Introduction to Statistical Mechanics
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Lecture - 32
Current Algebra

Okay, so let us continue with our discussion of what we have been calling second quantized formulation of many particle systems. So basically, the first quantized approach in also expressing various operators, in terms of operators that we are familiar with from elementary quantum mechanics, such as position and momentum whereas the second quantized approach basically rewrites these formulas in terms of creation and annihilation operators.

Where those creation and annihilation operators encode the statistics of the underlying particles in the system. So if you have a system of many fermions, then the creation and annihilation operators obey canonical anti-commutation rules, whereas when the system is composed of bosons, they obey canonical commutation rules.

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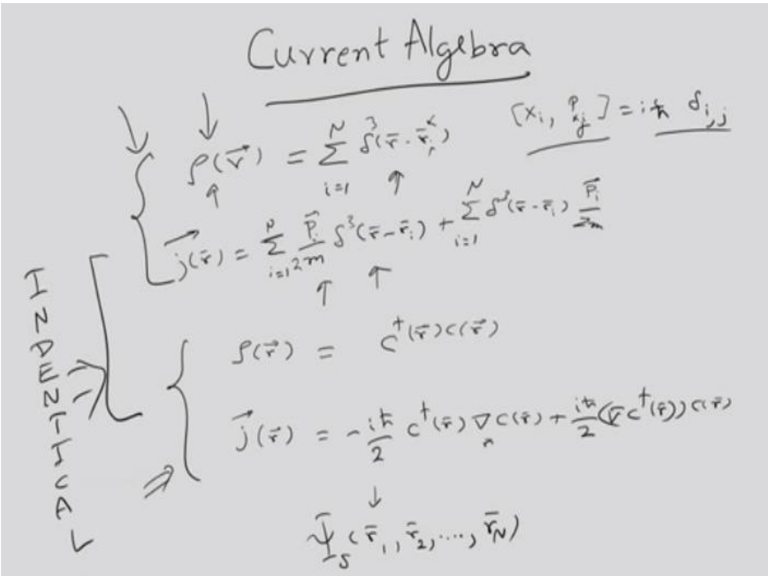

So if you recall that I started off with this definition of in the last class, I actually ended with this definition of the local density of particles. So if you have a bunch of particles the density of particle at position r is naturally defined like this, because you know if you have a discrete number of particles from 1 to n, so you know the density is 0 unless you are at the location of the particle.

So that is the reason why there is a delta function here and then you sum over all the locations of the particle and that gives you the local density. So similarly you can define current which is basically defined in terms of the, you know, what current is. It is basically I mean it is just the flux per unit time okay. So in other words dimensionally it has the idea of velocity just like this. So by then, remember that this would be the current.

And remember that because we want this to be self adjoined we are going to write this in this Hermitian form okay. So this is where I had stopped last class, where this is the first quantized or you know the traditional way of thinking about these quantities these operators. These are operators because in quantum mechanics ri is the position operator for the i-th particle and ti is the momentum operator and then you see you know that xi commutator Pxi is or Pxj is basically ih bar times Kronecker ij.

So in that sense these are indeed operators and so rho and j will not commute, because you know rho does not have a P in it, whereas j has a P in it, whereas rho has an r in it or xyz basically and thus px, py, pz and j, so you do not expect rho and j to commute. Alright so now, what I am going to do is, I am going to make the following assertion that this is, it is possible to write these two formulas in the second quantized language and it turns out that it is exceedingly simple to write down, meaning it is a very simple form namely this.

Well this is a vector, so it has an exceedingly simple form. So I am going to first write it down and then I am going to prove it okay. So this is how it works. So you see these two are identical. I mean these two are absolutely identical okay. So these two ways of doing it would be the first quantize or traditional way of doing of expressing the density and current in terms of momentum and position, whereas this approach is the creation annihilation way of representing the same quantities.

So now I have to prove this and remember how I have been proving this. So these are operators that act on many particle wave functions. So I have to first assume that there is such a wave function and then I acted on the many particle wave function and all I have to do is show that regardless of what this is. So for a general many particle wave function the action of rho on this, such a wave function is identical to the action of this rho on the same wave function, so similarly for the current j. So I am going to prove this now.

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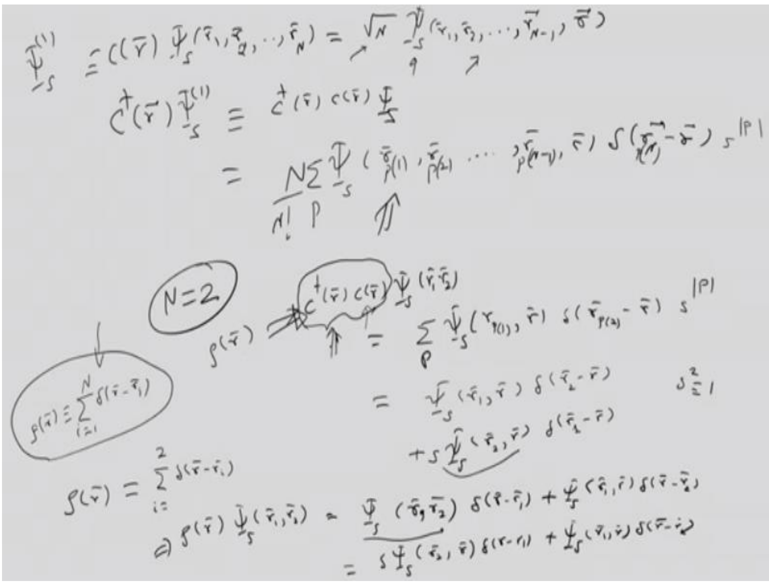

So remember that C of r acting on, if it is properly, let us assume is you start off with something properly symmetrized or anti-symmetrized as the case may be and then you know that I am allowed to do this so square root of n, because then you know you are annihilating. So you will be picking up the square root of n and then all you are doing is basically getting rid of one particle and the last one has gotten rid of.

In other words, you are forcing that rn to, you are freezing the value of rn to r. So now you see having done this, this is still anti-symmetric under the exchange of r1 r2 and so on or r symmetric with depending upon the value of this small s here. So this is properly symmetrized. So now I am going to create a particle you know on this new state. So I will let me call this psi 1s okay. So I am going to create a new particle on this. So this would be nothing but c dagger r cr acting on psi s.

So now this is going to be, if I create a particle you see the n - 1 particle, so I have to add one more particle, but then when I do that I have to, it will be square root of however many particles there are plus 1. So now there are n - 1 particles. So it is going to be n - 1 + 1 and square root of that. So that is basically another square root of n. So there was already a square root of n here. So those two get multiplied when you get n and then you will have to add a particle.

So remember here, there are only n - 1 particles. So adding a particle implies that I have to add the n-th particle back again. So I had annihilated the n-th particles. I have to put that back in and of course when I do that I will end up messing with the symmetrization. So I will have to properly symmetrize it, okay. So I will have to do that. So first I am going to, let me first add the particle and n-th particle again at location r. Now I have to kind of do this.

I have to take the permutation of all these particles you know and sum over all the permutation, then I have to put an s raised to this the output n factorial well. That is however many particles there are. So that is going to be my density okay. So let us work this out. I mean it is little bit tricky to appreciate this when there are so many particles. So let us do this assume that you start off with say a 2 particle system, a one particle wave function is a bit too simplistic.

So I am going to start off with let us assume capital N as 2 okay. So if it is 2, then it is easy to see what is going on. So then C dagger rcr acting on psi s with r1 and r2 only is going to be, so remember that if they are only 2 of them, this is going to be just 1. So it is 2 over 2 factorial which is exactly 1 and then it is just some over the permutation of 2 objects and psi s and this is going to be the permutation of, so if n is 2, there is going to be a permutation of 1 and r and delta of r permutation of the second one minus r.

So how do you permute? So there are only 2 permutations. So you either do this. So if p1 is 1 and p2 is 2, so that is basically the original unpermutted form. So that mod p is 0 there. So it is going to be basically 1 in that case. So in the other case, it is going to be s because if I interchange, it is going to be s. So sr2r this is going to be delta of r1 – r. So that is going to be my density. This is in the second quantized language.

So now I am going to see if this is the same as what you would get if I had thought of rho as. So this was the rho in second quantized language remember. So but then the traditional way of thinking of rho is basically this for 2 particles right. So now let us see if this has the same effect of by when you acted on a 2 particle system or 2. So you see the answer is s. So it is going to be psi of s acting on, for the first term it is going to look like r1 becomes r.

This becomes r2 and then you get a delta of r – r1 and then the other term is r2 becomes r where r1 remains as it is okay. So that is the effect of the density operator written in the traditional way acting on the 2 particle wave function. Now I have to convince myself that this is the same as what I have here. It is indeed the same, because remember that these 2, so these 2 differ by a factor of s and s squared is 1, because s is either 1 or -1.

So these 2 differ by a factor of s. So this is also equal to s psi of s r2r delta of r – r1 + psi of s r1r delta of r-r2. So you are done, but this is through an example of course, but you could prove it for the general case. So I will leave that to you as, you know you can do that by induction or any other method that you are comfortable with okay. So I am going to leave that to you as an exercise to try at home.

So the point is that the density operator in the second quantized formulation has this form, I mean just I want you to appreciate the simplicity of this way of looking at the density. See if you look at the density, that is written out in the conventional way. It has a huge number of variables, so it starts off with r1 r2 all the way up to rn. So there are n number of variables, where n could be macroscopically large.

However, all that complexity is hidden when you decide to write the density in terms of the creation and annihilation operators. They are all those the complexity of the information about the size of the system is subsumed into the definition of the creation and annihilation operators. So that is the simplicity of this approach, which enables you to do that and not only that the statistics all the particles that make up the system are also encoded in these operators themselves.

Whereas in the conventional way of doing things, the statistics of the particles is not apparent in the operators. It comes from the wave functions. It has to be imposed on the wave functions, whereas here it is already apparent, because it is encoded in the way in which you define your creation and annihilation operators. So you can do the same thing for current. So let me do that for current. This is my expression for current here.

So I am going to see if I can do the same thing for current. So let me do that traditional way of doing it first.

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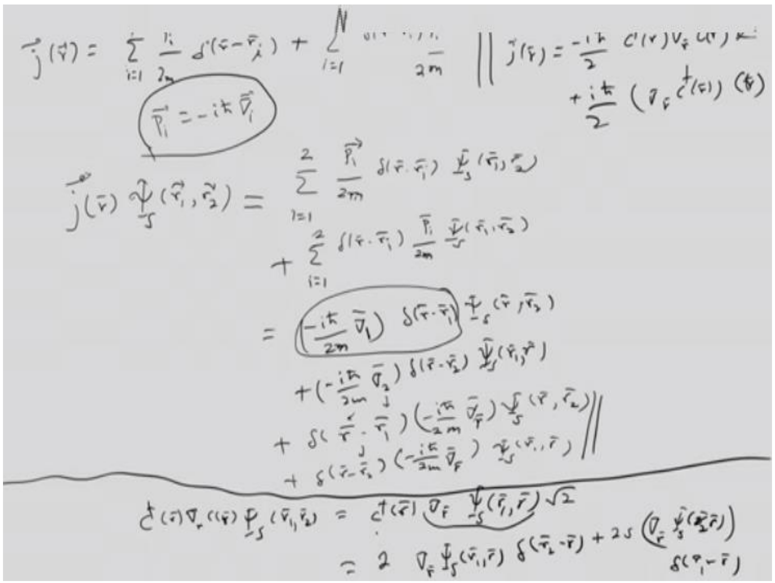

So remember that in the p and r language or the position momentum way of doing things that the current operator is defined in this fashion. This is the current density at location r for a system of n particles. Well, in the conventional way of doing things, it is not obvious from here whether the underlying particles are bosons or fermions and basically I have to specify that by examining the statistics of the wave functions that come along with these operators.

So now let me examine. So notice that a pi there is nothing but –ih bar grad i. So this is in conventional quantum mechanics, that is how you choose to represent the momentum operator. So now let me go ahead and examine the action of, so I am going to restrict myself to 2 particles again. So I have 2 particles and then I examine the action of the current density operator on a wave function of 2 particles okay. Let me work this out.

So if you work this out, how does this come about? So you see on the one hand, it is this. So if I look at the first term, it is 1 to 2. So if you expand this out, how does this look like? So if its i is 1, that means 1 is being forced to be r. So it is –ih power by 2 m del i acting on, but it also acts on well I am going to start with 1 okay. So that is how it looks like and the other terms are exactly the same, except that the momentum is to the extreme right okay.

So let us work this out and you see this term forces r1 to become r. So I am going to delete that r1 there and then this term forces r2 to become r. So I am going to delete r2 there and so you see this has a derivative on r2. Okay I am going to liberate this because it is not clear if further simplification is warranted, because remember that all I only show is that this is the same as, you know, if I choose to define. So this is the second quantized version would be –ih bar by 2m, you know, c dagger r cr + ih bar.

So it is without the m there. So the question is, is this the same as this I mean. I am just going to give you some a sketch of the proof, you can fill in the details later yourself. So you see just stare at this term, how does this look like? If you look at c dagger r grad cr, so if you act this on psi r1 r2, what is that going to look like. This is going to annihilate a particle. It is already properly symmetrized, so it could annihilate one or the other. So it is going to be del r.

So I am going to annihilate the last one as is the custom. So then you see once I do this, then I will be forced to add a particle, but then firstly I will have to put a square root of 2 because that is what it is and when I add a particle, there is another square root of 2. So the whole thing goes twice and I am going to add a particle. So that is going to be grad r psi s r1 r. So I am going to add a particle r2 – r, but then I have to permute.

So if it is r1 r2, but then if it is the other way around well the permutation would be with an s r psi sr r2 r del of r1 – r. So remember that you know, I can always rewrite this as twice this plus just the density. It is more like this. It comes from this. So this term is similar to that okay. So the reason is because r1 is forced to be r, so this is going to become r, so we follow this r1 becomes forced to be r and so I can do this. So this term is r2 is forced to be r.

So I can just get rid of this, put r there okay. So now you stare at this that apart from this factor of 2, so if I take a –ih bar by two, I will get the 2 is cancelled, but then you see the similar term comes from here also, because I can take this grad, act it on this or I can choose to act it on this. So when the grad acts on this okay, well I should keep this as 1, then you will see that it gets added up. So that 2 cancels out basically.

So I will allow you to work out the current part yourself. It is a little bit tedious, but you will see that it works out. So finally this term is going to look like this and so that is going to effectively verify the claim that these 2 formulas are absolutely identical okay, well perhaps without this m okay. So think of this as without the m because see dimensionally what is this. This is density times h bar K, so which is momentum. So yeah it is without the m okay.

So now it is going to work out. So without the m these 2 are equal okay. So once you convinced yourself of this, then you can also show that the densities and currents obey a certain, just like.

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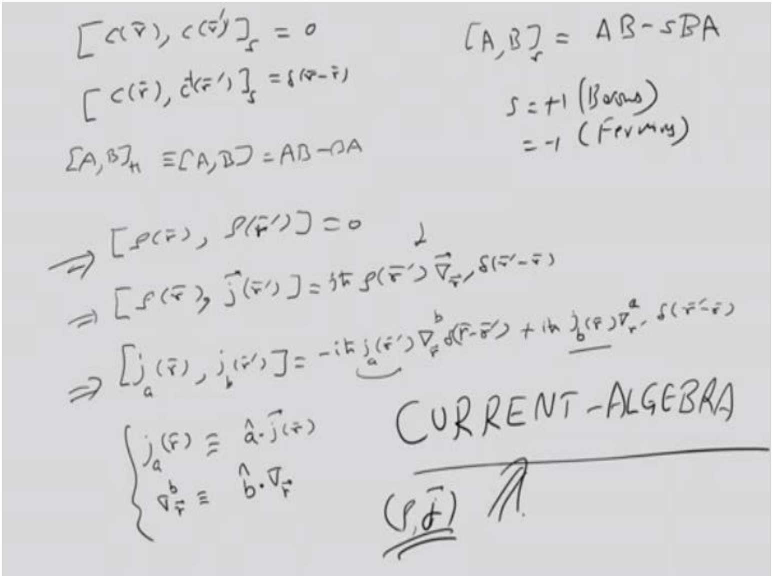

So remember that the creation annihilation operators obeyed this type of commutation rules, s commutation rules depending upon whether s is +1 or -1. So remember that A commutator B with subscript small s is basically AB – s times BA. So if s is +1, you are talking about bosons. If s is -1 you are talking about fermions. So s is +1, it is bosons, if s is -1, it is fermions okay. So you see it so happens that once you construct currents and densities in terms of the cs and c daggers, you will be able to convince yourself, allow you to do that as an exercise as well.

Keep in mind that I am not going to test you in the examinations or any of the tests or tutorials on any of these topics which have classified as advanced topics, because strictly speaking it is not part of statistical mechanics, but I am just teaching it to you because it is a natural continuation of you know where we are going to actually leave the course. So the natural next step is exactly what I have been teaching you.

So it is a natural continuation and it is something that you should pick up on your own. So it so happens that you can also convince yourself, that these rhos and j's you know obey certain commutation rules, the conventional commutation. So if I do not have a subscript, it means if I do not write anything below here, it really always means this AB – BA. So in other words if I do not write anything, I necessarily mean AB with a +1 there.

So that means the bosonic commutator. It is just the commutator the traditional not anti-commutator. So it is either a commutator or an anti-commutator. So if I do not specify, it always means a commutator. So you see it is going to obey these types of rules. So it is going to, you can convince yourself that it obeys. So the rho and j, they are commutators. They are conventional commutators, not anti-communicators just commutators.

Regardless of whether the underlying particles are bosons or fermions the simple commutator is expressible also in terms of rhos and js. So this is what is known as current algebra okay. So this is called current algebra and this is very important you know for the study of many topics in many body theory, specifically what is called bosonization, which I am not going to discuss at all in this course. Somewhere I tell you what all these symbols means, so just please be patient.

I am going to tell you what this means shortly okay. So what does all this mean? Firstly this is obvious what it means this is just the density, you take the density regardless of whether they are bosons or fermions, you just take the density at some other point. They commute. So in other words what does that mean physically? It just means that you can measure the densities of particles simultaneously at 2 different points and you know the measurements commute.

That means if you measure the density at a point first and measure the density of the system at another location but at the same time, you would not get a different state if you interchange the order in which you make the measurements. However, that s not true, when you are dealing with density and current. So when you are dealing with density and current, density and current do not commute, but however, their commutator is expressible in terms of densities and currents.

Specifically, in terms of this density here and here ja refers to the a-th component of the unit vector in the a direction, whatever that is. So that is what does the symbol? So if I take you know, if I write this I really mean this, you know in some b direction. So whatever it is that these types of commutators are going to come out in this fashion, the a-th AF component of the current density at location r, when you find the commutator of that with the b-th component of the current density at location r dash.

The commutator is going to be also expressible in terms of in this case only currents. So in this case it was density, it is going to be currents. So this is what is known as current algebra. So this is called current algebra, of course it should be called a density current algebra specifically, but it is called current algebra, because density can be thought of as the time component of a fore vector. So it is like a fore current rho and j together is like a fore current and so in that sense, it is called current algebra.

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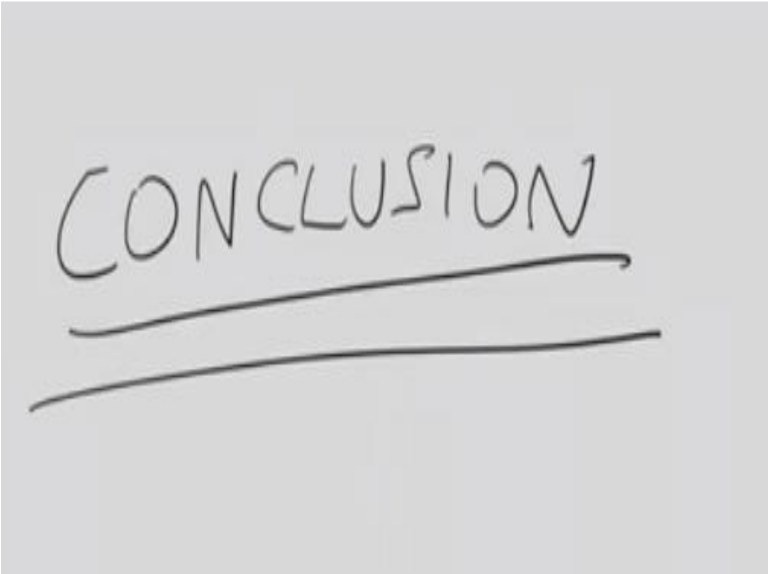

So I am going to stop this course right here. So what are the conclusions, what have we learned from this course? So what we have learned basically in this course is we have learnt that statistical mechanics is a study of systems with large number of particles, whether the underlying particles of a quantum mechanics or classical mechanics, it is possible to take into account the detailed dynamics of the underlying particles and predict the properties of macroscopic systems.

So in other words the macroscopic properties of large systems can be linked to the microscopic properties through these ideas. So also we do that by, not by necessarily solving the detailed dynamical equations for large number of particles, since that is not practical. We do that in a clever way in this course in statistical mechanics by side stepping those ideas by averaging over a whole number of microstates and as a result we will be able to compute the average macroscopic behavior of systems with large number of particles or subsystems.

So we were also successful in showing that the average is the whole story, so long as the system sizes are huge. So the fluctuations are suppressed and that is one of the important conclusions of this course and so as a result, we were able to apply such ideas to a whole bunch of you know a vast diversity of systems that are found in nature you know right from Fermi gases, Bose gases and classical you know ideal gas and Van Der Waal's fluid and even you know magnets like ferromagnets and paramagnets.

You were able to discuss you know Landau diamagnetism. We were also able to apply it to the other extreme namely astrophysical bodies like black holes and white dwarfs and so on. So we have done wonders in this course. We have studied you know subatomic structures using statistical mechanics like you know electrons in a metal all the way up to you know the degenerate Fermi gas in a white dwarf or we have studied the thermodynamics of black holes.

We have talked about polymers in the examples. So we have done lots in this course and I hope you enjoyed this course and you found it informative and it is, so if you take this course seriously do all the problems and you know attempt the examination seriously, you will be well equipped to be or to have a successful career in physics at an advanced level. So I thank you for registering for this course and listening to all my lectures. I hope you found them enjoyable thank you and bye-bye.