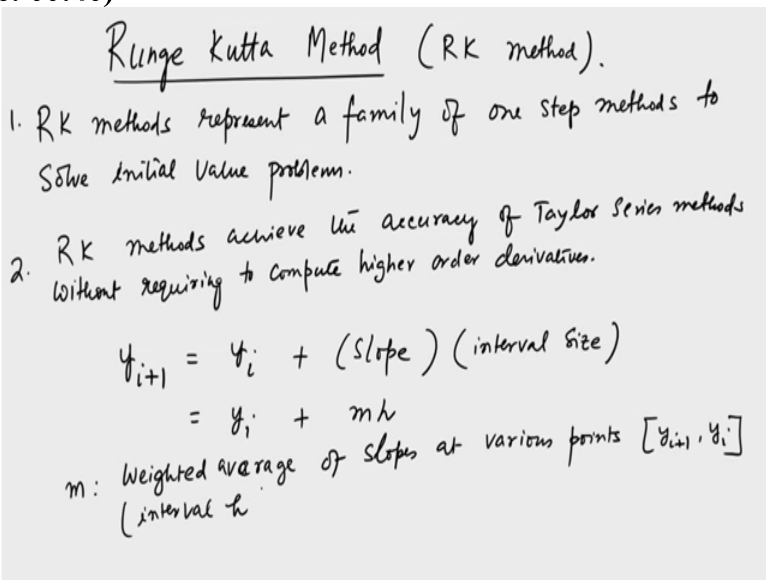

So, this is how it is written it is Runge and Kutta method okay and in short we call it as a RK method. So, RK methods represent a family of one step methods to solve the initial value problems. So, this is one important thing and the other important thing is that these are key methods achieve the accuracy of Taylor series expansion. We have seen that the Taylor series expansion give solutions of these differential equations of these initial value problems.

Taylor series methods without requiring to compute higher-order derivatives this is an important step because computing higher-order derivatives in a computer is actually introduces the process introduces a lot of error and it is if one can avoid it is best to avoid it. So, this method also gives you a meth we achieve the same accuracy of these Taylor series expansion without you know we do not have to calculate the higher-order derivatives.

In fact is the first order derivative that is that is important and sometimes you know the second order derivatives are needed. So, how do we calculate the solution of an initial value problem or a differential equation using this RK method. The formula is simple it is it is an iterative solution as we have seen earlier so this is equal to y i + a slope I will simply write that as a slope. We will see where the slope is calculated and this is an interval size okay.

And pretty much this is the formula that we have seen for calculating the solution of these initial value problems via Euler method or by the Heun's method that we have seen it is accepting that where you calculate the slope is important okay. And this slope let us call this as y i equal + m h and this m is the slope calculated at several points between the initial point i and the final point i + 1 but not only that this m consists of different weights of the slope taken at different points.

And these weights have to be decided a based on certain considerations which we are going to see here okay. So, m importantly is the weighted average of slope slopes are at various points in the interval y i + 1 and y i that is in the interval h.

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$$m = w_1 m_1 + w_2 m_2 + \ldots\ldots w_r m_r$$

w_i : Weights of the slopes at various points.

m_i are computed as follows.

$$m_1 = f(x_i, y_i)$$

$$m_2 = f(x_i + a_1 h,\ y_i + b_{11} m_1 h)$$

$$m_3 = f(x_i + a_2 h,\ y_i + b_{21} m_1 h + b_{22} m_2 h).$$

$$\vdots$$

$$m_r = f(x_i + a_{r-1} h,\ y_1 + b_{(r-1)1} m_1 h + \ldots\ldots b_{(r-1)(r-1)} m_{r-1} h$$

So, let us see that how you know this weights are calculated. So, this is in fact a one should not write it y but rather it should be written as x these are the points at which these are the slopes are calculated. So, this is on the x-axis alright so so we write this weighted slope as w 1 m 1 + w 2 m 2 + and so on and w r m r assuming that there are r points between x i + 1 and x i and each of width h I mean these the width between or the distance between the x i + 1 and x i is h.

So, w i denotes weights of the slopes at various points and m i's these m i's are computed as follows so m1 that the first slope is nothing but the slope at the initial point which is the function itself m 2 importantly it uses the slope m 1 it is an x i + a1 h and y i + a b11 m 1 h will specify what this a1 and b11 are and so on and this m 3 is the value of the function + a a 2 h and y i + b21 m 1 h + a b22 m 2h.

So the slope at the slope m 2 uses the value of the slope m 1 which is the value of the function and similarly the slope m 3 uses the value of m 1 and m 2 both and so on so there is a you can write it for the r m r which is our h slope which is equal to x i + a r - 1 h and there is a y 1 + a b r – 1,1 if you allow me to put a comma here because otherwise ok we can avoid putting that comma by just putting this into the inside a bracket.

So b r - 11 which is that 2 1 here and that is a m 1 plus and so on and then there is a b r - 1 r - 1 m r - 1 h and so on. So, it uses all the previous slopes that are computed and if you wish to write it like this you can write it in a compact form let me write that.

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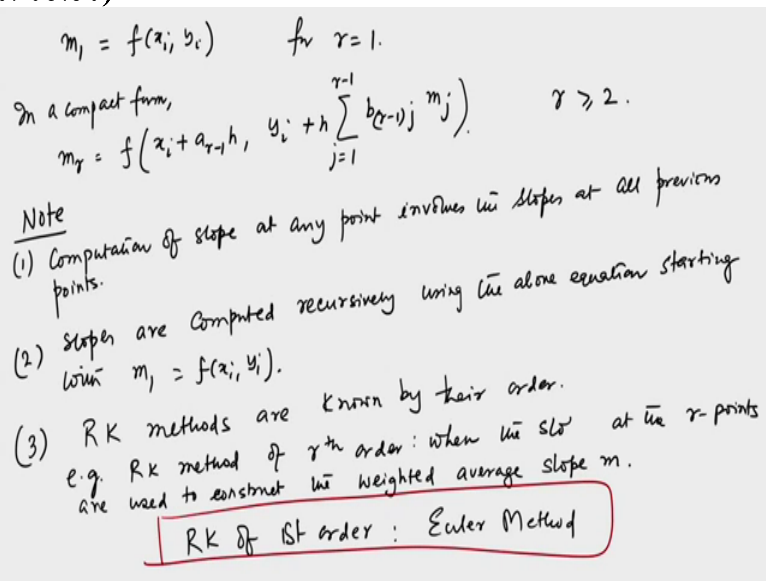

So, just you know once again writing m 1 so this is for r equal to 1 and m r in a compact form m r can be written as f x i + a r + 1 h and then a y i + h and there is a sum and there is a b r - 1 j m j j starts from 1 and goes up to r - 1 and this is the rh slope for r to be greater than equal to 2 so 2 onwards you can write this. So, note a few points may be noted here that one is that the computation of slope at any point involves the slopes at all previous points.

So, this is the difference between the Euler method or the Huen's method where you have chosen certain points and have calculated the value of the slope at those discrete points and they have no relationship with each other I mean maybe the you require some data but they are sort of independent where here the slopes in this RK method the slopes are all interlinked which means that slope at the rth point requires the value of the slopes at all r - 1 points that are there before ok.

And this slopes are computed recursively using the above equation starting with m1 which is equal to nothing but the slope at the initial point okay. And third the RK methods are known by their order we will soon see what it means that is there could be a second order RK method. The first order RK method is by default the Euler method. So, there could be a second order or there could be a third order or there could be a fourth order or even higher orders of RK method.

So, this is for example RK method of rth order when the slopes the slopes at the r points are used to construct the weighted average slope m okay. So, this is a the thing that we need to know about this computation of this RK method. And so as I said earlier that importantly I mean RK method of first order is called the Euler method you need to convince yourself but it is very easy. So, let me box this, so we will actually talk about RK methods of second order onwards.

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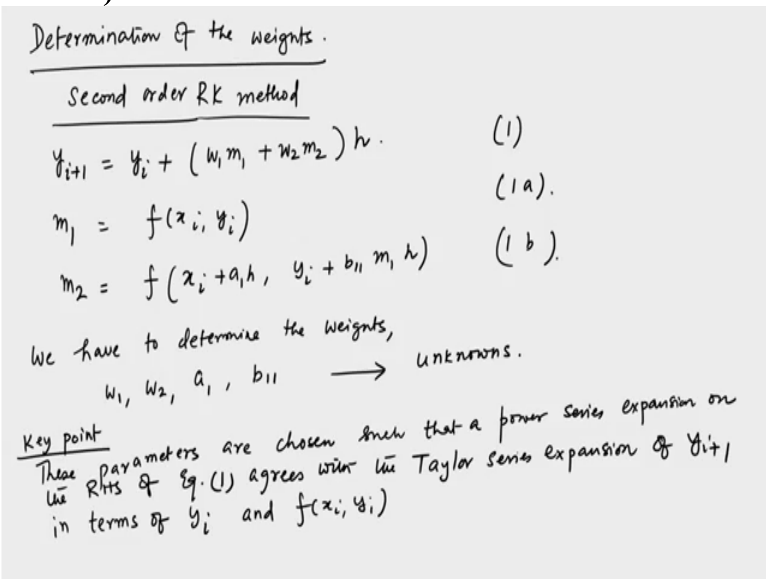

Now talk about determination of the weights which is the most important thing here okay. So, we need to calculate these a 1 a 2 b11 b12 etc and hence that those will help us to calculate the m 1 and m 2 and so on. So, basically we need to calculate the weights at different points and the number of points you choose that determines the order of the RK method. So, in that interval age if I choose a say a set of 3 points where I would calculate the slope then it is called as the third order RK method.

If we choose that number to be 6 then we will get a 6th order RK method okay. So, it depends on the number of points all right. So, let us talk about just a second order to begin with we will of course generalize it to higher orders or at least do third and fourth order. So, specifically let us talk about the second order for ease in derivation. So, second order RK method all right, so one has a y i + 1 equal to y i + w 1 m 1 + w 2 m 2 and h ok.

And so that is the two slopes at the two points that we have decided to compute m 1 is easy which is just the given problem itself that is your y prime which is nothing but if x i y i that is at the initial points. Let us call this as 1a and m 2 is f x i + a 1h we have written that earlier y i + b

11 m 1h and let us call it as 1b okay. So, we have to determine the weights so how many unknowns are there you have a w1 unknown under w 2 unknown a one unknown and a b11 unknown okay.

So these are unknowns, so the key method so or rather the key point here that we that the RK method uses is that these parameters what I mean is that these w 1 w 2 a 1 and b11 these parameters are chosen such that a power series expansion on the right-hand side of equation 1a power series expansion matches with this with this one equation 1 the right-hand side of equation 1 let us see what that is so agrees with the Taylor series expansion.

So, basically what is written here has to match with the Taylor series expansion of y i + 1 in terms of y i and f of x i y i ok. So, this is very important and this is how the derivation proceeds or rather the identification of the values of w 1 w 2 a 1 and b11 that will have to proceed by rather comparing or matching this equation 1 to the Taylor series expansion of y i + 1 in terms of y i and the slope.

(Refer Slide Time: 19:03)

Second order Taylor Series expansion,

$$y_{i+1} = y_i + y_i' h + \frac{y_i''}{2!} h^2 \qquad (2).$$

Given $y_i' = f(x_i, y_i) = f$

$$y_i'' = \frac{df}{dx} = f_x + f_y f$$

Putting in eq. 2

$$y_{i+1} = y_i + \underbrace{fh + (f_x + f_y f)\frac{h^2}{2}}_{(w_1 m_1 + w_2 m_2) h.} \qquad (3).$$

From eq. 1(b) Taylor Series expansion about (x_i, y_i).

$$m_2 = f(x_i + a_1 h, y_i + b_{11} m_1 h)$$

$$= f(x_i, y_i) + a_1 h f_x + b_{11} m_1 h f_y + O(h^2) \qquad (4).$$

And we know pretty well what that is so this is that the second order Taylor series expansion it is like y i + 1 it is equal to y i + y i prime h + a y i double prime by 2 factorial h square, let us call this as equation number 2 so this is given y i prime is nothing but f x i y i and for convenience let us write that simply as f and now we have y i double prime which we would write it as df dx so this is a function of both x and y so we will take a derivative with respect to x and also there will be a partial derivative with respect to y and so on ok.

So, this notation we are very familiar with and this is the double derivative which is fx + fy and value of the function so if we put all these things in 2 let us write putting in 2 so y i + 1 equal to y i + f of f h+ f of x + fy f h square by 2 okay let us call that 3. Now if you compare the right-hand side of 3 and the right-hand side of 1 let us just take you back to 1 so that is w 1 m 1 and w2 m 2 into h, so of course there is a y1 there y i there.

So we have a y i and then all these things so this must be that w 1 m 1 + w 2 m 2 + h because that is a expansion the power series expansion and this one is actually the Taylor series expansion taken up to the 2nd order. So, if we actually look at that slope m 2 so from equation 1b, m 2 becomes is m 2 is given as just to remind ourselves it is a 1h and y i + b11 b11 m 1 h and let us just call it as 4 if I expand this m 2 about x i y i of this do a Taylor series expansion I will get a f x i y i that is the first term.

And then it is a a 1h and then there is a del f del x which is we write it as fx + b11 m 1 h f y and + order h square ok so I do a Taylor series expansion about x i y i okay. So, that is your m 2 and if you substitute that let us just write this as equation 4 because this that was already written there so if I substitute equation 4 in 1 and replacing of course your m1 equal to f m 1 is nothing but f.

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Putting (4) in (1).

$$y_{i+1} = y_i + \left[w_1 f + w_2 f + w_2 a_1 h f_x + w_2 b_{11} f f_y \right] h + O(h^3).$$

$$= y_i + (w_1 + w_2) f h + (w_2 a_1 f_x + w_2 b_{11} f f_y) h^2 + O(h^3). \quad (5)$$

Comparing the RHS of (3) and (5), we can write,

$$\left.\begin{aligned} w_1 + w_2 &= 1 \\ w_2 a_1 &= \tfrac{1}{2} \\ w_2 b_{11} &= \tfrac{1}{2} \end{aligned}\right\} \begin{array}{l} 4 \text{ unknowns, } w_1, w_2, a_1, b_{11} \\ 3 \text{ equations} \end{array}$$

So, this is so putting 4 in 1 we get y i + 1 equal to y i + there is a w1 f remember m 1 is nothing but f a w 2 f + a w-2 a 1 h fx + w 2 b11 f which is m 1 and f y h and we are missing out terms which are of the order of h cube. If I rearrange this a little bit then this becomes equal to w 1 + w 2 f + sorry there is h also and there is a w 2 a 1 fx + w 2 b11 f fy into h square and then of course we am skipping terms from h cube onwards.

So, let us call it as equation 5 and if you compare the right hand sides of 3 and 5 then we can write so w 1 so you see that f h comes with a weight one okay so there is one here so we will call it so w 1 + w 2 which is the weight of fh term its equal to 1 now you see that h square comes with the fx + fy f ok fx + fy f divided by 2 is the h square term so h square term is this so

I can see that the fx so w 2 a 1 it is equal to half because there is a factor of 2 there in equation 3.

So, this w 2 a 1 is equal to 1/2 and similarly w 2 b11 is also equal to 1/2 because that is the coefficient of ffy all these are fine but then you land up with a problem that you have 4 unknowns namely a 1 a 2 sorry let us write it as the ones that are there before so w 1 w 2 and a 1 and b11 okay and only 3 equations. So, an unambiguous solution is not possible but then even if we do not get an unambiguous solution we get a family of solutions and this is what makes this method very powerful.

And one can actually look for different alternatives among the family of solutions in order to get the best solution or rather the ones that has the minimum error okay. So, one choice first choice say oh let us just do it in a different page,

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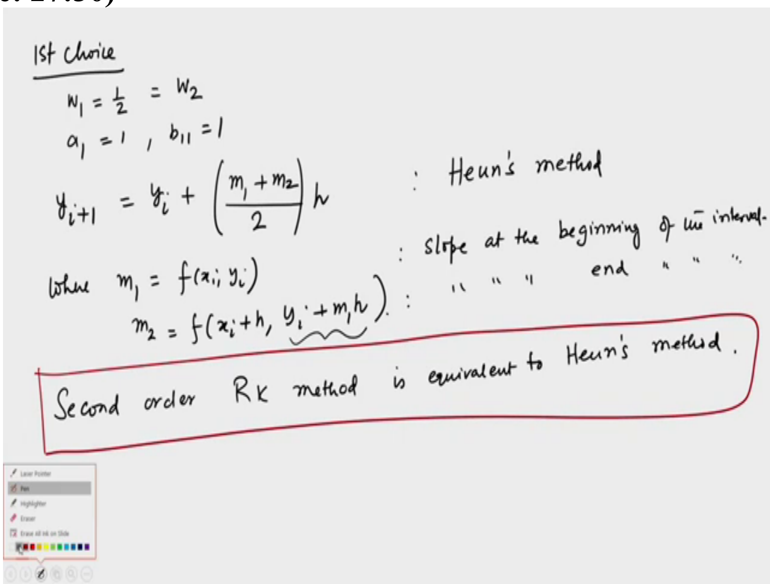

First choice and the first choice we can take as w 1 equal to half equal to w 2 and a 1 equal to 1 and b11 or equal to 1 okay. So, w 1 and w 2 to be 1/2 and a 1 equal to 1 and b11equal to 1 see what it gives so we get a y i + 1 equal to y i and it is a m 1 + m 2 divided by h I am so sorry m 1 by m 1 + m 2 divided by 2 into h if you recall this is nothing but the Heun's method where we have taken two points and the two extremities and taken their weights to be equal to half okay.

So we have taken the initial point i and the final point i + 1 calculated the slopes the slopes are called as m 1 and m 2 and while you iterate the solution the slope that is taken the effective slope that is taken is m 1 + m 2 divided by 2 okay we are of course just to remind you that this is equal to x i y i and m 2 requires the slope at a requires the slope of of m 1, I mean this low evaluated at the point x i y i and so this is at this point at the end point and the slope is needed here okay.

So, we need to calculate this value and put it back and calculate the slope etc and so on. So, this is the slope at the beginning of the interval and this is the slope at the end of the interval okay. And so we can make a comment that the second order RK method is Heun's method basically without the iteration, so if that is the thing let us look at one more choice.

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2nd choice.

$w_1 = 0, \quad w_2 = 1, \quad a_1 = \frac{1}{2}, \quad b_{11} = \frac{1}{2}.$

$y_{i+1} = y_i + m_2 h.$

$m_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_1}{2} h\right).$: Mid point method.

$m_1 = f(x_i, y_i)$

3rd choice.

$w_1 = \frac{1}{3}, \quad w_2 = \frac{2}{3}, \quad a_1 = \frac{1}{3}, \quad b_{11} = \frac{3}{4}$

$y_{i+1} = y_i + \left(\frac{1}{3} m_1 + \frac{2}{3} m_2\right) h.$

$m_1 = f(x_i, y_i).$

$m_2 = f\left(x_i + \frac{3}{4} h, y_i + \frac{3}{4} m_1 h\right).$

So let us take w 1 equal to 0 w 2 equal to 1 a 1 equal to 1/2 and b11 equal to 1/2 so then the second-order method gives y i + 1 equal to y i + m 2h and the m 2 is actually x i + h over 2 y 1 + m 1 by 2 into h ok. So, what it says is that it is called as the midpoint method. So, this we directly calculate a slope m 2 with m 1 being equal to so of course m 1 is nothing but equal to f x i y i. So, by calculating the slope at the midpoint we iterate the solution.

So we do not need to calculate it at the two extremities we can only calculate at the midpoint but for knowing the midpoint we need the two points at the extremities and so on. Let us take a third choice it here itself okay, this is called as a midpoint method and this third choice which we are going to do is called as a Ralston's method so let us see what that is. So, one can choose a w 1 equal to one-third a w 2 equal to 2/3.

So it is a and which fixes this a 1 equal to one-third and a b11 equal to 3, so w 2 being equal to 2/3 so for this will have to be 1/2 and this will have to be a 3/4 okay. So, b11 equal to 3/4 okay and this is known as Ralston's method let us see what the solution that I get is y i + 1 is equal to y i + a 1/3 m1 + a 2/3 m 2. So, it still is add the at 2 points but the slopes the weights have not been taken to be equal rather they have taken to be in the according to weights 1/3 and 2/3.

And so m 1 equal to again x i y i and m 2 is equal to f x i + a 3/4 h and y i + a 3/4 m1 h and so on okay. Now one can you know see that whether the first choice or the second choice or the third choice they give correct results and comparison necessarily needs to be done in order to understand that which of the rather which member of the family of RK method second order RK method methods give you the correct solution.

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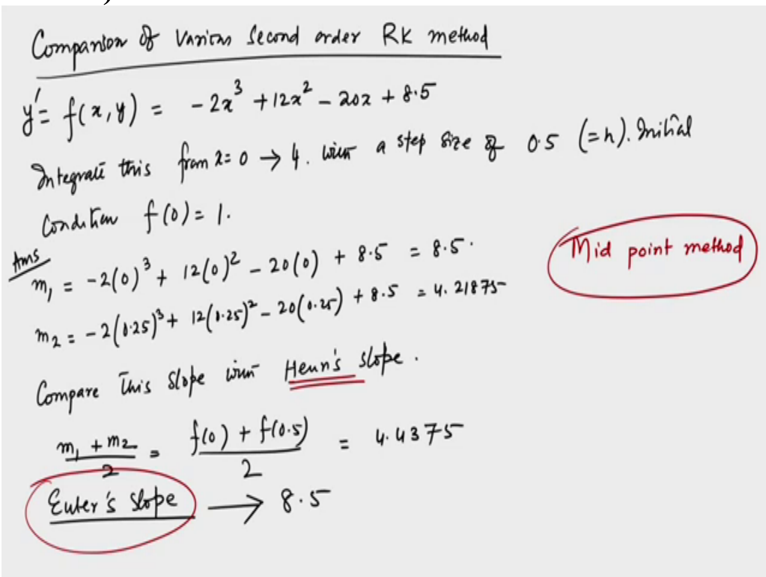

So, let us do the comparison of various and let us take this example so that f x we can write it y also but this is only a function of x + 12x square - 20x + 8.5 so you integrate this so what it means is that your y prime or dy dx equal to this so from 0 to say 2 for example or 4 whatever we are anyway not going to do that but we will show it for one step and then you can repeat the calculation. So, integrate this from x equal to 0 to 4 with a step size of 0.5 h.

So, this is equal to h and the initial condition being f of 0 is equal to 1 okay. So, let us do the midpoint method or rather this you know the Heun's method say for example and that is equal to so m 1 at the first this thing which is at x equal to 0 is - 2 into 0 cube + 12 into 0 square so you put 0 because 0 is the initial point and m1 is the slope at the initial point. So, we are giving the answer here 20 into 0 + 8.5 so this is equal to 8.5. so m1 equal to 8.5 so m2 which is to be calculated at the midpoint okay.

There is a midpoint method that we are talking about so this is let us just write it here and of course you can also do Heun which is easy to do so this is the midpoint method and so m 2 is equal to - 2 and we have to calculate it between 0 and 0.5 that is the interval so we will calculate it at 0.25, so this is 0.25 cubed + 12 0.25 square - at 20 0.25 + 8.5 that is equal to a 4.21875, so I get the slope so if we want to use the Heun's method then we will take the slope to be f m 1 + m 2 divided by 2.

So, average slope if you want which of course refers to the Heun's method which is equal to m 1 + so well I mean in the sense that, so let if you compare this slope itself with Heun which is nothing but m 1 + m 2 by 2 which is equal to f 0 +f of 0.5 by 2 so instead of 0.25 here you can put a .5 and then take a half of the slope, so this becomes equal to if you calculate it it becomes equal to 4.4375 and the slope of course in the midpoint method is different so this is the Heun slope ok.

And the slope becoming different the values will start getting different and of course the Euler slope to, Euler slope this is nothing but 8.5 okay. So, the Euler slope is very large whereas these two slopes the human slope or the midpoint slopes are somewhat having similar magnitude all right. So, what we have to do is that we have to suppose we are doing the midpoint method we have to substitute the slope into the iterating equation.

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Substituting the slope.

$$y(0.5) = 1 + (4.21875)(0.5) = 3.109375.$$

from $x=0$ to $x=2$

$$m_1 = 3.109375$$

$$m_2 = -2(0.75)^3 + 12(0.75)^2 - 20(0.75) + 8.5$$

$y(1)$ and so on.....

Ralston's Method

$$m_1 = 8.5$$

$$m_2 = -2(0.375)^3 + 12(0.375)^2 - 20(0.375) + 8.5$$

$$= 2.582031$$

Average slope $m = \frac{1}{3}(8.5) + \frac{2}{3}(2.582031) = 3.277344$

So, the value at 0.5 is 1.421875 and then one can multiply it by 0.5 one would get a 3.109375 okay. So, we got the value of y that is the solution at the first at the end of the first interval as 3.109375 all right. So, now we will take this value at this beginning of the interval so that is now the m 1 for the next interval so m one would be 3.109375, now we will have to calculate m 2, m 2 is again at the middle point of the next interval which is between 0.5 to 1.

So, we have to take a .75 and then calculate this here alright. So, if we use both then we can calculate y at one and then continue for the next one and so on, I think I have given you the hint how to calculate using the second order RK method with the midpoint have not discussed this Heun's method but you can also go and do the Heun's method just that we have to calculate the slopes at the beginning point and at the end point.

And then each will have a weight of half so it is m 1 + m 2 divided by 2. So, now let us come to the Ralston Smith, so m1 is calculated as 8.5 m2 now has to be calculated at the 1/3 point or rather the 3/4 point so if you divide your 0.5 into 4 equal intervals and then you take 3 of that which is .375 so it is a - I mean this is 0.375 whole cube + 12 into 0.375 - 20 into 0.375 so this square and so on.

And 8.5 so this becomes equal to 2.582031 so you have m 1 equal to this and m 2 equal to this 2.582031, so the average slope or rather the effective slope whatever you want to call this is equal to a 1/3 of 8.5 n + 2/3 of 2.582031and this is nothing but a 3.277344, so you see that we are the Heun's slope is 4.4375 the midpoint slope is 4.21875 and this is 3.277344. So, of course these are values the slope values differ and hence the final results will also differ I not proceed.

But you finish this at least do it till if you want from x equal to 0 to x equal to two and you can take the interval to be 0.5 and then at do a comparison of these different second-order methods for you to be convinced that among the family there would be some methods which are more efficient and will inflict less error in the final calculation of the initial value problem all right.

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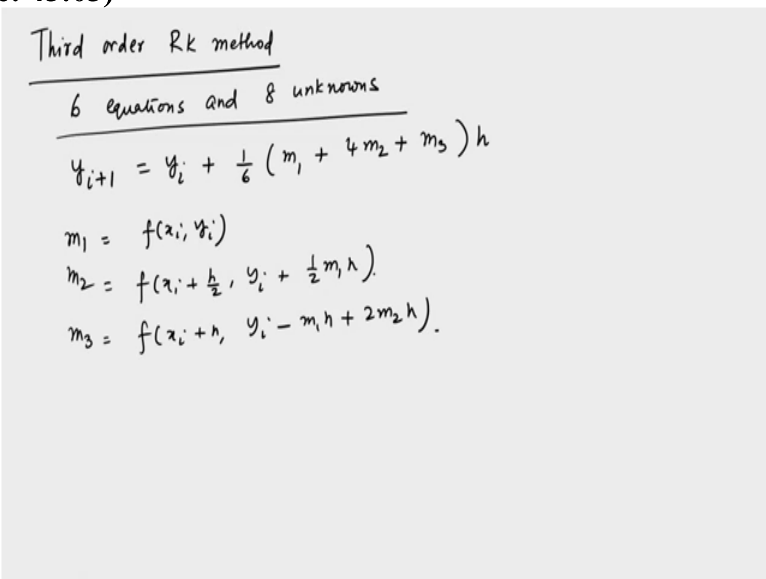

So, let us do this third order or rather we will we can do another problem of course but maybe we will catch up in the tutorial on these different second-order methods. But I hope you have gotten an idea that how to go ahead doing these these are all like Taylor series expansion but then you do not have to calculate the higher order derivatives and it is only the first order derivative that is needed.

So, coming to the higher order RK methods, so let us first discuss the third order RK method. So, we can perform a similar derivation as the second order RK method to arrive at results we will not give you proofs of that but it can be done. The only problem here it becomes a problem becomes worse in terms of the deficit of the number of equation as compared to the number of unknowns. Previously we had in the second order method we had 3 equations and 4 unknowns.

Here we have 6 equations and 8 unknowns okay so all right. So, we state the result without proof here it is a y i + 1 equal to y i + 1/6 m1 + 4 m 2 + m 3 into h where m 1 equal to f x i y i m 2 equal to f x i + h over 2y i + half m 1 h and m 3 equal to f x i + h y i - m 1 h + 2 m 2 h and so on okay.

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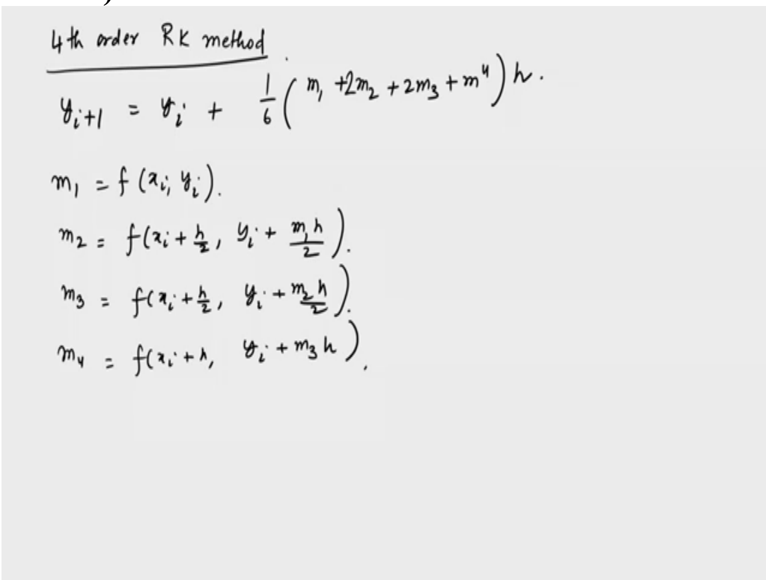

And similarly the fourth order RK method let us do it in a different slide. So, it is a following set of formula where you're y i + 1 it requires of course 4 slopes to be computed at 4 discrete points so it is a y i + 1 equal to y i + m1 + 3 is a 1/6 and there is a - m 2 + 2 m 3 + m 4 and h ok where m 1 is f x i y i m 2 is equal to f x i + h /2 y i + m 1 h over 2 m 3 is equal to f x i + h over to y i + m 2 h over 2 m 4 equal to f x i + h y i + m 3 h and so on okay.

So, this is the fourth order RK method and these the last two we have presented without any proof but let us do an example for the fourth order RK method. So, the two examples that we are going to do one for the second order and one for the fourth order okay.

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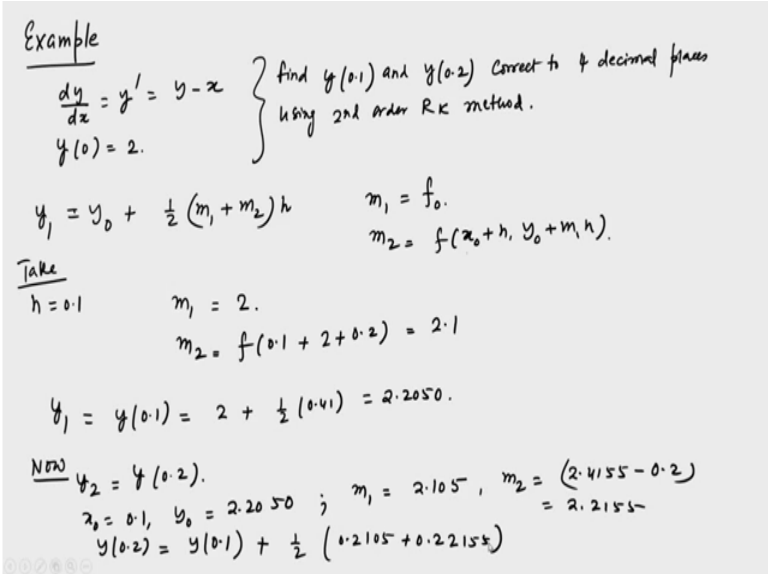

So the it is a dy/dx which is equal to y prime which is equal to y - x and the initial condition is given as 2, so the statement of the problem is that find y at 0.1 and y at 0.2 say correct to 4 decimal places to specify more clearly places using a second order RK method. This you have seen just for the practice I am including this so I will do a y 1 equal to y 0 + 1/2 m 1 + m 2 into h so we have m 1 equal to f 0 and m 2 equal to f x0 + h y0 + m 1 h.

So take h equal to it is not specified but we can take anything that we feel correct. So, this is m 1 equal to there is a value of the function, so that is equal to 2 and m 2 equal to f so this x 0 + h so there is this point 1 so 0 + 0.1 and this is like 2 + so this 2 into .1 that that is .2 and this is equal to a 2.1 if you calculate it so this is m 2. So, what I get is that y 1 equal to y 0.1 equal to 2 which is y at 0 and then 1/2 into 0.41 which is equal to 2.2.050 calculating it up to 4 decimal places.

Now we will call this the next interval as a y 2 which is equal to a y at 0.2 so there of course the initial things are taken as 0.1 y 0 that is f which is 2.2050 now y 1 becomes equal to y 0 and m 1 becomes equal to 2.105 and m 2 becomes equal to when you calculate it becomes equal to 2.4155 – 0.2 and we can calculate it and put it there okay. So, this y 2 becomes equal to which is y at 0.2 it becomes equal to y at 0.1 + 1/2 and .2105 multiplied by h so we can we can multiply it by h and this becomes equal to so this becomes equal to 2.2155.

So, this when it gets multiplied with h which is 0.1 it becomes this okay so this is equal to 2.42210 okay all right.

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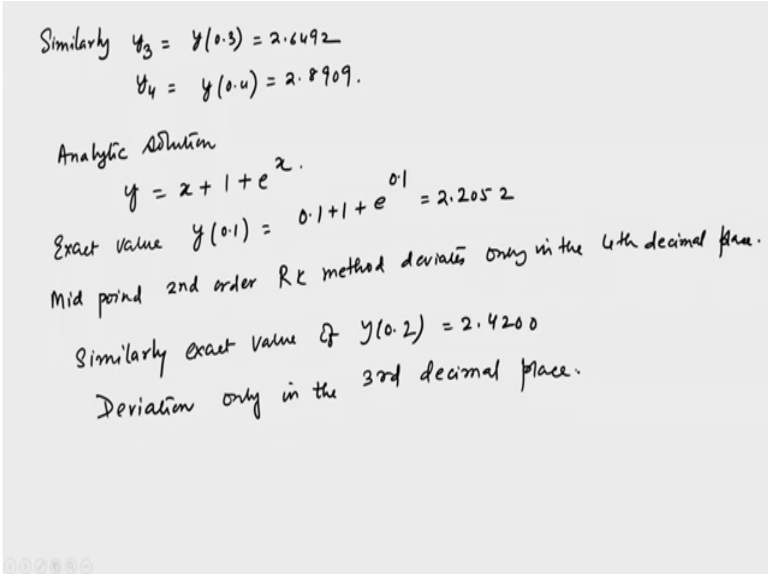

So, this is y at .2 and so on this is what is needed so y at .1 and y at .2 similarly we can calculate y 3 equal to y at .3 that becomes equal to 2.6492 and y 4 equal to y at 0.4 equal to 2.8909 okay. So, basically we are using this the midpoint method and to calculate these slopes and so on. So, the analytic solution of this is so y equal to x + 1 + exponential x so the exact value y of 0.1 equal to 0.1 + 1 + exponential 0.1 it becomes equal to 2.2052 whereas what we got is not too bad it is equal to 2.2050 where it is 2052.

So, the midpoint method in this particular case midpoint second order RK method deviates only in the fourth decimal place. So, similarly of course the exact value of y 0.2 is equal to 2. 4200 whereas we got 4210 so that is in the third decimal place, so deviation only in the third or decimal place all right. So, let us just look at a fourth order method and then we can close this discussion on the RK method.

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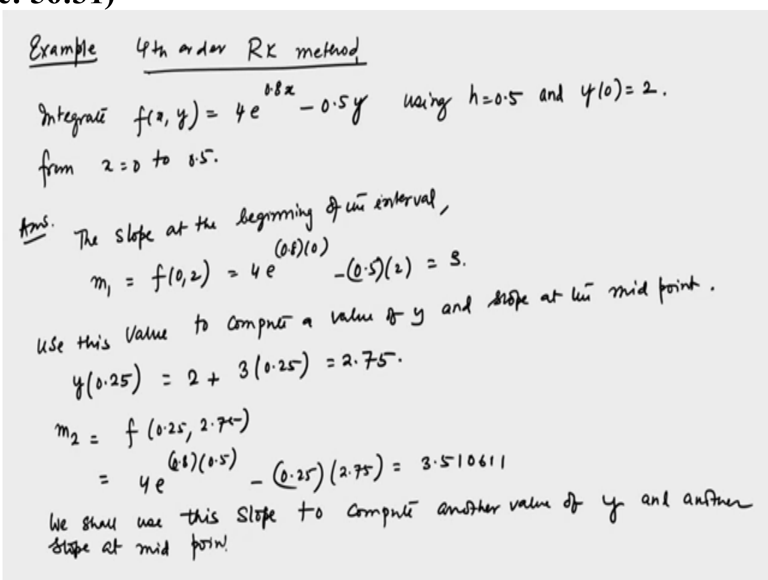

So, the its integrate f xy it is equal to 4e 0.8x - 0.5y using step size of 0.5 initial value y 0 equal to 2 from x equal to 0 to 0.5, so to begin with the slope at the beginning of the interval this is equal to m 1 equal to f 02 which is for e 0.8 into 0 -0.5 into 2 which is equal to 3. So, use this value to compute a value of y and slope at the midpoint. So, this slope has to be calculated at the midpoint so alright.

So y at the first interval because you are going from so h is equal to so the at the middle of the interval it is 2 + 3 into 0.25 which is equal to 2.75 okay. So, this is this is the value at the midpoint so that that is the slope and hence your m 2 will be f of 0.25 and 2.75 so this is nothing but a 4e .08 into 0.5 - .25 into 2.75 it is equal to 3.510611 okay. Now we will use this slope to compute another value of y and another slope at midpoint all right.

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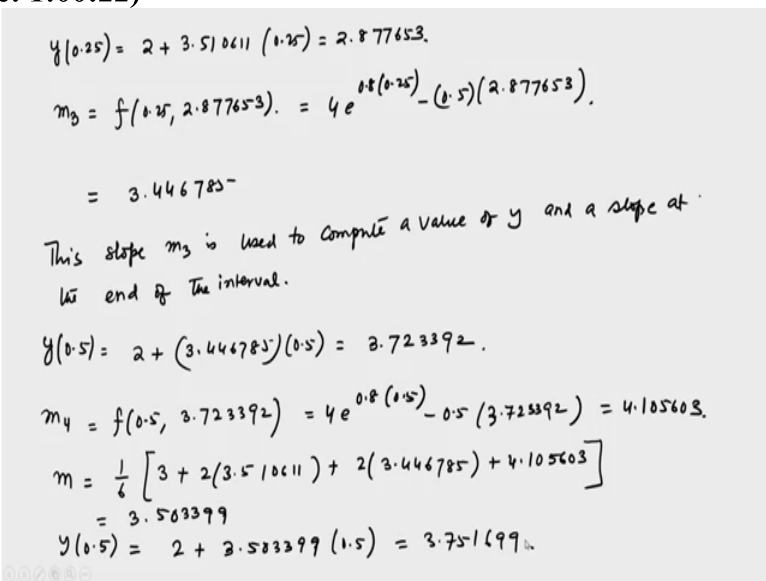

$y(0.25) = 2 + 3.510611\,(0.25) = 2.877653.$

$m_3 = f(0.25, 2.877653). = 4e^{0.8(0.25)} - (0.5)(2.877653).$

$= 3.446785$

This slope m_3 is used to compute a value of y and a slope at the end of the interval.

$y(0.5) = 2 + (3.446785)(0.5) = 3.723392.$

$m_4 = f(0.5, 3.723392) = 4e^{0.8(0.5)} - 0.5\,(3.723392) = 4.105603.$

$m = \frac{1}{6}\left[3 + 2(3.510611) + 2(3.446785) + 4.105603\right]$

$= 3.503399$

$y(0.5) = 2 + 3.503399\,(0.5) = 3.751699$

So, what we do is that we calculate this y at 0.25 which is 2 + 3.510611 and multiplied by 0.25 so that becomes my m 2 so it is 877653 and hence the m 3 is equal to 0.25 and 2.877653 and this is nothing but equal to 4 e to the power .8 into .25 – 0.5 into 2.877653 and that is nothing but if you calculate it, it is equal to 6785, so we calculated we have calculated this slope m 3.

Now this slope m 3 is used to compute a value of y and a slope at the end of the interval okay. So, y at 0.5 is equal to 2 + 3.446785 multiplied by 0.5 which becomes equal to 3.723392 and so on, so m 4 it is equal to f 0.5 3.723392 which is equal to 4e 0.8 0.5 – 0.5 3.2 sorry 72 723392 alright, so if you calculate that that becomes equal to a 4.105603. So, I have calculated 4 slopes at 4 discrete points and these are going to be used to update the solution at the end of the interval.

So the m the effective m or the net m is 1/6 3+2 into 3.510611 + 2 into 3.446785 + 4.105603 and so on. And then if you calculate this slope it becomes 3.503399 so using the fourth order method I get the solution at the end of this interval as 2.2 + that is y 1 that is y at 0 or basically at x equal to zero which is a value to that is given 3.503399 multiplied by 0.5 and this becomes equal to 3.751699 and so on okay.

So by and large I have given you an idea that how these initial value problems are solved. We have first discussed Euler method which is calculating the slope and updating the solution by the slope computed at the initial point. The Heun's method was one better than that in which the slope at both the initial and the final points were calculated and the average of the two were taken in the sense that average means just multiplying each of the slopes by a factor of half and then that is considered as the effective slope.

In the RK method we saw that we can actually divide in the interval h we can take several points and in those several points there is no reason for us to take an a priori weight we can take these weights to be you know some values which satisfy certain equations and then we can build up actually a family of methods higher order second order and higher order methods and second order is easier to handle for us.

So, we have seen three kinds of methods one is of course the midpoint method the other is the Heun's method which is which already we have learnt and then the Ralston's method and then it is turns out that amongst these we have a very large family of methods it is not the list is not exhaustive. And then we have looked at the at least the formula working formula for the third order RK method and the fourth order RK method.

And here we do an example of the fourth order RK method in which the slopes at four different points that to be computed which are pretty easy to do and then these slopes have been used in order to update the solution at the end of the interval okay. And then you have to take all these input for the next interval and then again update the solution in the sense that this value of y at 0.5 will be actually taken if you have the range to be say from x equal to say 0 to 2 not 0.5 as we have said here.

If you have that range then you can actually update the solution taking this as the initial solution and then or calculating again m1, m 2, m 3, m 4 and then put these weights such as it is shown here and then update the solution to get a y at 1 and so on okay. These RK methods are very popular numerically a lot of you know work actually use these methods some either second order or fourth order methods in order to calculate the solution. We will see some practical applications of in the next discussion.