Numerical Methods and Simulation Techniques for Scientists And Engineers

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Lecture 15
Heun's method

So, far we have seen Euler's method. Euler's method is a is a simplest one-step method which calculates the solution of a differential equation given by say y prime equal to f of xy. And it retains only the first 2 terms of the Taylor series expansion and not only that Euler's method takes the slope at the beginning point or at the point which is the left extreme point the slope is calculated and hence the solution is iterated for the next interval with again the solution that we obtained from the first iteration as the leftmost point. And then again the slope is calculated there and that is how the solution goes on.

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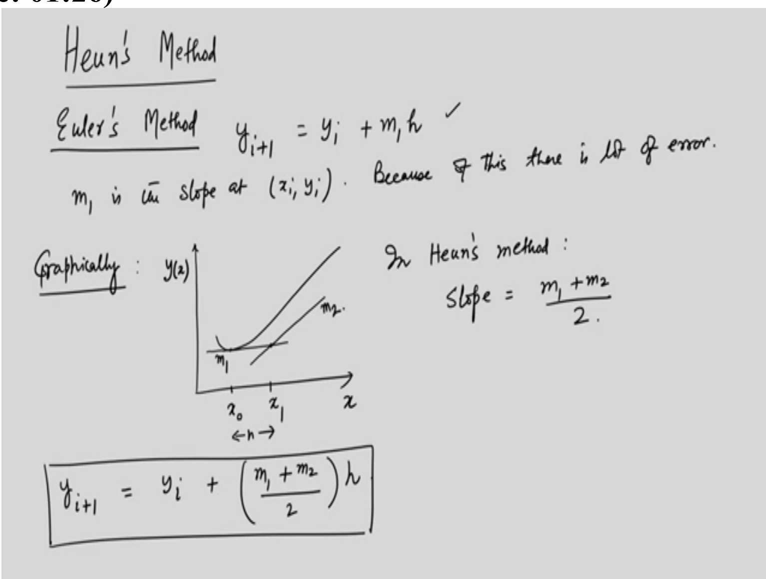

Here we want to learn a method which is called as a Heun's method and we will show that this gives a reasonably better estimate of the solution of the differential equation by taking the mean of the slopes at the leftmost point and the rightmost point of an interval okay. So, it does not only use the slope at the leftmost point which is used by the Euler's method here it is calculated at both the points and that is how it would proceed.

So in Euler's method we have learned that it is y i + 1 it is equal to y ii + m1 h this m1 this subscript 1 is introduced here which was not introduced earlier because we want to say that this is the first point on the Left point and so that is how this is calculated. So, this is the solution of the differential equation. So, m1 is the slope at x i y i and because of this really there are or there is lot of error okay.

And we have seen how bad those errors are but nevertheless it was worth learning about the Euler's method because it is the first method or the you know the first approximation that one can do in order to arrive at the solution of the differential equation. So, what Heun's method does is that it tells that we will take 2 slopes one at the beginning point and other at the end point and this slope here would be considered as a the mean of the 2 slopes that we get.

So, you might wonder that just taking the mean of the 2 slopes will just alter this value but then we will see later that it actually is one order better than the Euler's method and how is it that we will see in a while. So, let me tell you graphically what Heun's method is so we have f of x or rather let us call it as y as a function of x and this is of course x and suppose we have a solution of this type okay.

And this is your x 0 and at a distance h we have the next point x 1 and of course this value is nothing but your y of x 0 and this value is nothing but that but then what happens is that in Euler's method we calculate the slope at this point and take this point as if it is the y value that is add the value of x equal to x 1. So, we take a slope here and let us say we take a slope here okay so this is called this slope as m1 the flat one and call this slope as m2.

Then what it is done in this Heun's method, so in Heun's method the slope is equal to m1 + m2 by 2 okay. So, the solution is that is y i + 1 it is equal to y i + m 1 + m 2 by 2 into h ok, so this is the Heun's formula and you can distinguish it from the Euler's method because this involves your slope at the first point but this involves this mean slope of the 2 points in a particular interval. So, this is the main working formula of Heun's method and so let us just do this derivation more carefully.

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$y'(x) = f(x,y).$

$m_1 = y'(x_i) = f(x_i, y_i)$

$m_2 = y'(x_{i+1}) = f(x_{i+1}, y_{i+1}).$

$m = \frac{m_1+m_2}{2} = \frac{y'(x_i) + y'(x_{i+1})}{2} = f(x_i, y_i) + f(x_{i+1}, y_{i+1}).$

$$\boxed{y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right].}$$

y_{i+1} can be found using. Euler's method.

$y_{i+1} = y_i + h f(x_i, y_i). \longrightarrow$ predictor.

$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{Euler})\right]. \rightarrow$ Corrector.

This is called as a one-step predictor-Corrector method.

So, let us say given this equation y prime of x equal to f of x y so we obtain m 1 equal to y prime of x i that is the first point which is nothing but equal to f x i y i ok and m - that is a y Prime at x i + 1 and this is equal to f x i + 1 y i + 1 ok, so these are the 2 slopes evaluated at the points x i and x i + 1 ok and the responding values are given by the differential equation which is y prime equal to f.

So y prime x i equal to f of x i y i and y prime of x i + 1 is f x i + 1 and x y i + 1 alright. So, m being the mean of the 2 slopes or the arithmetic average of the 2 slopes this is equal to so this is like a y prime x i + y prime x i + 1 by 2 and this is nothing but f of x i y i + f of x i + 1 y i + 1 okay. So, the Heun's method would give you a result which goes as y i + 1 equal to y i + h over 2 and then f x i y ii and f x i + 1 y i + 1 ok.

Now this is of course the same formula that we have written earlier ok. Now there is apparently nothing wrong with this but however you will face difficulty because you see that the y i + 1 is there on both left and right hand sides of the equation ok. So, that tells you that if you do not know y i + 1 how would you calculate y i + 1 because it requires that input. So, that tells that we can find y i + 1 can be found using Euler's method.

And this called as the predictor so y i + 1 equal to y i + h fF x i y ii so this is called as a predictor. So, once I get the predictor on the right hand side I can find out the character which is y i + 1 + h by 2 f x i y ii + f of x i + 1 y i + 1 I will write it as Euler alright. So, this is the formula for the so this is the corrector step of the method. So, once again let us go back to this equation this equation where you have a y i + 1 which necessitates you to know the value of y i + 1 anyway because it is on the right hand side.

So you cannot solve this equation unless you make a guess for this y i + 1 and y make a guess because Euler's method is available in order to calculate this y i + 1 which is simply this formula which is just using the slope at the initial point. So, y i + 1 equal to y i + h f x i y i that we call it as the y i + 1 Euler and that so the top step or the step before the last but 1 step is a predictor method where the y i + 1 is actually calculated using the values of x i and y i which are already known.

And correction is used or a corrector method is used you calculate the f y i I mean y i + 1 by calculating the y i + 1 from the Euler method and putting it back here. So, let us see this so this is called as let me write it here itself this is called as a one-step predictor-corrector method okay.

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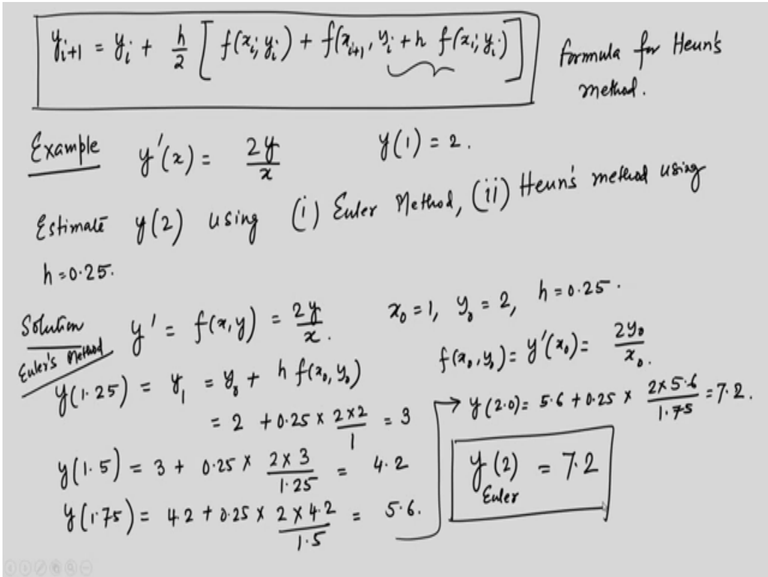

So, let us see that so using this one-step predictor corrector method the value or rather the estimate of the solution is obtained as h over 2 f of x i y ii + f of x i + 1 y I + h f of x i y i ok. So, that is the that is your y i + 1 which is obtained from the Euler method ok. So, let us give an example so this is your pretty much the formula for the Euler method I am sorry a formula for the Heun's method not Euler method we have discussed earlier pardon me okay.

Let us give an example of this usage of the Heun's method so a y prime of x is say for example 2 y over x and the initial condition is given as y of 1 equal to 2 okay. So, the question is that estimate y 2 that is y at x equal to 1 is given and you have to find y at x equal to 2 using Euler method which we have already learnt and Heun's method using h equal to 0.25 all right. So, this is the interval size that is given and you are asked to calculate why the value of the solution of this equation or rather this the value of the solution at x equal to 2 by using both the methods and using this interval size of 0.25.

So, let us do the solution carefully so y prime equal to f of x y equal to 2y over x ok so x 0 is given as 1 y 0 is given as 2 and h is given as 0.25 ok. So, y at 0.1 .25 in Euler's method, so this is Euler's method so this is equal to using the value of y at let us call this as y1 which is equal to y0 + h f x 0 y 0 all these things are known because your f x 0 y 0 is nothing but y Prime at x 0 which is equal to 2 y 0 by x 0 y 0 is given as 2 so this is 4 over 1 which is 4.

So this is equal to 2 that is given then the step size is 0.25 and then it is 2 into 2 divided by 1 and that is nothing but 3 so this is 4 into 0.25 which gives you 1 and then 2 + 1 equal to 3. So, that is the y at 1.25. Similarly y at 1.5 will be nothing but this value at 3 + 0.25 that is the step size multiplied by 2 into 3 divided by 1.25 okay. So, basically that is your so that will be 2y 1 by x 1 y 1 is already computed as 3 so there is 2 into 3 and then divided by x 0, x 0 is nothing but the at the initial point it was 1.25.

So this is 1.25 that gives you a 4.2 I am not too worried about the significant digits neither I am putting the decimals but you can do it when you do a real problem because in the computer there is certain amount of accuracy that needs to be given. So, we let us do it for 1.75 and that is equal to this 4.2 + 0.25 this is 2 into this was 3 so this is like 4.2 and because x is 1.5, so this becomes equal to 5.6 okay.

I will take it here from so y at 2.0 it is nothing but this 5.6 there is a value at one just preceding our step and then as a .25 multiplied by 2 into 5.6 divided by 1.75 and that gives you a 7.2, so the value that you get from Euler's method, so let us write it as just our this is equal to 7.2 at x equal to 2 remember this value will remind you so that is just using the Euler's method.

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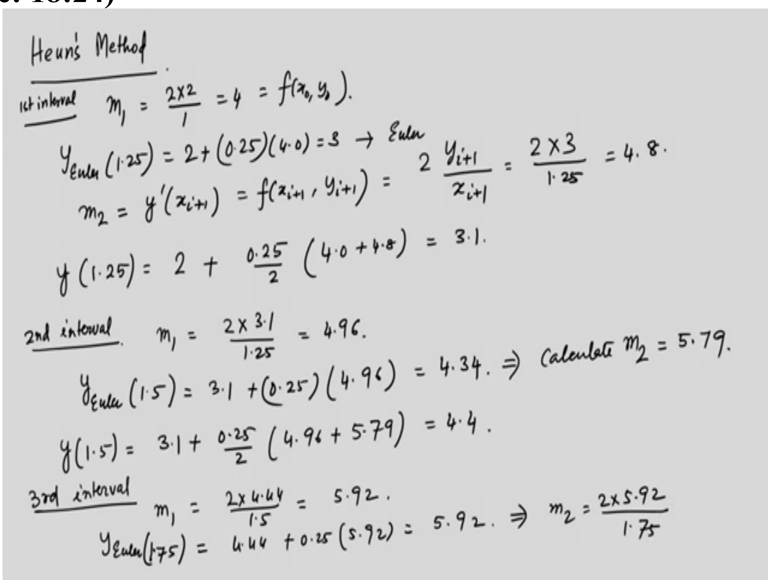

Let us use the Heun's method so m 1 equal to 2 into 2 by 1 which is equal to 4 which is equal to f x 0 y 0 now I will use the Euler at 1.25 as 2 + 0.25 into 4 equal to 3 so this is the Euler value anyway we know the value at this thing now m2 is calculated using this value of y from the Euler method. So, this is the predictor step and the corrector would be so simply y at x i + 1 which is nothing but f x i + 1 y I + 1 so this is equal to 2y i + 1 x i + 1 so that is the value of this and this is equal to 2 into 3 divided by x i + 1 is nothing but 1.25.

So this is the character method which gives you 4.8 so in the first step at 1.25 the solution comes as the value at 1 value of the function at 1 or the solution this is this and then there is a 4.0 + 4.8 which is equal to 3.1 not too much of improvement. But i it was 3 earlier now it is 3.1 and this is the first interval. So, let us go to the second interval that is between 1.25 and 1.5 and we will have to calculate these things.

So, second interval gives you m1 which is nothing but into 2 into 3.1 divided by 1.25 which is equal to 4.96 ok so that is your m1 again I have to find the predictor value at 1.5 this is equal to the value at 1.25 which is 3.1 we have already calculated this value and then it is a .25 multiplied by a 4.96 so that is a predictor value giving you a 4.34 and immediately the corrector value can be obtained by using this 3.1 + .25 divided by 2 and we have a 4.96 and 5.; so this is your 4.96 and there is a 5.79 because this is your the m 2.

So, m2 would be calculated again so calculate m 2 and m 2 comes out to be 5.79, so this is 5.79 this is equal to 4.4 third interval now I will write down the results third interval 1 has m1 equal to 2 into 4.44 divided by 1.5 this is equal to 5.92 so why Euler predictor 1.75 this is equal to 4.4 four + 0.2 5 into 5.92 this is equal to 5.92 and this gives m 2 equal to 2 into 5.92 and 2 into 5.92 2 and divided by 1.75 and this is equal to 6.77 of course you should check all these values by explicitly calculating

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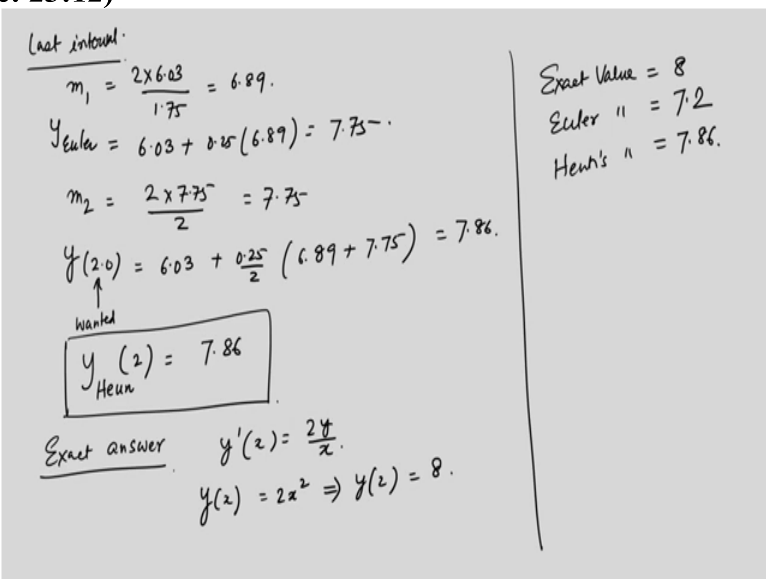

And for the last interval your m 1 equal to 2 into 6.03 divided by 1.75 this is equal to 6.89 there could be minor mistakes in the calculation in putting the number so please check as you see this so the Euler value is equal to it is 6.03 + 0.25 into 6.89 which gives us 7.75 so the m 2 becomes equal to 2 into 7.75 divided by 2 is 7.75 so that is the value so there is a 7.75 there and then y at 2. .0 that is the value that you want wanted this is equal to 6.03 + 0.25 divided by 2 6.89 + 7.75 it is equal to 7.86 C if you go to slides back you got it as 7.2.

So y Heun at 2 it is nothing but 7.86 okay the exact answer let me give you here it is y prime of x equal to 2y over x so y of x sorry y of x equal to 2x square that gives y at 2 equal to 8 so the exact value is 8, Euler give 7.2, Heun's is 7.86, so this is much closer to the exact value okay. So, let us do the error analysis for this Heun's method.

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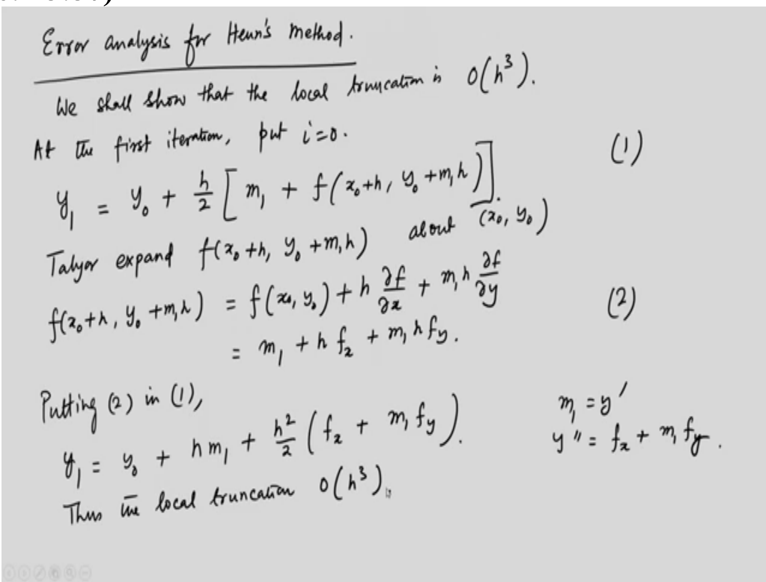

So, it is a second-order method we will show that so what you miss out is terms which are of the order of h cube h being the interval step size okay. So, we will show that the local truncation is of the order of h cube okay. So, at the first iteration put y equal to z put i equal to 0 i equal to 0 so I have y 1 is equal to y 0 + h by 2 as you have seen it is m 1 + f of x 0 + h y 0 + m 1 h so this is the Huen's formula and we want to find out the correction due to the or rather the error introduced due to the local truncation.

So do a Taylor expansion so this f of x 0 + h y 0 + m1 h equal to f of x 0 y 0 + h del f del x + m1 h del f del y that is the Taylor expansion key retaining up to first order. So, we are doing it about x 0 y 0 that point so we get a f x 0 y 0 and then we take a h del f del x and then m1 h del f del y so this can be written as in our shorthand notation m 1 this we know that this is equal to m 1 and this is h f sub x sub except script x denote that it is a del f del x + the m 1 h f of y ok if you write this as equation 1 and write this as equation 2.

So, if you put 2 in the right hand side inside the bracket for the fx 0 + h y0 + m 1 h, so your y1 becomes equal to y 0 + h m1 because there are 2 m1's coming from one m 1 is already there and one m 1 coming from here, so that becomes h m 1 and then you have a h square by 2 and then you have a x fx + m 1 fy ok. So, you see that clearly the h square term is taken into account missing only the h cube term onwards.

So of course your m1 equal to y prime and y double prime equal to f of x + m1 f of y so that is the things that are substituted so the local truncation is of the order of h cube ok.

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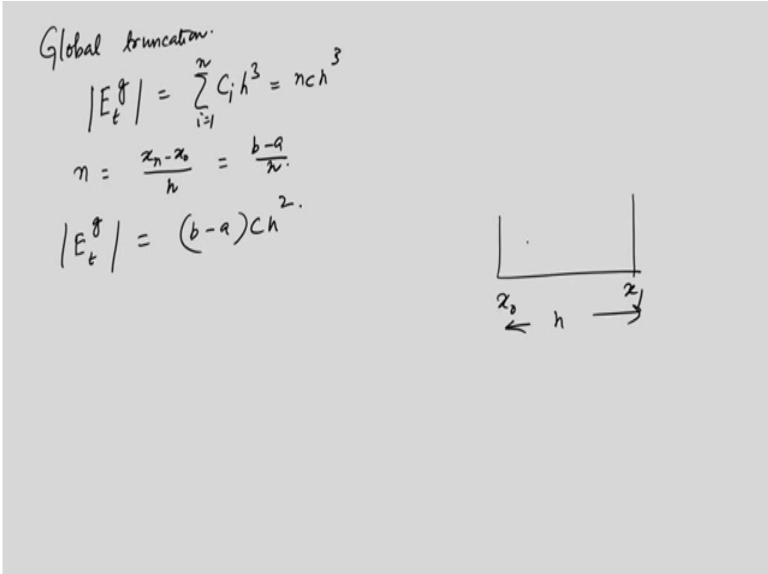

So, the global truncation E tg this is equal to we have i equal to 1 to n C 1 rather CI h cube remember this was h square for the Euler's method and his n C h cube and what you have for n is the total interval x n – x 0 we might have written b - a divided by h is equal to b - a divided by h. So, our E tg this goes as b - a C h Square and so on okay. So, that is the global truncation error and so we have looked at the 2 simplest methods rather the one is the Euler method and the other which uses Euler method and does a better approximation of the solution of the differential equation by introducing this Heun’s method.

Now what happens if we actually calculate the slopes at each one of the points in a given interval suppose you have an interval which is this between x 0 and x 1 that is the interval okay and this is of width h and there are lots of points in principal infinite number of points between this. And we are either only calculating the slope at this point the initial point and so that is Euler’s method and all we are calculating it at these 2 points and taking a mean.

But what about if we take a number of points in between and take appropriate weights of that those slopes and then put it into the solution just the way we are doing it. And so basically you will not have m 1 or m 1 and m 2 divided by 2 but then you will have some weights m 1 with another some other weights m 2 some other way it is m 3 with some other waves m 4 and m 5 and so on this is called as a Runge-Kutta method.

And this runge-kutta method is quite popular and widely used so we will have to learn this Runge-Kutta method. And so there are certain orders of the Runge-Kutta method and of course here in the class we cannot show you anything more than maybe fourth order Runge-Kutta method but that should be good enough for one to understand that how this existing solution by the Euler method or by the Heun's method can be actually improved by using the Runge-Kutta method. So, we will take that up next.