Numerical Methods and Simulation
Techniques for Scientists And Engineers
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Lecture 14
Solution of differential equation, Taylor

So, let us now look at the numerical methods of solving differential equations. This is mainly an iterative method by which we solve a differential equation could be ordinary differential equation or partial differential equation. And these iterative methods can be a one-step method or a multi-step method, let me explain what they mean.

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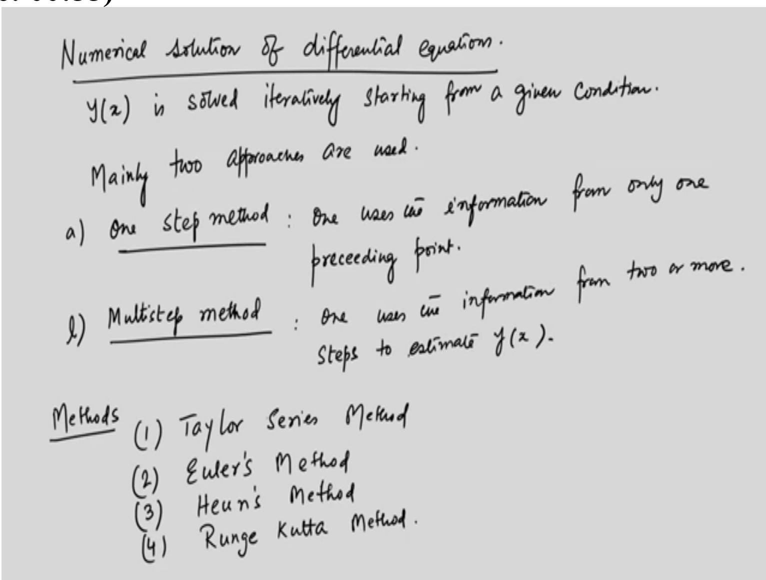

So, this is numerical solution of so as I said that y as a function of x is solved iteratively starting from a given condition and mainly 2 approaches are used. So, this is a is that one step method so what happens is that one uses the information from only one preceding point. So, every time it is iterated you need only the solution that you have obtained at the previous point and all other earlier solution can be discarded.

So, this is called as a one-step method and similarly a multi-step method one uses the information from 2 or more steps to estimate y of x okay. So, we would you know look at a few methods of solving these differential equations by this iterative procedure and the main methods are one let us call it a Taylor series method. Two what is called as the Euler method a very closely linked one to the Euler is called as a Heuni methods it is like Euler method.

Let us write Euler's method, Heuni method and 4 the most commonly used and the most important one is called as a Runge-Kutta method okay.

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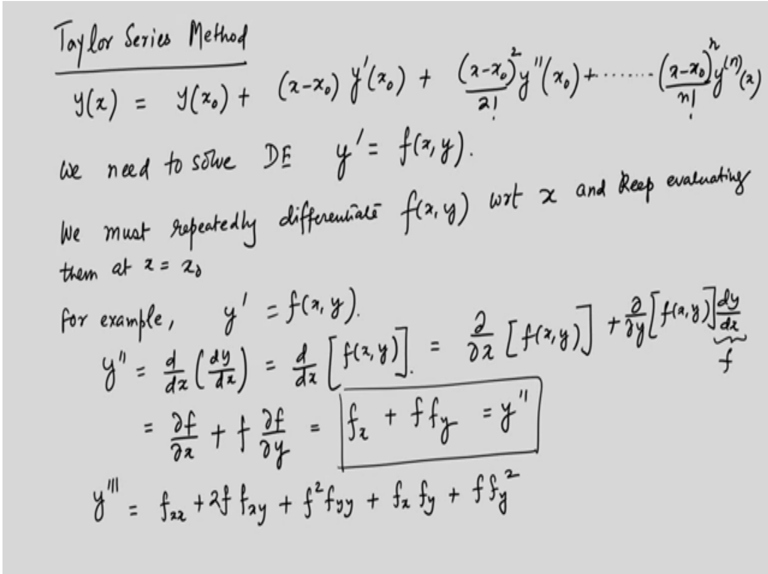

So, we will see them 1 by 1 but before that let us at least establish the formalism that lead to the approximate solutions of these differential equations. So, let us start with the Taylor series method okay. So, what we do in Taylor series is that we expand a function let us call it f of x or y of x about a point x 0 in this particular fashion so it is x – x 0 and then y prime at x 0 + x – x 0 square by 2 factorial which is nothing but 2 it is a y double prime evaluated at x0 and so on.

And then it is x – x 0 whole to the power n by n factorial I am writing it in this particular fashion this n inside the bracket and as a superscript it denotes that that is a any derivative of y with respect to x and so on. So, y we write down this we write down this because we need to solve the differential equation we keep writing it as de as an abbreviation it is y prime equal to f xy. So, you need to solve this equation and the solution would appear as y as a function of x and we claim that a Taylor series solution would represent a valid solution to this differential equation.

Let us see how so we basically need to evaluate higher-order derivatives y prime y double prime while triple prime and so on. So, we must repeatedly differentiate f of xy with respect to x so wrt is with respect to and keep evaluating them at x equal to x 0 ok. So, this is the whole idea so this is little cumbersome process no doubt about it because you have to evaluate all higher-order derivatives and let us see how we do that.

So, for example you have a y prime equal to f of xy so a y double prime is nothing but a d dx so this means a d2y dx2 which means a ddx of dy/dx which is a ddx of so this is that and since dy/dx is nothing but f xy so we will write a ddx of f xy so let us write it with a square bracket here alright. So, this is nothing but equal to del del x because it is a f is a function of both x and y while you are taking a derivative with respect to x only so it is a del del x of f of xy + del del y of f of xy and a dy/dx rather del y del x or dy/dx is fine because y is a function of x ok.

So this can be written as a del f del x now I explicitly not writing the functions f as a function of x and y but I mean the same thing. So, this first term is del f del x the second term is f because this is nothing but f so it is a f del f del y that is the second term and this can be further written as f x + f f y all right where the subscript x denotes that it is a derivative taken with respect to x. So, my y double prime is this so the second derivative is nothing but this quantity okay.

And similarly a third derivative can be found by taking 1 more derivative with respect to x so this is f xx which is a del 2 f del x2 and then there is a f rather 2f fxy that is a mixed derivative so that del f del x and del f del y and then there is a f square you need to work this out carefully this is a del 2f del y2 then there is a fx fy, so this del f del x and del f del y these 2 are multiplied and then ffy square so this is a del f del y and and taking a square of that.

So, these are the higher derivatives which can hence put into this let us call this as equation 1 and then y as a function of x could be calculated.

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Example $y' = x^2+y^2$ $\quad y(x=0)=1 \Rightarrow y(0)=1 \Rightarrow y'(0)=1.$

$y'' = 2x+2yy' \qquad y''(0)=2.$

$y''' = 2+2yy''+2(y')^2 \qquad y'''(0)= 2+2\times1\times2+2=8.$

The Taylor Series solution of the DE is

$$y(x) = y(x_0)+(x-x_0)y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots$$

$x_0=0$ $\quad y(x) = 1+x+x^2+\frac{4}{3}x^3+\cdots$ ← Taylor Series Solution.

Error in Taylor Series

Error $\sim (x-x_0)^{n+1}$ If $|x-x_0|$ becomes large, error also becomes large. In the interval $[a,b]$ when $|b-a|$ is large, the method is inadequate.

Let us take an example of this and let us have this y prime equal to x square + y square and this is what you have to solve ok. And the initial condition is that the y at x equal to 0 its equal to 1 and this is written as y 0 equal to 1 ok, so that is the initial condition being given so since our y prime is given a y double prime is equal to 2x + 2y y prime and because at y at x equal to 0, y equal to 1, so y so what is y prime so this is this gives okay so a y prime which is equal to x square + y square.

Since y at x equal to 0 equal to 1 y prime at x equal to 0 will not write x equal to 0 but we will simply denote it by 0 in the bracket that is equal to 1 as well. So, y prime 0 equal to 1 now y double prime 0 this is equal to because we have to in the Taylor series if you remember that we have to evaluate these derivatives at a given point which is at the initial point the point that is specified in the problem.

Here the specified point is at x equal to 0 ok and this is equal to y prime equal to this is fine and then y double prime equal to 0 is nothing but you put x equal to 0 and y equal to 1 and y prime equal to 1 you get a 2 okay, y triple prime which is nothing but 2 + 2y y double prime + 2y double prime Square so a wide triple prime evaluated at x equal to 0 is nothing but so this 2 and then y is equal to 1 y double prime is 2.

So it is 2 + 2 into 1 into 2 this is at 0 and then + 2y prime square so that is equal to 2 again so this gives the 8 okay. So, the Taylor series solution for the differential is y at x is y at x 0 + I am just writing the general solution once more y prime evaluated at x 0 x - x 0 whole square by 2 factorial y double prime evaluated at x 0 + x – x 0 cube 3 factorial and y triple prime x0 and of course all these other terms which we have not calculated as yet.

So, we will keep this solution up to this and this is nothing but this is equal to because y at so x 0 is 0 okay and y at x 0 is already given which is equal to 1 x 0 is of course 0 so this is x and y prime at x 0 y prime at 0 is 1, so it is x + it is x square y double prime at x 0 is nothing but 2, so this, this 2 will cancel with the 2 factorial, so this is 1 + x square and then it is a 4/3 x cube and so on okay that is the solution of this equation.

So this is y as a function of x so this is the solution so a Taylor series solution and of course you understand that there will be error because we are truncating the series after the third derivative or the third term rather the fourth term. If you include y the point about which it is expanded and this will introduce error. And these errors are you know has to be these errors have to be calculated and let us discuss a little on the error.

Error in Taylor series method well I even a write-in you know in a continuous fashion all these things they this looks like n so should not confuse it is a series that I wanted to write okay. So, this is the thing that we have to calculate and of course this the error will of course go as if we stopped after nth term here we have of course stopped after the third term. If we stop after the nth term the error is going to go as x – x 0 whole to the power n + 1 okay.

And if x – x 0 becomes or it is a mod of that rather that is the difference between where you want the solution numerate I mean to numerically compute and the point x 0 suppose you want x to be far off from the origin that is x equal to 0 then of course this term will become large. So, if it becomes large then of course the error also becomes large. This of course restricts the utility of the method.

And so the utilities says that if in the interval a to b in the interval a b when b - a is large the method is inadequate.

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The accuracy can be improved if we increase the number of subintervals in the whole interval $[a, b]$. Divide into $[a, x_0], [x_0, x_1], [x_1, x_2] \cdots\cdots [x_n, b]$

Compute $y(x_i)$ successively using Taylor series expansion. Here $y(x_i)$ is used as the initial condition for finding $y(x_{i+1})$.

$$y(x_{i+1}) = y(x_i) + y_i'(x_i)(x_{i+1} - x_i) + \frac{y_i''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots\cdots \frac{y^{(m)}(x_i)}{m!}(x_{i+1} - x_i)^m$$

Here $x_{i+1} - x_i = h$ $\quad$ h: width of the interval.

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2!} y_i'' + \cdots\cdots \frac{h^m}{m!} y_i^{(m)}$$

y_{i+1} is used as the initial condition for y_{i+2}.

$$y_{i+2} = y_{i+1} + y_{i+1}' + \frac{h^2}{2!} y_{i+1}'' + \cdots\cdots + \frac{h^m}{m!} y_{i+1}^{(m)}$$

$\vdots$ and so on.

So, the question is what to do about that okay it is very clear that the method is inadequate because of this fact that suppose you want to calculate at x equal to 10 whereas x 0 is 0 okay and then you know this as you miss powers of these x – x 0, so the leading order power that you would be missing is x – x 0 whole to the power n + 1. And then of course this becomes you know the method becomes inadequate and 1 has to have a remedy or at least have an idea that how much it is accurate and so on.

Let us talk about the remedy so the accuracy can be improved if we increase the number of subintervals in the whole in the whole interval a b okay. So, do not take the entire interval at 1 go rather this interval from a to b you divide into sub intervals several sub intervals so that you go from a to x 1 x 1 to x 2 and so on. So, let us divide into a x 0 x 0 x 1 and x 1 x 2 and then go and do it between say x n and b.

And that way and apply this Taylor series in each of these sub intervals or other these small smaller intervals that we have. So, compute yx i successively using Taylor series expansion okay. So, this you do it and so basically here y at x i is used as the initial condition for finding y x i + 1 okay so let us write it x i + 1 equal to y x i + y prime evaluated at x i x i + 1 - x i, so you take equal intervals of equal width is what is meant here and y so this is y prime.

So this y double prime at x i by 2 factorial and x i + 2 - x i sorry x i + 1 x i whole square and so on. So, here x i + 1 - x i equal to h where h is the width of the interval okay. And so basically this will go all the way up to y to the power m xy that is a ma the derivative and m factorial x i + 1 - x i whole to the power m. And so this will be written as this we write it little shorthand notation as y i + 1 which is nothing but equal to y evaluated at x i + 1 it depends on the value at so this will write it as y i y evaluated at i and + hy i prime + h square by 2 factorial y i double prime and so on.

And then go all the way it is h to the power m by m factorial y i to the mth derivative of that, so this is now this y i + 1 which is evaluated for the first interval say for example this is used as the initial condition for y i + 2. I hope this is clear that you divide the entire interval into various sub intervals and apply the Taylor series expansion each one of those sub intervals and use the initial value for the next sub interval as the one that you have obtained from the first sub intervals the result obtained in the first sub interval.

So, now again you write down y i + 2 which is equal to y i + 1 + hy i prime i + 1 + h square so if this is not each cross this is only the so it is h square by 2 factorial y i + 1 double prime and so on and then you have a h m to the power m factorial y i + 1 m and so on okay. So, this way what happens is that your sub interval size when it becomes too small then and you actually iterate the solution over the sub intervals and then you get a much better result for the solution of the differential equation.

Let us take an example we cannot sit and do it for a large number of intervals but at least 2 intervals we can show.

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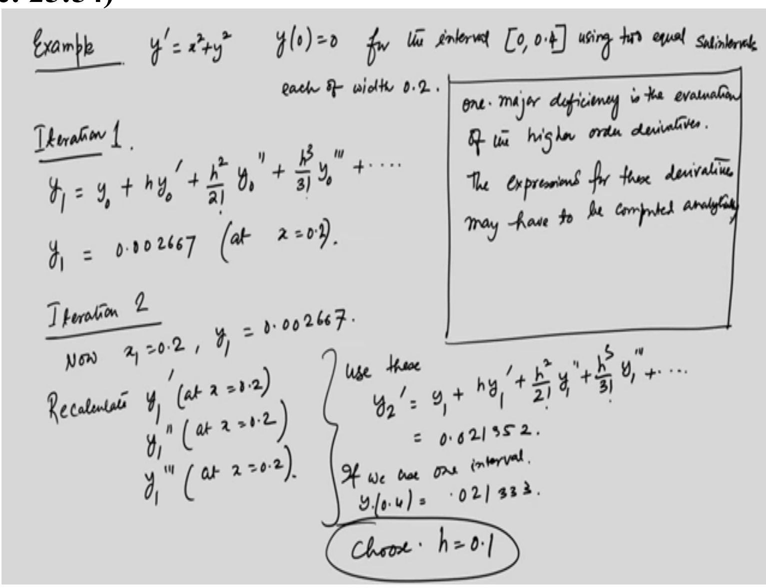

Take an example ok so the same example y prime equal to x square + y square now take y at 0 equal to 0 and for the interval is 0 and 0.4 using 2 equal subintervals of well this is not there is no sub intervals each of width each of width 0.2 okay. So, let us do it so we have this interval 0 to 0.4 and we want to break this interval into 2 equal intervals so 0 to 0.2 and 0 to 0.2 to 0.4 and use the Taylor series expansion and see that what we get okay.

So this is iteration 1 so y 1 equal to y 0 + hy 0 prime + h square by 2 factorial y 0 double prime and so on h cube by 3 factorial y0 triple prime and so on. So, you can calculate this and put x equal to 0 and remember that y at x equal to 0 is equal to 0 so if you do that then what you get is that 0.002667 at x equal to 0. So, this is remember that this is the solution at x equal to 0 so this is your y 1 sorry it not x equal to 0 which is 0.2 so that is the solution at 0.2 use this solution for the next interval from .2 to .4 ok.

So now so this is well I should not say iteration but I should write that maybe it is iteration because I have divided into 2 intervals so this is now iteration 2 so now x 1 equal to 0.2 and y1 equal to 0.002667 okay. So, now recalculate y 1 prime at x equal to point to y 1 double prime at x equal to .2 y 1 triple prime at x equal to .2 and so on okay. Because you have this you can do it analytically so 1 can do it and what 1 gets is the following for y to use these to get y 2 equal to y 1 + hy 1 prime + h square by 2 factorial y 1 double prime + h cube by 3 factorial y 1 triple prime and so on ok.

So this if we truncate up to this it becomes 0 2 1 3 5 2 and if you use 1 interval between this what one gets is that why at 0.4 equal to .021333 so this using 2 intervals you get the result different in the fourth decimal place and you know you can improve the accuracy if you choose a smaller one let me give you as a home assignment choose h equal to 0.1 where you have to do it 4 times but you see that how accurate that becomes and also do it you know exactly by using analytic methods the methods that you were aware of in your from your mathematical physics or mathematics course you can do it and get the exact result and compare with this.

So one you know major deficiency or we can write it here. So, let us make a box of this so one major deficiency is the evaluation of the higher-order derivatives. And you must have gotten a feeling that not only they have to be computed by hand you could also compute using the methods of derivatives that you are aware of but numerically. But those numerical methods or even the analytic methods of computing higher order derivative especially for complicated functions f of xy it is complicated.

Because you are taking a derivative with respect to x whereas this could be a function of several variables x y z and so on I mean here we are simply talking about x and y. So, these evaluation of the derivatives are this thing, so the expressions for these derivatives may have to be to be computed analytically that is what I mean is that you may actually need to have them first on pen and paper and then you code it there.

And as we have already said that numerically evaluating derivatives is always a risky procedure and its prone to a lot of errors because of the fact that one usually divides it with a small number that h being small the method has you know as its own deficiency. So, let us now talk about Euler method ok. So, this is about the Taylor series method it is not very useful but a very innovative you simply write down the Taylor series expansion evaluate all the derivatives and calculate you know the the derivatives at the initial point that is given and that will do the job.

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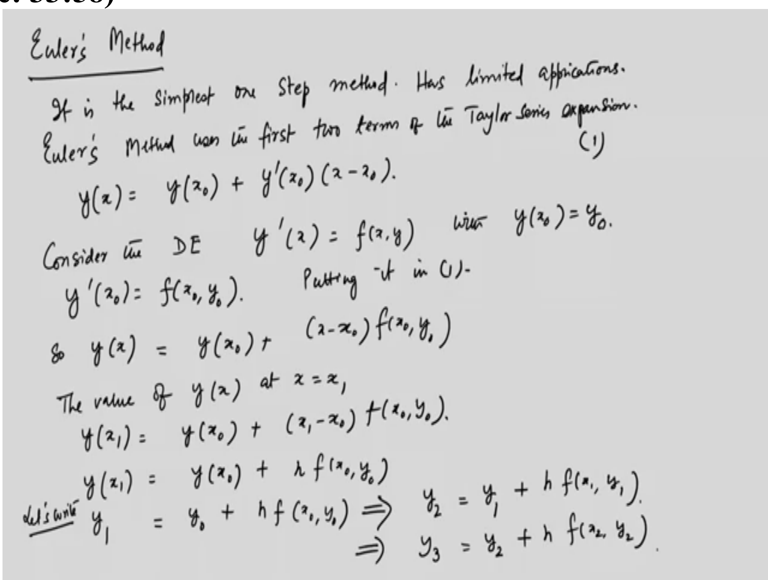

So, Euler's method is the simplest one-step method okay and of course it has also limitations but we studied it for the reason that it is the first method that or rather it is a method that is used in all the you know higher order methods or more efficient methods. So, it is important for us to learn this Euler method and we say that is a simplest one-step method has limited applications these are the disclaimers but yet we need to learn it.

And what it does is that it uses Euler's method uses the first 2 terms of the Taylor series expansion all right. So, what it does is the following that it tells that the solution is of this form where x 0 is the initial point and the slope or the derivative calculated there and then it is a x – x 0 ok. So, consider this DE to be solved y prime x equal to f x y with the initial condition as y at x 0 is nothing but y is 0.

So, y y prime at x 0 is nothing but f x 0 y 0, so this is the because the solution the equation is y prime equal to f so y prime at x 0 is nothing but f at x 0 and y 0, so y of x is nothing but y at x 0 + x – x 0 f of x 0 y 0 we simply put in place of so let us call this now as equation 1 and so we put it in 1 all right. So, this is your equation that the or rather this is the solution that you get, so this is, so the value of y at x at x equal to x 1 say specifically you are required to find the value of y at some x equal to x 1.

So, y at x 1 is y at x 0 + x 1 – x 0 f of x 0 y 0 okay if you take this value to be different from x 0 if x 1 is different from x 0 by an amount h we get y at x 1 is simply equal to y at x 0 + h f of x 0 y 0 ok simple enough and quite intuitive but of course has its limitations as we will see. So, let us write this as y 1 equal to y 0 + h fx 0 y 0 this allows us to write for the next interval y 2 equal to y 1 + h f x 1 y 1 and this further allows us to write y 3 equal to y 2 + h x 2 y 2.

So, just by knowing for a given interval just by knowing the solution at a subsequent point one can I mean just by knowing the solution at the proceeding point one can know the solution at the subsequent point by taking just the first 2 terms of the expression of this Taylor series expansion.

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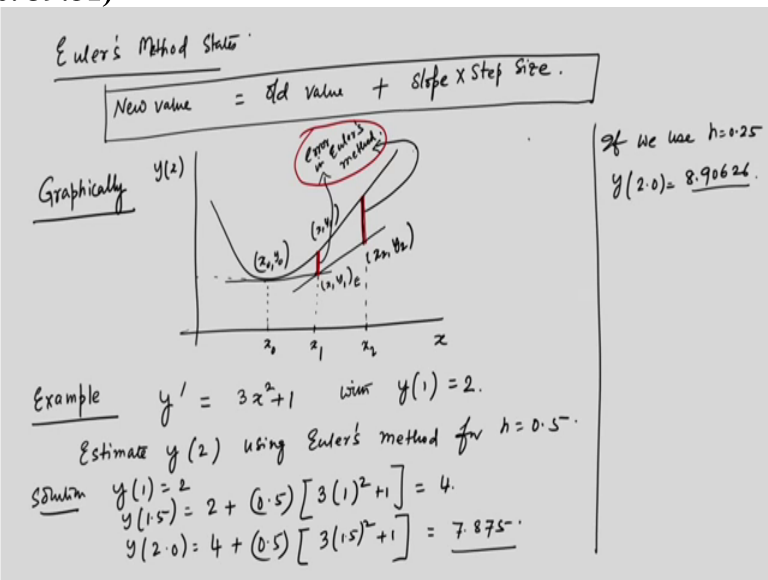

So, we can write this as so Euler's method states that a new value what we mean by new value is a new value of the solution is equal to old value of the solution that is y is function of x + slope that is your f which is a value of the function with y prime equal to f and then the step size that one uses okay. So, one can iterate this over the intervals or rather by breaking down or splitting down the interval into several sub intervals and can do that every time using this formula get a new value of the root of this equation or rather the solution of this differential equation.

Let us see graphically what it means say the solution is like this okay. So, this is y as a function of x and this is x and say this is my x 0 that is the starting point or the initial condition that is given okay. So, this value is of course is x 0 y 0 that is the value of the function okay and what I am supposed to do is that I'm supposed to draw a slope here okay and go to a point x 1 which is at a distance h apart and get the solution here, so this solution is so this point is understood by this method as x 1 y 1 ok.

But actual is this value x 1 y 1 ok so this is my x 1 y 1 actual but this is the by this Euler's method that is the x 1 y 1 that one1 gets x 1 y 1 let us call it Euler so we will write E here and so on. So, you are missing this much so this is the error in Euler’s method. so, because you are drawing a tangent there or a slope there and then you are calculating coming to the point x 1 and you are thinking that that is the solution and so on.

And then again you draw a slope here okay and come to a point x 2 again equidistant and you think that this point is your x 2 y 2 sorry and again you are making a big mistake of living out this thing so this is again the error in the Euler's method okay. So, these are the errors and these errors as you see that these errors are growing and these errors would grow if you keep you know doing this and this a function is a complicated function that is your final solution of the differential equation has a form like this.

And you know if it is deeply rises as x increases then you will miss out more and more and things like that okay. So, these are graphically this is what it means and these are the things that you are missing out let me draw it with a different colour so that it becomes you know more so this thing is what you are missing out okay. And this is what you are missing out all right these are the error which are creeping in at every stage as you are doing it from an interval say some you know a to b here of course we have taken a to be that x 0 and things like that.

So let me give an example so let us take an example of y prime equal to 3x square + 1 with given as the initial condition is given as y at x equal to 1 is equal to 2. See the main merit of this Euler's method is that you do not have to calculate higher order derivative the first derivative is going to be fine so that way is a simpler method but of course you are seeing that it introduces large errors as we go ahead with the you know and the procedure.

So the question is that estimate y at 2 using Euler's method for h equal to 0.5 okay so you are given at x equal to 1 so you go from x equal to 1 to 1.5, 1.5 to 2 so there are only 2 intervals that are to be used so it is just like this what is shown here. So, we already know so the solution is that we already know that y at 1 equal to 2 y at 1.5 used the formula which is 2 that is the old value and then this h which is the step size and then one has to calculate the value of the function or the this derivative which is already given.

And you have to calculate it at 1 so 3 into 1 square + 1 so this is like 4 okay. And similarly for the next one 2.0 one gets a 4 + again 0.5 is h so that is the step size and this value is 3 into 1.5 square + 1 so you see the since it is using the first derivative and first derivative is there in the equation itself in the first order differential equation that you are solving then this becomes 7.875 okay.

So, if you use if we use h equal to .25 that is further reduced the interval from.5 to .25 one gets y at 2.0 to be 8.905626 you should check this value and you would understand that just by taking the interval to be large which is 0.51 gets a value which is 7.875 with its almost it is equal to 9 so that is a large error.

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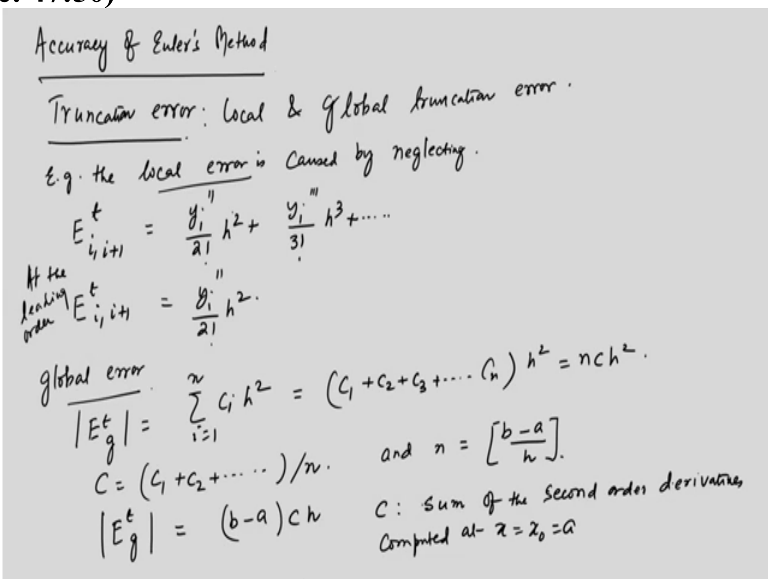

So, let us discuss on the accuracy of Euler's method, so as usual the accuracy is affected by 2 sources of this error and the 2 sources are at the round of error and the truncation error. So, of course the truncation error is dominant because you are truncating the Taylor series after the first term. So, because you are doing that the error is introduced at the second step onwards which is a double derivative.

So and every time you are missing the double derivative onwards so they kind of you know add up as you go ahead with these every interval from one interval to another like here we have missed a double derivative at this interval at 1.5 and then again we have missed a double derivative from at the 2 level to that at x equal to 2 and then this has a cumulative effect okay of these truncation.

And let us just write these truncation errors as these are well local and local and global truncation errors. So, what I mean by local and global is the following. So, the local one is at a given step the error that you pick up and global is the cumulative effect of all that. So, for example the local error is caused by neglecting this term onwards which is let us call it as i and i + 1 that is a truncation and this is y i double prime by 2 factorial h square + y i triple prime by 3 factorial h cube and so on.

So this is the leading order let us take the leading order at the leading order why leading order because h is small h is supposed to be small so this is equal to i i + 1 equal to y i double prime by 2 factorial h square so the global one is the sum total of all of them so let us write E t g for the global which is equal to sum over C i h square i equal to 1 to n and this is like C 1 + C 2 + C 3 and so on all the way up to C n h square equal to nC h square.

So, C equal to like C 1 + C 2 + so on divided by n and n equal to the total subintervals number of subintervals that one has used and so this E t g this is nothing but b - a does the total interval C into h were seized those coefficient. So, C is nothing but the sum of the all the second order derivatives computed at x equal to x 0 which is a that is the left you know interval extremity of the interval that is given to us.

So let us see that so this is just the second derivatives sum of the second derivatives that is what is your C is.

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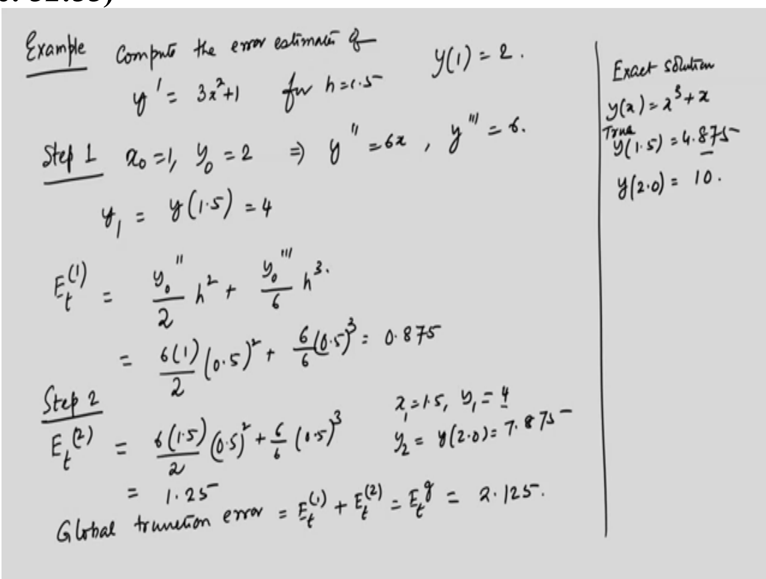

Let us see an example is giving you know simple examples so the step 1 is x 0, so what you have to do is that you have to let us compute the error estimates of y prime equal to 3 x square + 1 for h equal to 0.5 so that is the error that we calculate off the last problem. So, x 0 equal to 1 y 0 equal to 2 that is given the same problem as earlier so y at 1 equal to 2 so y double prime equal to 6x so y1 equal to y into 1.5 which is equal to 4 as has already been calculated.

And y triple prime is equal to 6 so et 1 that is the first step is y 0 double prime by 2h square I am taking also the third term h cube so this is like 6 into 1 Cal evaluated at x 0 which is equal to 1 by 2 into 0.5 square + 6 by 6 into 0.5 whole cube and this is like nothing but 0.875 that is the error at the first so this is step 1 and so this is the step one and E t 2, let us write it at step 2. This is again that so now our this thing is x equal to 1.5 y 1 equal to 4 y 2 equal to equal to y 2.0 equal to 7.875 and so on.

So, this is equal to 6 into 1.5 by 2 0.5 square + 6 by 6 0.5 cube so this is equal to 1.25 so these are the local truncation errors that step 1 and step 2. So, the global truncation error is its equal to E t 1 + E t 2 and this is what is we call it as E tg this is equal to just sum of both of them and that becomes equal to 2.125 okay. So, if you are interested in the exact solution which can be done analytically. So, y of x equal to x cube + x so y at so 2 values y at 1.5 is 4.875 how much did you get you got 4 as opposed to 4.875 and y at 2.0 which is what you need you got it as 10 whereas you got it as 7.875.

And if you reduce the interval you got about 9 but the exact value is about 10. So, these are the ones that are shown graphically by this red line so these are the values so this is like a larger value which is I mean 7.875 as opposed to 10 is the real value that you have it here and whereas you know you got a value which is much lower than that. And similarly this was 4.875 so anyway so these are the errors that are creeping in at every stage of your iteration of every interval of your iteration and that is why it has limited applicability.

But nevertheless it is a very intuitive method just uses 2 terms of the Taylor series expansion and they you do not have to do anything analytically because the differential equation itself which is y prime equal to f has a value that is given. So, one has to need to really calculate the value of the function and use it you know use the formula that your old new value equal to old value plus the slope into the step size and keep iterating these solutions make the interval smaller and smaller and it is likely that your accuracy of the method or the solutions that you obtain would improve.