# Introduction to Statistical Mechanics
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## Module No # 02
## Lecture No # 09
## Low and high temperature equations of state

Okay, welcome back! so if you recall in the last lecture I had stopped at a stage where I derived the specific heat of an ideal Fermi gas.

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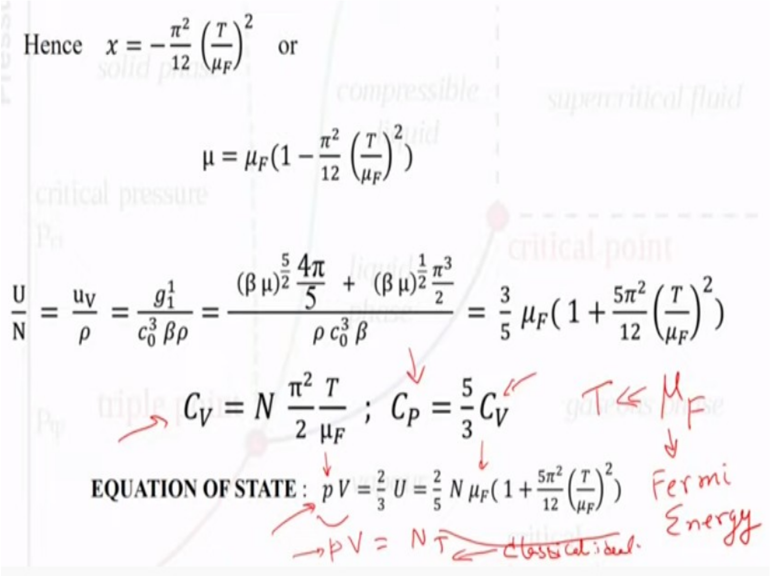

Hence $x = -\frac{\pi^2}{12}\left(\frac{T}{\mu_F}\right)^2$ or

$$\mu = \mu_F\left(1 - \frac{\pi^2}{12}\left(\frac{T}{\mu_F}\right)^2\right)$$

$$\frac{U}{N} = \frac{u_V}{\rho} = \frac{g_1^1}{c_0^3\,\beta\rho} = \frac{(\beta\mu)^{\frac{5}{2}}\frac{4\pi}{5} + (\beta\mu)^{\frac{1}{2}}\frac{\pi^3}{2}}{\rho\, c_0^3\,\beta} = \frac{3}{5}\mu_F\left(1 + \frac{5\pi^2}{12}\left(\frac{T}{\mu_F}\right)^2\right)$$

$$C_V = N\,\frac{\pi^2}{2}\frac{T}{\mu_F}\ ;\ C_P = \frac{5}{3}C_V$$

$T \ll \mu_F$

Fermi Energy

EQUATION OF STATE : $pV = \frac{2}{3}U = \frac{2}{5}N\mu_F\left(1 + \frac{5\pi^2}{12}\left(\frac{T}{\mu_F}\right)^2\right)$

$\rightarrow pV = NT \leftarrow$ classical ideal.

So the answer we obtained where these so the specific heat at constant volume was found to be proportional to temperature and so there was a relation between specific heat at constant pressure and constant volume which was of this nature. So and also we derived the equation of the state of an ideal Fermi gas at low temperature. So recall that this is applicable at low temperatures by low temperatures I mean temperatures small compared to μF and if you remember what μF was,

μF was the chemical potential at absolute 0 so that is also known as Fermi energy so the chemical potential at absolute 0 is known as Fermi energy. So the specific heat at constant volume is proportional to temperature at low temperatures and this is a hallmark of basically what is called electronic contribution to the specific heat of a solid. So the electronic contribution comes from the fact that fermions for solid contains fermions I mean it contain electrons has a you know in their orbitals of the item that make up the solid contain electrons.

And the electrons contribute to the specific heat of this solid and their contribution is governed by Fermi Dirac statistics so we have been successful in showing that the specific heat contribution of the electrons to the solid is proportional to the temperature at low temperatures, okay and also at low temperatures we can derive what is known as the equations of state and you can see that at low temperatures unlike so there is a dramatic departure from a classical ideal gas.

So if you recall, the classical ideal gas was $pV = NT$ so this is for a classical ideal gas this is regardless of what temperature I am talking about this is always true so but then you can see that as temperature is lowered as the pressure so if you fix the volume and fix the number of particles as temperature is lowered the pressure actually falls of a classical ideal gas but not so for a gas made of fermions.

So if you have a quantum gas made of fermions and so even though it is ideal it still as this dramatic departure from a classical ideal gas and origin for this I have already explained it is because of Pauli exclusion principle. So basically what was happening here is that if you start at 0 temperature for example for a classical ideal gas at absolute 0 the pressure is strictly 0 but the for a Fermi gas at absolute 0 the pressure is not 0 for the simple reason that you know we have already shown that pV is a so regardless of whether it is Bose gas Fermi gas or classical ideal gas $pV = 2/3\ U$.

So it is just so happens that at absolute 0 that total internal energy of a classical ideal gas is 0 because basically at 0 temperature all molecular motion ceases and so the molecules basically come to a standstill and they can do that because they are classical particles so it can simultaneously the positions being independent of time and velocity is being 0. And however that is not true for quantum gas so what happens is that at absolute 0 the system is in the ground state and then because the Fermions they obey Pauli’s exclusion principle which means that you cannot really populate or the 0 momentum state with more than two particles.

So if you have a macroscopic number of particles in your system then you are forced to well you can at most accommodate two particle in the 0 momentum state and immediately you will have to go to the next higher state which will bump up the energy because if it was strictly 0 momentum state and you know that energy is $p^2/2m$ and you can still get total energy is 0 if those electrons were there in the 0 momentum state.

But then given that only two electrons at most can be accommodate in the 0 momentum state so we have a macroscopic number of electrons. So the question is where do you accommodate them, so we start accommodating them in the higher and higher momenta, so by accommodating them in higher momentum states we are gradually increasing the energy of the system so when you reach a when you exhaust all your electrons you would have reached a very high level of energy and that is called a Fermi level, so the Fermi energy.

So now you can calculate the total energy contained in that system which will be enormous and that is the reason why pressure is also significant because it is going to be proportional to the Fermi energy and the Fermi energy is typically huge for a Fermi gas at absolute 0. So this is so there is a name for this type of pressure it is called the degeneracy pressure and it is so happens that this degeneracy pressure is actually the reason why white dwarfs the stars you know there are stars in the sky that are known as white warps.

So white dwarfs are stars where the nuclear fusion has ceased so all the nuclear process is that for example out own sun is still burning bright because the nuclear fusion process that it is taking place because it is releasing the huge amounts of energy which is contributing to the pressure that counteracts the gravity that is trying to shrink the stars into a point. So the gravity is always present but then the pressure that responsible for counteracting the gravity is not going to be always present, it is going to be present so long as the those nuclear reactions take place.

But then you know that how nuclear reaction works, lighter elements get converted to heavier elements and so the so on and hence so forth. So once you start running out of lighter and lighter elements the nuclear reactions so there is a fusion reaction slows down and the stage is reached where it is ceases altogether. So the moment nuclear fusion ceases then there is no you know the pressure that you know contributed by those reaction also becomes 0.

So now when no pressure is available to counteract gravity, so gravity takes over and the star shrinks and shrinks. So now the big questions is that does it shrink to a point immediately or is there some mechanism some other mechanism besides nuclear fusion that takes over that prevents these star from collapsing to a point so it so happens that there is mechanism and that was a famously discovered by the great Indian astrophysicist S Chandrasekar on his voyage from India to England on a ship.

So he thought about what could be those mechanisms and he knew about quantum mechanics and Fermi gas and that sort of thing and you realize that the mechanics could be precisely that degeneracy pressure. See once the nuclear reaction ceases then all the matter tries to collapse to a very small region of space due to gravity and all the atoms get squeezed together so tightly that the electrons dislodge from the atoms so at this they become part of the collective.

So they start becoming like a gas so there will be a gas of electrons which propagate through the entire star so that is precisely what a Chandrasekar studied is imagined that there is gas of electrons which have been dislodged from the atoms which are propagating to the entire star, so there is going to be a degeneracy pressure. So if you try to compress them any further because there is this pressure due to the Fermi level, the Fermi energy of those electrons causes them to resist that shrinking any further.

So there is going to be a stage where the radius of the star reaches an equilibrium value where the pressure, the degeneracy pressure which is trying to prevent the star from shrinking any further is cancelled by the or it is counteracted by the gravitational pressure that is trying to shrink the star, so there is a gravitational force which is trying to pull it in and there is counteracting pressure force due to degeneracy the Pauli exclusion principle which is trying to counteract that when the two balances and equilibrium and that equilibrium is reached for a certain radius of the star and in fact more importantly for a certain mass of so Chandrasekar derived is famous Chandrasekar limit where he shows that there is an interesting upper limit to the initial mass of the stars.

So this mechanism, this equilibrium mechanism is mathematically feasible only if the mass of the initial star was not too high. So if the mass of the initial star was too high his equations predicted that there would be some instability and so Chandrasekar famously said that you know it, so this mechanism cannot support this star so the star would collapse to a point and okay i do not want to spend too much more time on the Chandrasekar limit right now because I am going to be discussing probably the detailed derivation of the Chandrasekar limit little later.

So this is just a preamble for you to understand the importance of the equation of state of a ideal Fermi gas. So it has a huge implication to varies feels of some feels lf physics such as astrophysics and so on. Okay so now let us get to the you know counter part of what we have been studying namely the Bose gas. So we have studied the ideal Fermi gas now let us go and see if we can study the ideal Bose gas.

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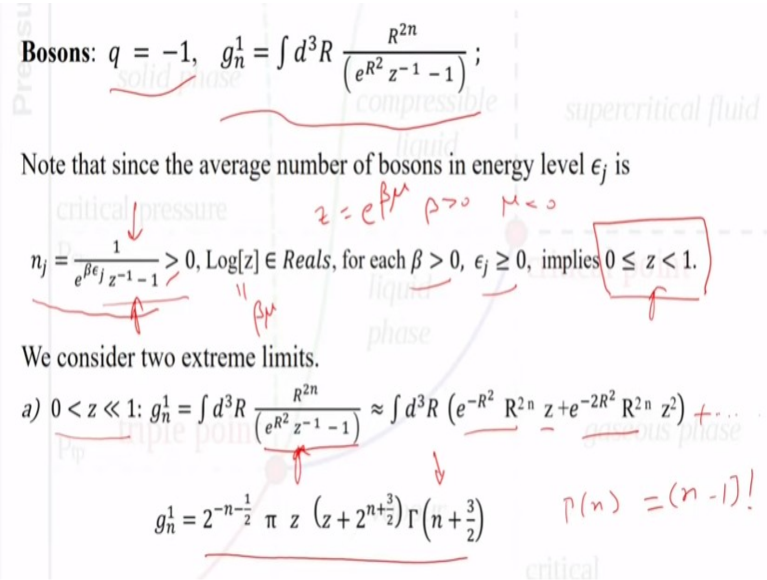

**Bosons**: $q = -1, \quad g_n^1 = \int d^3R \, \frac{R^{2n}}{\left(e^{R^2} z^{-1} - 1\right)}$ ;

Note that since the average number of bosons in energy level $\epsilon_j$ is

$z = e^{\beta\mu} \quad \beta > 0 \quad \mu < 0$

$n_j = \frac{1}{e^{\beta\epsilon_j} z^{-1} - 1} > 0$, Log[z] ∈ *Reals*, for each $\beta > 0$, $\epsilon_j \geq 0$, implies $\boxed{0 \leq z < 1}$.

$= \beta\mu$

We consider two extreme limits.

*a)* $0 < z \ll 1$: $g_n^1 = \int d^3R \, \frac{R^{2n}}{\left(e^{R^2} z^{-1} - 1\right)} \approx \int d^3R \left(e^{-R^2} R^{2n} z + e^{-2R^2} R^{2n} z^2\right) + \ldots$

$$g_n^1 = 2^{-n-\frac{1}{2}} \pi z \left(z + 2^{n+\frac{3}{2}}\right) \Gamma\left(n + \frac{3}{2}\right)$$

$\Gamma(n) = (n-1)!$

And that has its own interesting features as you will see, so unlike the Fermi gas where there is a pressure even at absolute 0 that is not going to be the case in the ideal Bose gas but then, so even though you know its similar to classical ideal gas in that sense but there are there is a still there is a difference between the classical ideal Bose gas and the difference in what is known as Bose Einstein condensation which is peculiar to ideal Bose gas.

So let me go ahead and discuss these issues so if you recall for Boson’s this q which was the parameter that determined the quantum statistics was actually going to be -1 we had to select it to be -1. So if you remember if it,if we select q to be +1 we would be describing fermions and then you know just coincidence q to 0 I would be describing classical Boltzmann gas. So now given that q is -1 corresponds to bosons and now I am called upon to evaluate such an integral so if you recall this parameter n has only two interesting use one of them is 0 the other is 1.

So when it is 0 this integral would be corresponding to basically the total number of particles in the system as a function of z which is the fugacity. So I will be discussing or the density as it were so density of the particle related to fugacity is obtained by evaluating this integral for $n = 0$. So the integral would correspond to the total internal energy of the system or which studying the energy density of the bosons as a function of z which is fugacity.

But then besides this integral we had something else we had observed namely in this ratio that I am talking about here there is

$$n_j = 1/(e^{\beta\varepsilon j} z^{-1}-1)$$

this ratio this has a physical meaning namely it corresponds to the number of Bosons in energy level $\varepsilon_j$ and given that this is average number of Bosons in that energy level it has to be greater than 0 because it cannot be certainly less than you can be 0 but it cannot be less than 0.

So what is should be demanding is that the number of Bosons in energy level $\varepsilon_j$ had better the greater than or equal to 0 and given that $z = e^{\beta\mu}$ and $\beta$ and $\mu$ are real numbers then you can see that $\log z = \beta\mu$ which is of course also real. And since $\beta > 0$ and $\varepsilon > 0$ and this as to be valid so you can convince yourself that all this put together means z as to be between 0 and 1, okay.

So it cannot be more than 1 so what does that imply so that implies this is $\beta > 0$ that implies that $\mu < 0$. So in a Bose gas typically the chemical potential is always less than or equal to 0 okay so that is the same as saying z is between 0 and 1. So now let us consider two extreme limits, one extreme limit is where z is really close to 0 z is always greater than 0 of course but let us imagine that it is very close to 0.

So if it is very close to 0, so recall that if z is very close to 0 this number is huge so the leading contribution that is in the denominator so the leading contribution is going to be proportional to z and then I can do my you know expansion in powers of z and the term proportional to z will involves this integration over this variable and the term proportional to $z^2$ will involve an integration of that sort.

So when I do that so on and hence so forth so when I do this integration I end up with this answer so you can see that it is proportional to z and this the another term is proportional to z $^2$ and so on. Alright so this is my gamma function if you recall you know if n was the integer when it is because n is either 0 or 1. So if n was integer it would (n-1)! but then this is n + 3/2 so you have use an appropriate generalization of the factorial function which is what is called the gamma function. So I urge you to look it up if you do not know what it is we have encountered this before by the way alright.

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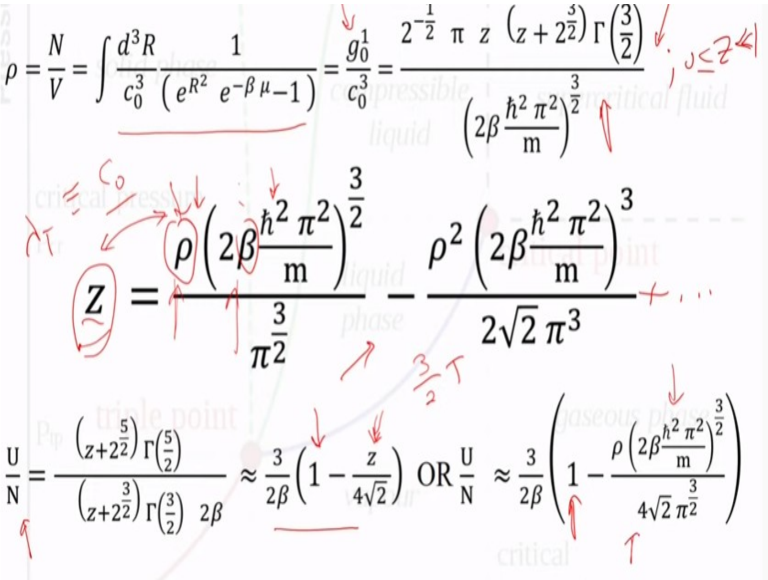

$$\rho = \frac{N}{V} = \int \frac{d^3R}{c_0^3} \frac{1}{\left(e^{R^2}\ e^{-\beta\mu} - 1\right)} = \frac{g_0^1}{c_0^3} = \frac{2^{-\frac{1}{2}}\ \pi\ z\ \left(z + 2^{\frac{3}{2}}\right) \Gamma\left(\frac{3}{2}\right)}{\left(2\beta \frac{\hbar^2 \pi^2}{m}\right)^{\frac{3}{2}}}\ ;\ 0 \le z \ll 1$$

$$Z = \frac{\rho\left(2\beta \frac{\hbar^2 \pi^2}{m}\right)^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} - \frac{\rho^2 \left(2\beta \frac{\hbar^2 \pi^2}{m}\right)^3}{2\sqrt{2}\,\pi^3} + \dots$$

$c_0 = \lambda_T$

$\frac{3}{2} T$

$$\frac{U}{N} = \frac{\left(z + 2^{\frac{5}{2}}\right) \Gamma\left(\frac{5}{2}\right)}{\left(z + 2^{\frac{3}{2}}\right) \Gamma\left(\frac{3}{2}\right)\ 2\beta} \approx \frac{3}{2\beta}\left(1 - \frac{z}{4\sqrt{2}}\right) \text{ OR } \frac{U}{N} \approx \frac{3}{2\beta}\left(1 - \frac{\rho\left(2\beta \frac{\hbar^2 \pi^2}{m}\right)^{\frac{3}{2}}}{4\sqrt{2}\,\pi^{\frac{3}{2}}}\right)$$

So now let us I am going to make use of this so remember that I told you if n is 0 ,I will be using that for evaluating the density of bosons. So the density of the bosons is going to be this and if you recall $c_0$ is my notation for the thermal wave length which is more commonly denoted by $\lambda_T$ okay so most of the other books would call it $\lambda_T$. I have started off writing $c_0$ I do not want to change my notation midway so I am continuing with $c_0$ so that is basically thermal wave length alright.

So now the density of Bosons is going to be this and I put in my expression for $c_0$ and I get this for the density of bosons as a function of chemical potential assuming the chemical potential as very close to 0 and by the way what is that correspond to sorry the fugacity so this is only this only correct when the fugacity is very small compared to 1 and greater than 0 in other it is very close to 0 and positive.

So what does that mean in terms of the chemical potential, so if the fugacity is very small it means that the chemical potential is large and negative because remember that the fugacity is nothing but $e^{\beta\mu}$ so since $\beta$ is positive and z fugacity as to be really small that means that $\mu$ is large and negative. So remember that you know all the expressions that we should be handling should have the number of particles and volume fixed so rather than having a situation where the fugacity or the chemical potential is fixed.

So even though this method kind of naturally allows you to study a system where the number of particles is fluctuating that is particles are being exchanged with the environment in addition to energy so but I want to you know the less usual situation the more usual situation is when only the energy is being exchanged with the environment whether the number of particles in the system is fixed.

So if I want to go back and forth between two pictures it is going to be possible in natural way it is going to be possible to do that with the same formulas would describe either situation only if, if you remember there was a condition that is going to be those two pictures are going to be equivalent only if you are working in the thermodynamic level. So in other words only it only works for large systems where you can ignore fluctuations we have been we have discussed that earlier okay.

Given all this caveats I can go ahead and invert this relationship and write down the expression for the fugacity in terms of density of the particles. So given that fugacity is small that is the same as if the temperature is fixed that is another way of saying that the density is small so it is a dilute system. So I am looking at a dilute limit of a Bose gas so it is also it is also the as so if you fix the density z has to be very small so if z is very small that is two ways of doing it either may $\rho$ very small or $\beta$ very small.

So how do you make $\beta$ very small you make $\beta$ very small by making temperature very large so z very small has two different interpretations one is the interpretation that z is very small means $\rho$ is very small because we store related and $\rho$ is very small is the same as studying a dilute limit so that means the number density is very small means they are very few Bosons in your volume V.

Alternatively if you do not want to do that you can fix the number of Bosons in your system to be as many as you want but then you have to pay the price that if you want to make z very small you have to make β very small which is the same as studying a very high temperature limit of the Bose gas. So that is also the classical limit so you can see the dilute limit is the same as the classical limit.

So you can see that that is also the same as h bar tends to 0 so if you choose not to do either of these, you fix ρ, fix β and you still want z to be small the only other way is to make $\hbar \rightarrow 0$ which of course you cannot do in nature but I mean mathematically you can see then that is equivalent. So z being very small is the same as either thinking of dilute limit where the ρ is very small or if you do not want to do that you think of β is very small which is the same as high temperature or the classical limit or alternatively the other version of the classical limit this $\hbar \rightarrow 0$ where ρ and β are fixed, alright.

After that long preamble so let me get to the thing which I want to study so I am going to find the energy per particle so I have to divide the expression between the total energy of the bosons divided by the total number of particles. So when I do that I get this expression and remember that z is really small so because of that the leading contribution to the energy particle is precisely the classical result 3/2 T.

So remember that the energy per particle of a classical ideal gas is 3/2 T because its U = 3/2 NT but per particle is 3/2 T. So that would be the leading contribution here and there is quantum correction caused by the fugacity and which can of course be expressed in terms of density. So this is the classical limit of a quantum Bose gas okay semi classical limit the strictly classical limit would be this this would be the leading correction to the classical limit.

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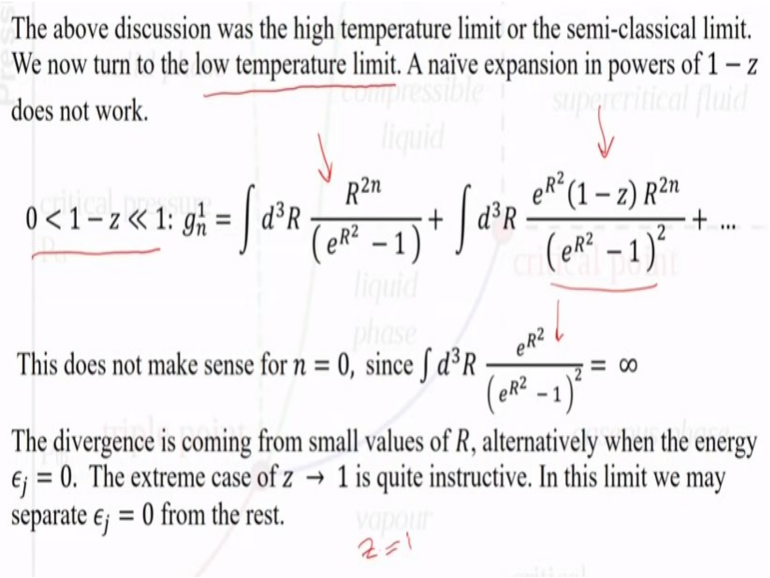

The above discussion was the high temperature limit or the semi-classical limit. We now turn to the low temperature limit. A naïve expansion in powers of $1-z$ does not work.

$$0<1-z\ll 1:\ g_n^1=\int d^3R\,\frac{R^{2n}}{\left(e^{R^2}-1\right)}+\int d^3R\,\frac{e^{R^2}(1-z)\,R^{2n}}{\left(e^{R^2}-1\right)^2}+\dots$$

This does not make sense for $n=0$, since $\int d^3R\,\frac{e^{R^2}}{\left(e^{R^2}-1\right)^2}=\infty$

The divergence is coming from small values of $R$, alternatively when the energy $\epsilon_j=0$. The extreme case of $z\rightarrow 1$ is quite instructive. In this limit we may separate $\epsilon_j=0$ from the rest.

$z=1$

Alright so now let us discuss the other extreme so remember I have told you that z the fugacity is always between 0 and 1 so what I have studied is one extreme limit where z is close to 0. So I can study the other extreme where z is close to 1 of course you might be wondering why am I studying various limits why not leave z as it is the answer is that of course the general expression are already there but the point is they are not illuminating they do not teach you anything it some ugly looking formula that you have to stare at and believe.

So by studying limits what you are doing basically is that you are trying to get a feel for the subject you are trying to see what happens if I you know increase this parameter keeping the others fixed what type of physics am I recovering from that or it does the system go to this situation or that situation so that is what you are trying to get a deeper understanding of those formulas themselves so it is not physics it is not fixed it not just about deriving formulas it is about getting a feel for what those formulas mean practically.

Alright so in order to do that I just finished studying the limit where z was close to 0 now I am going to study the other limit where z is close to 1 so that is same as saying you know this 1 - z is close to 0. z is close to 1 as same as saying 1 – z is close to 0 and remember 1 – z is positive because z is less than 1 okay so now if I go ahead and naive expand in powers of 1 – z you know I just remember that I have to evaluate this and if I just decided to think of 1 – z as a small quantity and try to expand in powers of that as we disappointed because of course I am going to get this which makes some sense.

But I am going to also get start something bigger in the denominator and that is going to cause divergence for small values of R. So all the higher terms are going to diverge so implying that there is some kind of an singularity at z = 1 is a some kind of singular point that I cannot really expand and I mean it is a false assumption to say that you can expand in power of 1 – z so that would be possible only if your answer and analytic function of 1 – z.

So what this is telling you is that it is not which is by all this coefficient are diverging. So now the reason why it does not work is because so this is trying to tell you something that when R is small so when the z is close to 1 the R = 0 is going to be problematic.

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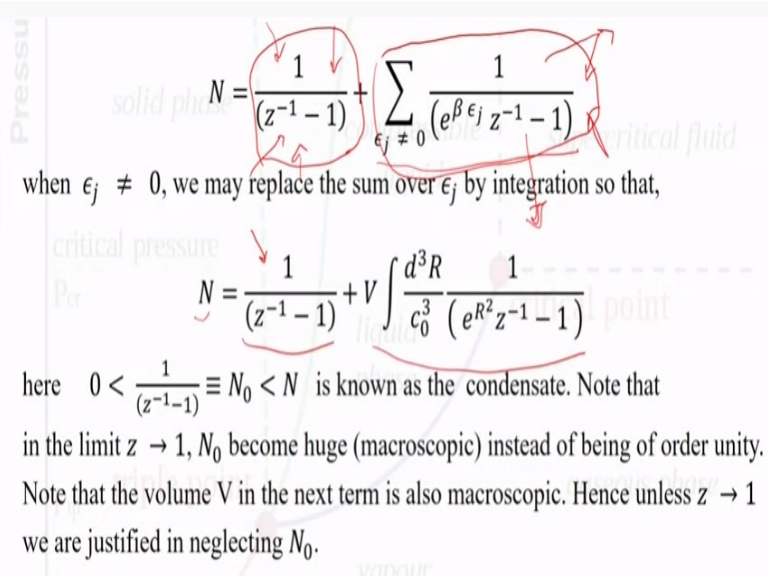

$$N = \frac{1}{(z^{-1}-1)} + \sum_{\epsilon_j \neq 0} \frac{1}{(e^{\beta\epsilon_j} z^{-1} - 1)}$$

when $\epsilon_j \neq 0$, we may replace the sum over $\epsilon_j$ by integration so that,

$$N = \frac{1}{(z^{-1}-1)} + V\int \frac{d^3R}{c_0^3}\frac{1}{(e^{R^2}z^{-1}-1)}$$

here $0 < \frac{1}{(z^{-1}-1)} \equiv N_0 < N$ is known as the condensate. Note that in the limit $z \to 1$, $N_0$ become huge (macroscopic) instead of being of order unity. Note that the volume V in the next term is also macroscopic. Hence unless $z \to 1$ we are justified in neglecting $N_0$.

And in fact what you can do is you can start off with remember that this was really the expression for the total number of particles so I am kind of deliberately separating out a term which corresponds to $\epsilon_j = 0$ so why am I doing that? I am always allowed to do the mathematical identity I can always do this because put together sum over all $\epsilon_j$ and the reason why I am doing this is because you see when z is close to 1 this is going to be divergent see if z is not close to 1 this is some number of the order of unity.

So if z is between 0 and 1 so suppose z is half this some uninteresting number of order unity but however this is proportional to the size of the system because this is if the rest of the energy level this is sum over the rest of the energy levels. So if z is not anywhere close to 1 then I might as well you know ignore this in favor of this, separating without this term does not really give me anything I mean there is no mistake or benefit to doing this.

So if Z is not close to 1 but the point is that as Z starts approaching 1 see there is going to be a stage which is reached where this quantity is going to be as important as this quantity. So as you approach z tends to 1 this was huge because it was proportional to the size of the system there is a sum over all the energy levels except one of them. So it is still proportional to the size of the system and this was going to be of order unity for the most part except when z tends to 1 it is also going to start increasing until the stage is reached when these two becomes comparable.

And that is the situation I am interested in when z is incredibly close to 1 so much so that this is of the same order the first term which I have separated it out is this the same order is the rest which is proportional to the size of the system. So in other words when $\varepsilon_j$ is not 0 there is no singularity in the denominator now I can impunity replies the summation with integration because with the confidence that it is not going to be mathematically ill-defined there is no divergent because now R is not 0 so it is safe to do this thing ,this correspondence.

So now it is so happens that this has a particular name that we can assign it to our notation we denote this by $N_0$. $N_0$ is what is known as the number of Bosons in the condensate so remember that is some kind of a number because if this is the total number of particles and this is that total number of particles you know not including $\varepsilon_j = 0$ and this by implication as to be number of particles with $\varepsilon_j = 0$.

So that is called the condensate so that means the if there are macroscopic number of particles with $\varepsilon_j = 0$ that is what known as the condensate or in that phenomenon is known as Bose Einstein condensation.

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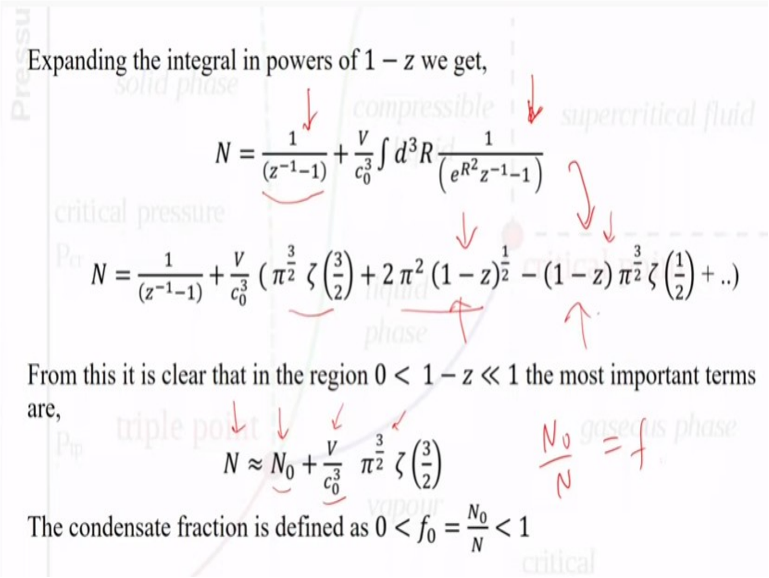

Expanding the integral in powers of $1 - z$ we get,

$$N = \frac{1}{(z^{-1}-1)} + \frac{V}{c_0^3}\int d^3R \, \frac{1}{\left(e^{R^2} z^{-1}-1\right)}$$

$$N = \frac{1}{(z^{-1}-1)} + \frac{V}{c_0^3}\left(\pi^{\frac{3}{2}}\,\zeta\left(\frac{3}{2}\right) + 2\,\pi^2\,(1-z)^{\frac{1}{2}} - (1-z)\,\pi^{\frac{3}{2}}\,\zeta\left(\frac{1}{2}\right) + ..\right)$$

From this it is clear that in the region $0 < 1 - z \ll 1$ the most important terms are,

$$N \approx N_0 + \frac{V}{c_0^3}\,\pi^{\frac{3}{2}}\,\zeta\left(\frac{3}{2}\right)$$

$$\frac{N_0}{N} = f$$

The condensate fraction is defined as $0 < f_0 = \frac{N_0}{N} < 1$

So now let us see what we can do with these formulas and so it is so happens that now I can evaluate this integral by expanding of course not in integer powers of 1 – z but then I have to you know in appeal to my knowledge of this poly log function, so I am gonna skip this derivation so you can just do it yourself on some you know some symbolic algebra package like mathematica and you get your I mean if you do it without any further assumptions it becomes the poly log function.

And so it is $Li_{3/2}(z)$ and then you expand that poly log in powers of 1 – z and you will see that mathematica or any of those symbolic packages will tell you that the leading term is this the next leading term is non-analytic in 1 – z which is why you are getting all those things you will have it is when you try to do a name Taylor series. So the name Taylor series does not work in this case because the dependence is actually non-analytic in 1 – z so because they are no, it is not just integer powers half integer powers as well okay.

But so happens that you can write down the series, now when z is close to 1 what this is telling you is that you can regardless of whether analytic or not you can simply throw away all these term. So when z approaches 1 so remember that z approaches 1 is the interesting limit because z is breathtakingly close to 1 it is only then when this is microscopic I mean even is z this slightly far away from 1 abruptly drops to some quantity of order unity so it has to be extraordinarily close to 1 otherwise what we are saying will be uninteresting.

So if z is extraordinary close to 1 we can certainly drop all this in which case we end up with this result that the total number of particles is the total number of particles in the condensate which is remember now microscopic times the volume of the system times the universal constant and of course it involves the thermal wave length here. So it so happens that this is now we can write down an expression for this ratio $N_0/N$ which is the fraction of Bosons in the condensate you know as a ratio of total number of Bosons in the system.

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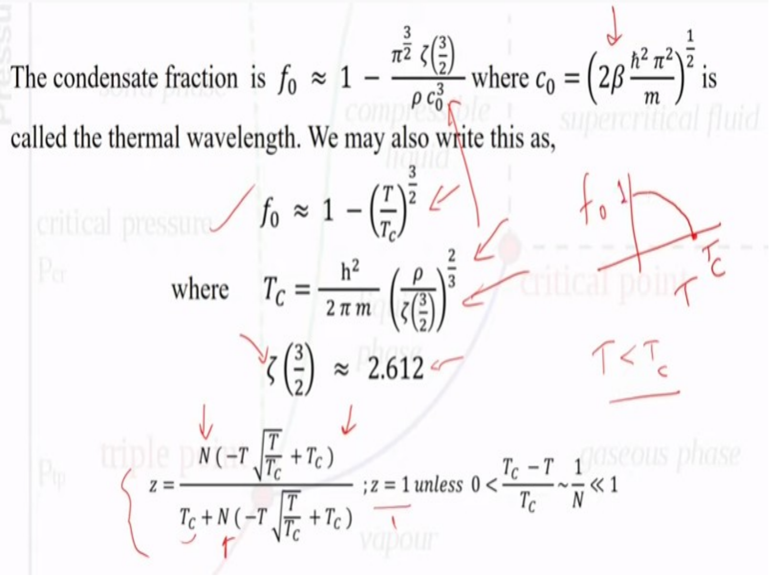

The condensate fraction is $f_0 \approx 1 - \frac{\pi^{\frac{3}{2}}\,\zeta\left(\frac{3}{2}\right)}{\rho\, c_0^3}$ where $c_0 = \left(2\beta\frac{\hbar^2\pi^2}{m}\right)^{\frac{1}{2}}$ is called the thermal wavelength. We may also write this as,

$$f_0 \approx 1 - \left(\frac{T}{T_C}\right)^{\frac{3}{2}}$$

$$\text{where} \quad T_C = \frac{h^2}{2\pi m}\left(\frac{\rho}{\zeta\left(\frac{3}{2}\right)}\right)^{\frac{2}{3}}$$

$$\zeta\left(\frac{3}{2}\right) \approx 2.612$$

$$z = \frac{N\left(-T\sqrt{\frac{T}{T_C}} + T_C\right)}{T_C + N\left(-T\sqrt{\frac{T}{T_C}} + T_C\right)}\; ; z = 1 \; unless \; 0 < \frac{T_C - T}{T_C} \sim \frac{1}{N} \ll 1$$

So you can see by definition that fraction has to be between 0 and 1 and then you can derive a formula for that condensate fraction as it were and you can express it in terms of that temperature because remember that this was thermal wavelength which was dependent on temperature in this fashion and you can go ahead and re-express the condensate fraction in terms of temperature and then you can write it in this compact and appealing form where it the condensate fraction becomes 1 minus that temperature in units of some characteristic temperature called $T_c^{3/2}$ .

So this is a very classic standard result in Bose Einstein condensation and this characteristic temperature scale it depends on the density of Bosons in your system and this is known as the Bose Einstein condensation temperature. And if you are wondering about this so funny symbols this is nothing but a ζ function having you do not have to strictly know what it is because after all I got it from simply using my symbolic package to you know expand it is in power of 1 – z in it just gives me this funny looking numbers and they are not funny to a mathematician they know very well they called zeta functions.

And but then you know if you want to know numerically what it is 2.612 etc., I mean it is approximately 2.612. So as a result the Bose Einstein condensation temperature is the density of Bosons raise to two thirds times of whole bunch of constants. So you can see that I can also express my z in terms of the temperature and this is very clear that z is 1 unless T starts to approach $T_c$ so far so long as $T < T_c$ so this only works at low temperatures so low compared to $T_c$ small formalism.

So when temperature is less than $T_c$ immediately remember that this N is huge, because N is huge this thing less important so immediately this becomes 1. So it only this is going to break down only when you really close to $T_c$ . So for the most part z is 1 so I am just a consistency cross check so you do not pay too much attention to this important thing is this one. This is the condensate fraction as a function of temperature.'

So you can even plot this so it is going to look like this I forgot to plot it, so I can plot it out right now so it is going to look like this. So this is my $T_c$ and this is 1 so when at absolute 0 it looks like this then it drops and then at $T_c$ it becomes 0.

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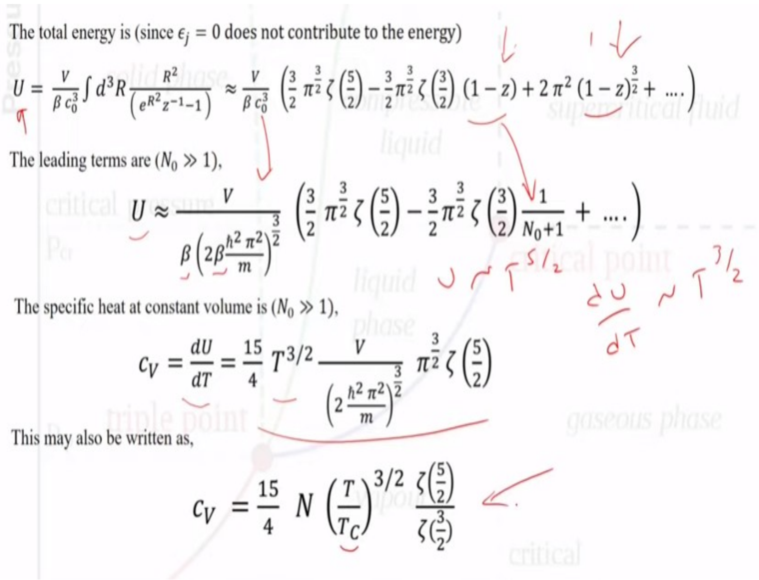

The total energy is (since $\epsilon_j = 0$ does not contribute to the energy)

$$U = \frac{V}{\beta c_0^3}\int d^3R \frac{R^2}{\left(e^{R^2}z^{-1}-1\right)} \approx \frac{V}{\beta c_0^3}\left(\frac{3}{2}\pi^{\frac{3}{2}}\zeta\left(\frac{5}{2}\right) - \frac{3}{2}\pi^{\frac{3}{2}}\zeta\left(\frac{3}{2}\right)(1-z) + 2\pi^2(1-z)^{\frac{3}{2}} + \ldots.\right)$$

The leading terms are ($N_0 \gg 1$),

$$U \approx \frac{V}{\beta\left(2\beta\frac{\hbar^2\pi^2}{m}\right)^{\frac{3}{2}}}\left(\frac{3}{2}\pi^{\frac{3}{2}}\zeta\left(\frac{5}{2}\right) - \frac{3}{2}\pi^{\frac{3}{2}}\zeta\left(\frac{3}{2}\right)\frac{1}{N_0+1} + \ldots.\right)$$

$U \sim T^{5/2}$

$\frac{dU}{dT} \sim T^{3/2}$

The specific heat at constant volume is ($N_0 \gg 1$),

$$C_V = \frac{dU}{dT} = \frac{15}{4}T^{3/2}\frac{V}{\left(2\frac{\hbar^2\pi^2}{m}\right)^{\frac{3}{2}}}\pi^{\frac{3}{2}}\zeta\left(\frac{5}{2}\right)$$

This may also be written as,

$$C_V = \frac{15}{4}N\left(\frac{T}{T_C}\right)^{3/2}\frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)}$$

So now you can just repeat this activity for, so remember we just did this for that total number of particles so the density, number density so the particle now you can do the same thing for the energy density of the particles, the similar Taylor series, then here too you get a non-analytical coming to first term appears analytic in 1 – z where the very next term is not so even here you cannot really do a naïve Taylor's series.

So now you can rewrite this in terms of remember that z is breathtakingly close to 1 so that is amplified by this correspondence which is just $1 - z = 1/(N_0+1)$ where $N_0$ is the number of Bosons in the condensate which is enormous and because of that 1 – z is incredibly timing which is why I am a entitled to ignore them. So I am entitled to ignore this term and etc., so as a result I am able to write down my energy in the system and related to temperature.

So long as temperature is small compared to TC, so when temperature is small compared to TC the enormous number of Bosons in the condensate and I am able to do this so if I know the relation between the total internal energy and temperature I can write down the specific heat at constant volume remember that volume is fixed in my analysis till now. So by definition I am calculating the specific heat at constant volume.

So when I do that I get a result which says that it is a this specific heat at constant volume is a proportional 3/2 T . $T^{3/2}$ because remember that there is a $\beta^{3/2}$ there and there is a beta so the energy density is actually proportional to temperature raise to 5 halves, okay. So and dU/dT is proportional to $T^{3/2}$ so that is how it works okay.

So now I can rewrite it in terms of I mean I can get rid of all this horrible looking constants and re-express it in terms of this nice thing which we have called a condensation temperature and there is this interesting looking formula that emerges which says that specific heat of an ideal Bose gas at low temperatures is proportional to temperature raise to 3 halves measured in unit of the condensation temperature.

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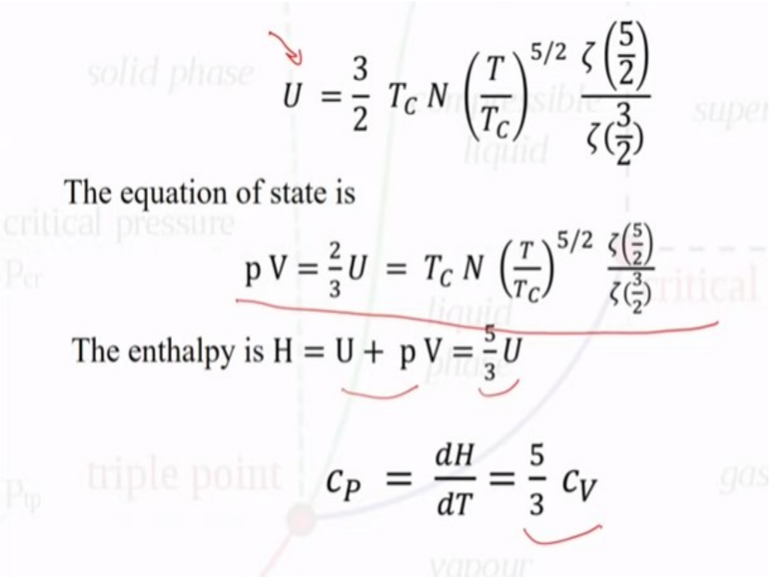

$$U = \frac{3}{2}\, T_C\, N \left(\frac{T}{T_C}\right)^{5/2} \frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)}$$

The equation of state is

$$\underline{\mathrm{p\,V} = \frac{2}{3}U = T_C\, N \left(\frac{T}{T_C}\right)^{5/2} \frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)}}$$

The enthalpy is $\mathrm{H} = \mathrm{U} + \underline{\mathrm{p\,V}} = \underline{\frac{5}{3}}U$

$$c_P = \frac{dH}{dT} = \underline{\frac{5}{3}\, c_V}$$

So as a result I can do the same thing with my total internal energy, I can re express it in terms of $T_C$ rather than those ugly looking constants which now you know total internal energy it allows me to express or write down my equation of state. So my equation of state is this pV =(2/3)U regardless of whether I am studying classical gas or Bose gas or Fermi gas so long as ideal and as a result the energy versus momentum dispersion is $p^2$ / 2m in all three of them, all three cases and lastly I am working in 3 dimensions that means my gases occupy 3 dimensions of space.

So if somehow they occupy only two dimensions this is not going to be valid anymore this was specific heat at constant volume. But then if I want to get specific heat at constant pressure what I should be doing is calculating what is known as enthalpy so if you remember I have explained to you earlier there is a formula for that which is used as pV and that is going to be five thirds U and then the rate of change of enthalpy with temperature will give me specific heat at constant pressure that is five thirds the specific heat at constant volume.

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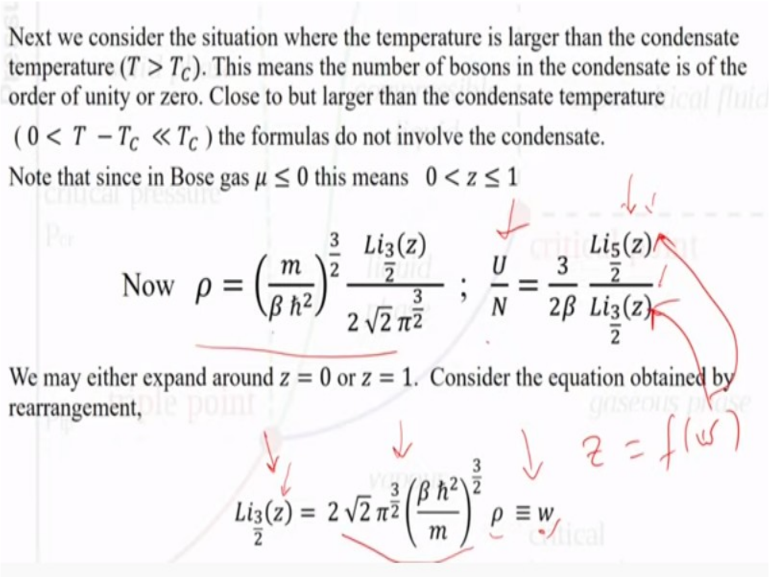

Next we consider the situation where the temperature is larger than the condensate temperature ($T > T_C$). This means the number of bosons in the condensate is of the order of unity or zero. Close to but larger than the condensate temperature ( $0 < T - T_C \ll T_C$ ) the formulas do not involve the condensate.

Note that since in Bose gas $\mu \leq 0$ this means $0 < z \leq 1$

$$\text{Now} \quad \rho = \left(\frac{m}{\beta\,\hbar^2}\right)^{\frac{3}{2}} \frac{Li_{\frac{3}{2}}(z)}{2\sqrt{2}\,\pi^{\frac{3}{2}}} \quad ; \quad \frac{U}{N} = \frac{3}{2\beta}\,\frac{Li_{\frac{5}{2}}(z)}{Li_{\frac{3}{2}}(z)}$$

We may either expand around $z = 0$ or $z = 1$. Consider the equation obtained by rearrangement,

$$Li_{\frac{3}{2}}(z) = 2\sqrt{2}\,\pi^{\frac{3}{2}}\left(\frac{\beta\,\hbar^2}{m}\right)^{\frac{3}{2}} \rho \equiv w$$

$z = f(w)$

So now just for sake of completeness I am going to study the case, so this was when the temperature was less than $T_C$ when there are was a macroscopic number of Bosons in the condensate, now imagine that temperature is larger than $T_C$ but not so large that that the system approaches the classical limit where if it approaches the classical limit then of course it is we already studied.

So now imagine that the temperature is slightly more than $T_C$ in which then all of the sudden there are absolutely no Bosons in the are there will be only you know of order unity Bosons in the condensate so the number of bosons in the condensate is no longer macroscopic is only a few so now I can go ahead and study, I want to study what happens when the temperature is slightly above $T_C$.

So in order to do that, if you remember the general formula where these so that density was related to z in this general fashion and the energy per particle was related to z also in this general fashion. So now what I should be doing formally is getting rid of this z and expressing it in terms of ρ and T and putting that back in here and the energy is going to be function of temperature and density of Bosons.

So as a result I am being called upon to invert this relation which tells me that z is related to so it is called w times this whole bunch of numbers I mean rho times the density of Bosons times the whole of things like this I have given it as new name I've called it w so the. So now I am being called upon to invert this so write z as a function of w because then I can go ahead and substitute that here and there then I will be getting because w is ρ and temperature so I will be expressing energy per particle in terms of density and temperature rather than this unusual looking object called fugacity, okay.

So question is how would I invert this so this looks formidable because this poly log function is a little bit unfamiliar to me inverting this is going to be even more unfamiliar.

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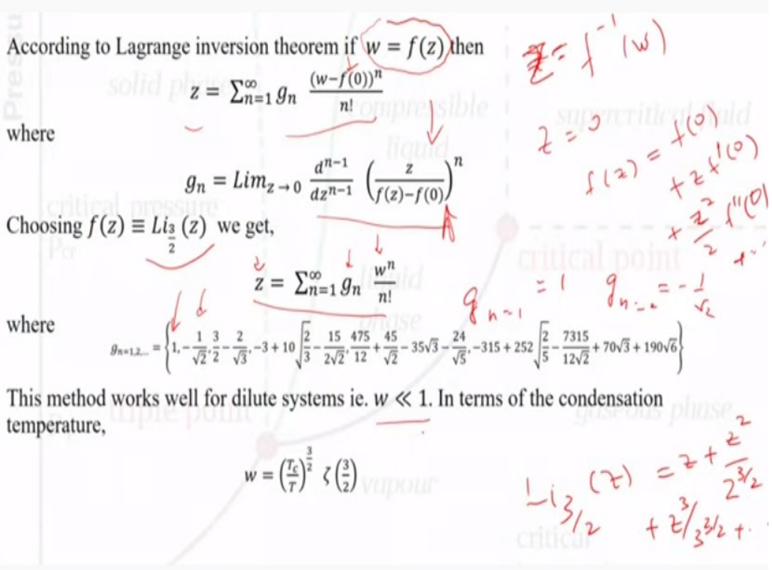

According to Lagrange inversion theorem if $w = f(z)$ then

$$z = \sum_{n=1}^{\infty} g_n \frac{(w-f(0))^n}{n!}$$

where

$$g_n = Lim_{z\to 0} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z}{f(z)-f(0)}\right)^n$$

Choosing $f(z) \equiv Li_{\frac{3}{2}}(z)$ we get,

$$z = \sum_{n=1}^{\infty} g_n \frac{w^n}{n!}$$

where

$$g_{n=1,2,\dots} = \left\{1, -\frac{1}{\sqrt{2}}, \frac{3}{2} - \frac{2}{\sqrt{3}}, -3 + 10\sqrt{\frac{2}{3}} - \frac{15}{2\sqrt{2}}, \frac{475}{12} + \frac{45}{\sqrt{2}} - 35\sqrt{3} - \frac{24}{\sqrt{5}}, -315 + 252\sqrt{\frac{2}{5}} - \frac{7315}{12\sqrt{2}} + 70\sqrt{3} + 190\sqrt{6}\right\}$$

This method works well for dilute systems ie. $w \ll 1$. In terms of the condensation temperature,

$$w = \left(\frac{T_c}{T}\right)^{\frac{3}{2}} \zeta\left(\frac{3}{2}\right)$$

$z = f^{-1}(w)$

$z = 0$

$f(z) = f(0) + z f'(0) + \frac{z^2}{2} f''(0) + \dots$

$g_{n=1} = 1 \quad g_{n=2} = -\frac{1}{\sqrt{2}}$

$Li_{3/2}(z) = z + \frac{z^2}{2^{3/2}} + z^3/3^{3/2} + \dots$

But fortunately there is a general method for doing that and so that is go by the name of Lagrange’s inversion theorem. So it is nice to know this things because you never know when you will use them because may be in this example is not that important but there will be stage where you know you might be we might require it if you do not know where it is you would not be able to use it.

So the Lagrange inversion theorem is very interesting what it says is that so imagine that there is a equation like this which I want to invert so that means w = f(z) and I want to write z in terms of w so I want to inverse so I want to write $z = f^{-1}(w)$. But imagine this is possible to do only if so this is going to be possible only if f is analytic in z at some value of so let us assume that z is equal to 0 that you are talking about.

So imagine that f is analytic around 0 so what that mean is that I can do a Taylor series around z. So this so in other words the implication is f can always be written like this so if cannot be written like this method does not work, so if imagine that f can be written like this. So if f can be written like this then it is so happens that Lagrange inversion theorem guarantees that z can be written in terms of w in this fashion.

So you can write z explicitly as

$$z = \Sigma_{n=1}^{\infty} g_n \, (w\text{-}f(0))^n/n!$$

which can be calculated, so Lagrange inversion theorem gives a formula for that coefficient as something like this, it is the limit as $z \to 0$ is n – 1 th derivative versus ratio. So now I can go ahead and applied to poly log function so remember that I wanted to invert this $w = Li_{3/2}(z)$ and question is how do I invert that? I use the Lagrange inversion theorem.

And I am able to write when f is 0 poly log is 0 because remember poly log is

$$Li_{3/2}\,(z) = z+z^2/2^{3/2}+z^3/3^{3/2}+...$$

So z is 0 this whole thing is 0. So as a result this f(0) is 0 so I can write z in terms of w in this fashion where these coefficients can be computed explicitly but unfortunately you cannot compute this nth term.

You can just list them just one by one n = 1, this is for n = 1 this for n = 2 so called $g_n = 1$ is 1, $g_n = 2$ is $– 1 /2^{1/2}$ like that. So I have listed whole bunch of them I mean I did not do by hand of course I use my favorite package this is called mathematica it is just turns how the all this numbers. So clearly this method works certainly well for dilute systems.

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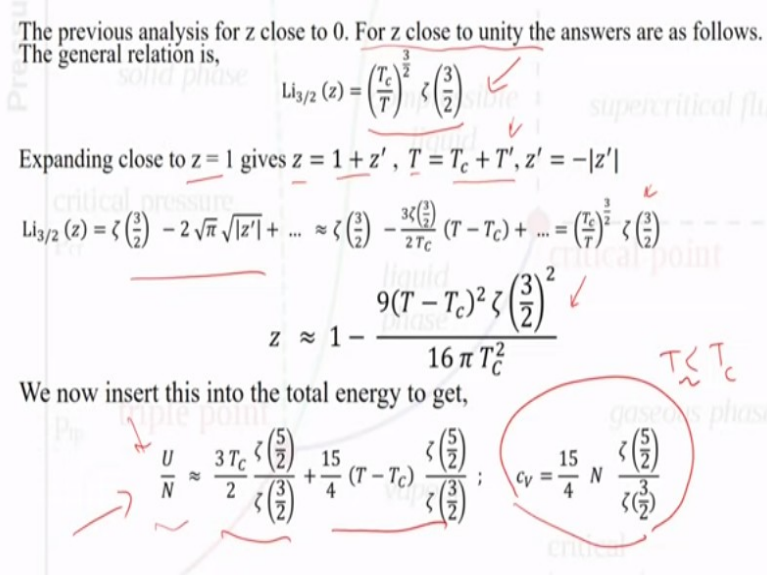

The previous analysis for z close to 0. For z close to unity the answers are as follows. The general relation is,

$$\mathrm{Li}_{3/2}(z) = \left(\frac{T_c}{T}\right)^{\frac{3}{2}} \zeta\left(\frac{3}{2}\right)$$

Expanding close to z = 1 gives $z = 1 + z'$ , $T = T_c + T'$, $z' = -|z'|$

$$\mathrm{Li}_{3/2}(z) = \zeta\left(\frac{3}{2}\right) - 2\sqrt{\pi}\sqrt{|z'|} + \ldots \approx \zeta\left(\frac{3}{2}\right) - \frac{3\zeta\left(\frac{3}{2}\right)}{2T_C}(T - T_C) + \ldots = \left(\frac{T_c}{T}\right)^{\frac{3}{2}} \zeta\left(\frac{3}{2}\right)$$

$$z \approx 1 - \frac{9(T - T_c)^2 \zeta\left(\frac{3}{2}\right)^2}{16\pi T_C^2}$$

$T \lesssim T_c$

We now insert this into the total energy to get,

$$\frac{U}{N} \approx \frac{3T_C}{2}\frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} + \frac{15}{4}(T - T_C)\frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} ; \quad c_V = \frac{15}{4} N \frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)}$$

So in which case w becomes this and so for dilute systems we already know how it works. But what is more interesting is this, so if z is close to unity then how does it works so the general relation is that so when z is close to unity I can always write like this okay so that is what this says. So even though it works well for dilute system it doesn't mean that it does not work at all when w is close to 1. So it kind of works even otherwise so when w is this and $\mathrm{Li}_{3/2}$ is w I can go ahead and write this relation.

So now if I expand close to 1 so I write as z as 1 + z' and T as close to $T_C$ so remember T is larger than $T_C$ but close to $T_C$ and because z is always less than 1, z' has to be negative okay so it is negative and small compared to 1 because then 1 – z is close to 1 and less than 1. Alright so now if I expand in powers of z' and also in powers of p' so then I get this expression okay so and z becomes so I can rewrite as z in terms of T – $T_C$ and that allows me to cut a long story short I end up getting relation for the number of so the total energy per particle to be this.

So why do i do all this that means seems like a lot of work for what reason I have not told you yet. The reason is the following, now that I have spent so much of effort and got an expression for the total energy per particle for temperature slightly more right than the condensation temperature. Now I can go ahead and ask myself what is the specific heat of this Bose gas slightly more than the condensate temperature and that comes out by just differentiating this with respect to temperature and get this answer.

And this answer is actually the same as what it is for T slightly less than $T_C$, T close to $T_C$ but less than $T_C$ so it is so happens that the specific heat across the condensation boundary is continuous so the specific heat does not change abruptly okay. So only the total internal energies changes abruptly the specific heat does not change okay.

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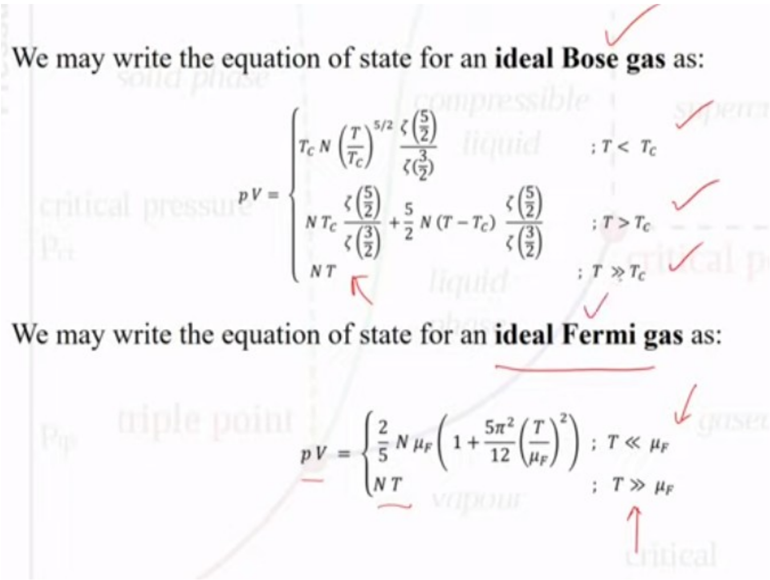

We may write the equation of state for an **ideal Bose gas** as:

$$pV = \begin{cases} T_C N \left(\frac{T}{T_C}\right)^{5/2} \frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} & ; T < T_C \\ N T_C \frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} + \frac{5}{2} N (T - T_C) \frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} & ; T > T_C \\ N T & ; T \gg T_C \end{cases}$$

We may write the equation of state for an **ideal Fermi gas** as:

$$pV = \begin{cases} \frac{2}{5} N \mu_F \left(1 + \frac{5\pi^2}{12}\left(\frac{T}{\mu_F}\right)^2\right) & ; T \ll \mu_F \\ N T & ; T \gg \mu_F \end{cases}$$

So now we can go ahead and write down you just let me consolidate my formulas here so one can use this U that I have derived so remember that pV=2/3 U in general so I can use that to write down my pV the equation of state for a Bose gas and Fermi gases follows. So when T is less than $T_C$ my pV is related to temperature in this fashion, when T >$T_C$ is related to temperature and this fashion and when T >>$T_C$ it is just the classical ideal gas okay.

So for an ideal Fermi gas we can write the equation of state in this fashion and for T << μF if this and for T >> μF it is again the classical ideal gas. So this is a nice consolidated table of formulas for the equation of state of an ideal Bose gas and an ideal Fermi gas okay. So I am going to make use of this in next I am stop here now. So in the next class I am going to make use of these two ideas and you now persuade you to believe what is known as the virial expansion of the equation of the state which is by useful because in situations.

Remember we have studied only ideal gases so if you go ahead and study a gas where there are interaction between molecules then it is no longer clear because that is a very complicated problem in general. So if the molecules interact with each other then it is a hard problem and but however you can make some head way into analyzing that through what is known as the virial expansion of the equation of state which is motivated by starring at this ideal results we obtained so far you know by consolidating them which is the classical ideal gas and the quantum Bose gas and quantum Fermi gas.

So by studying these three equations of state we can kind of guess what the form of the equation of the state might be in general so that I called the virial expansion I am going to use that later on and to motivate you know study of the non-ideal classical fluid which is known as the Van der waals fluid. So I am going to come to that gradual thing okay I am going to stop here now and hope you will join me for the next class this is going to be all over real expansion Van der waals fluid and so on, okay thank you.