Introduction to Statistical Mechanics
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Lecture - 29
Creation and Annihilation of Fock Space - II

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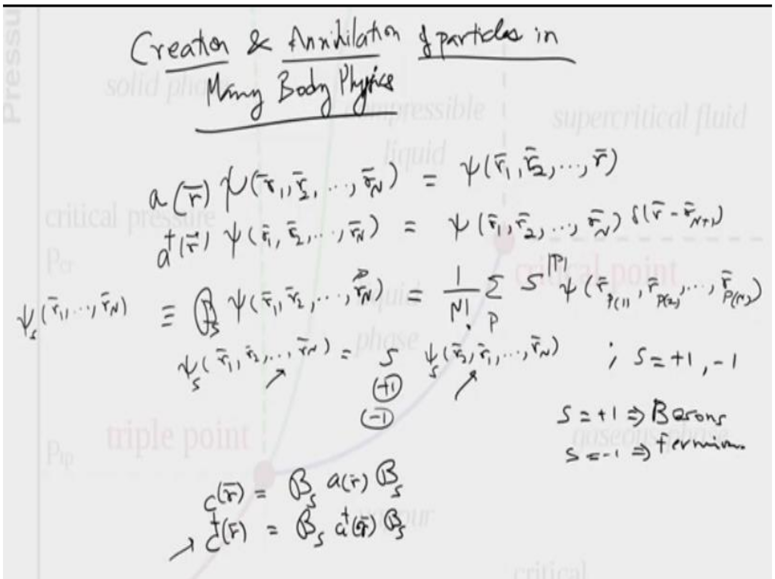

So let us continue this discussion of the definition and properties of creation and annihilation operators in many body physics. So specifically, I am talking about creating quantum particles, such as bosons or quantum particles such as fermions. So I want to describe operators that create fermions or create bosons at a particular point in space. So if you recall that I had stopped at a stage where I defined a certain operator called a of r, which has the property that if you take a many-body wave function, which need not be properly symmetrized or anti-symmetrized.

It is some general wave function. So what it does is basically it just kills the last coordinate. So this would be called an annihilation operator, but it does not necessarily annihilate a particle of any particular statistics either bosons or fermions. So in some sense, it just annihilates, but it is the incomplete description of the system, because I have to construct operators that actually describe creating fermions or bosons from this operator called a of r.

So similarly you can think of a creation operator something, which creates one more particle, but then it creates it at some location rn + 1 okay. So that is what it is. So then I also defined basically the symmetrization operator. I think I have been calling it something well maybe I have called it b of s. So this has the property that if you take psi of r1 r2 up to rn, so what it does is, it basically permutes. P is the given permutation of the numbers 1, 2, 3 up to n.

So mod P is the order of that permutation and I am required to sum over all possible permutations. So this is going to look like P of 1 r of P of 2 and r of P of n okay. So now this object is basically properly symmetrized or anti-symmetrized version of the original wave function, which does not have any particular statistics. So this for example, this wave function has the property that if you interchange 1 and 2 for example, it could be any pair of coordinates.

But suppose I choose to exchange 1 and 2, what is going to happen is, it is going to have this property okay, so which you can easily verify. So this is by construction. So the symmetrized or anti-symmetrized, so if s is 0, this is not -1. I mean I instinctively wrote fermions. So strictly speaking this is not really, I mean this would be -1, s is either +1 or -1 okay, not 0 or 1. So if s is +1, what you are describing is that s equals +1 means bosons okay, bosons and s is -1 means fermions okay.

So that is easy, because see just s okay. So if it is just s. So if s is +1, I am describing bosons. If s is -1, I am describing fermions. So if s is +1, so you see the wave function does not change sign when you interchange r1 and r2. So if I interchange r1 and r2 and if s is +1, it does not change sign, but if it is -1, it changes sign. So -1 describes fermions and +1 describes boson. So this is just a recap I mean this is what I did till now okay.

So I hope you will just recall what I did in the last class. So this is what I did and also I showed that you can define something called the creation of a fermion or a boson in terms of these operators. So you just sandwich this a of r in between two symmetrization operators and that is guaranteed to describe the annihilation of a boson or a fermion depending on the value of s. So similarly you can define the creation operator of a boson or a fermion by sandwiching A dagger in between two symmetrization or anti-symmetrization operators.

Now you know what I am going to do now is, I am going to give you a very simple specific example with 2 particles to convince you that this kind of a definition does indeed do what I claimed it does. So let us see if I can show that. Before I do that let me do the general case.

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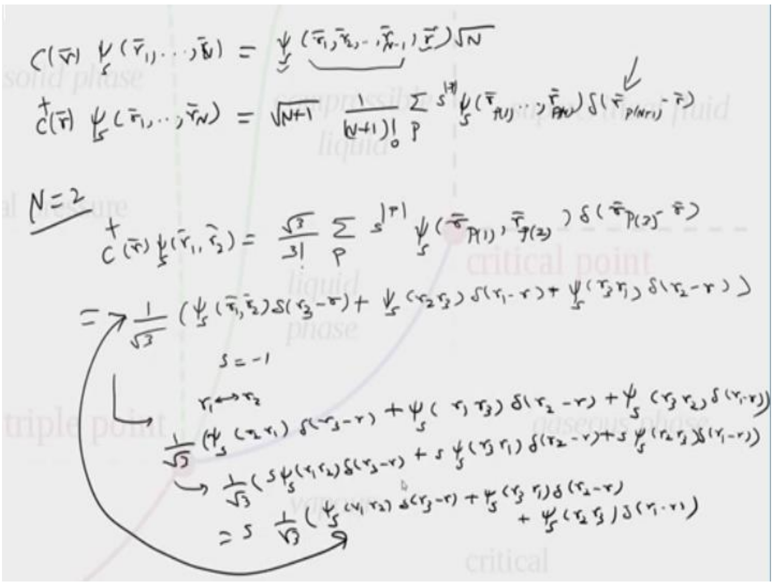

So C of r acting on a properly symmetrized wave function will give you another properly symmetrized wave function okay, but then it is going to be a function of n variables plus some variable, which is kind of frozen. So this part is frozen. So you do not have to touch this. So what this does is it anti-symmetrizes if s is -1, and symmetrizes if s is +1 only these number of 1 less variable okay times square root of n.

So similarly the creation operator acting on the already properly symmetrized wave function is going to, so this is a little bit complicated, but you know what it is. So it is square root of n + 1. So this is familiar to you from you know your quantum mechanics of harmonic oscillators and that sort of thing. These factors are familiar to square root of n if you annihilate and if you square root of n + 1 if you create okay.

So because you have added one particle, what you have to do is basically do that okay. So now let me specialize to the case n is equal to 2 and let us see if this makes any sense. So now I am going to create a particle n equal to 1 would be unconvincing, because it is too simple. So that is the reason why I have chosen the next the simplest non-trivial case, which is n equal to 2. So this is just for illustration.

So now if I do this, this is going to come out as square root of 3 over 3 factorial at times sum over all the permutations of s raised to mod P psi of s r of permutation of 1 permutation of 2 times Dirac delta permutation of 3 – r. So remember that once I put in an additional variable namely the 3 you see earlier you had only 2 variables r1 and r2. Now if I create a particle, I am necessarily introducing one more coordinate, which is dynamical in nature, which is r3.

So now I have to make sure that the 3-particle system now continues to describe the system of 3 bosons if it earlier described a system of 2 bosons or 3 fermions, if it earlier described a system of 2 fermions. So in order to do that, I have to properly symmetrize here. See the reason why that is not important to do when you annihilate, the moment I put a subscript s here implies that even if I freeze one of the variables, it is anyway s symmetric under the exchange of any pair of this.

Because it is s symmetric under the exchange of any pair, so even if you freeze out one of them it is going to be symmetric under the exchange of any of those other pairs, but not so if you add a particle. So if you add a particle, then you have to take the trouble of actually anti-symmetrizing or symmetrizing, you know, all over again, because you cannot really, you know this gets left out. This variable gets left out and even though mutually these are all properly symmetrized, the last one that enters is left out.

So you have to include that in the symmetrization and this is how you do it okay. So expanding this out, what this comes out to be is, I am going to allow you to simplify it. So I am going to tell you the simplified version of this. So after simplification, it is going to come out as 1 over square root of 3 times psi of s r1 r2 times Dirac delta r3 - r + psi of s r2 r3 r1 - r + psi of s r3 r1 delta r2 - r okay. So now you can see that this does describe a system of 3.

So if suppose s is 1, you can see that if I exchange r1 and r2 so this goes to this okay and this becomes this and this becomes itself okay. Actually I missed an s here. So it is I think I have to put an r3 r2, there has to be an s. So if I take r1 r2, it is going to give you an s factor outside, but then if I take, no, even then it is correct sorry, r3 r + okay. Let us do it for s equals 1, it is fairly obvious, but let us do it for s equals -1.

So in that case, what is going to happen here, is that suppose I interchange r1 and r2, so it is going to become psi of s r2 r1 r3 r + psi psi of s r1 r3 delta r2 r + psi of s r3 r2 then delta r1 – r. So now you see this factor is 1 over square root of 3s into psi of s r1 r2 delta of r3 – r, well regardless of whether s is 1 or -1 and this is s into psi of r3 r1 delta r2 - r and this is s times psi of s r2 r3 into delta of r1 – r. So now this s comes outside. So that is the key thing.

So s comes outside and then what remains is this okay. Now you compare this quantity with this quantity, you will see that they are equal, except for this s. So it is the same thing. So that means this wave function if you interchange r1 and r2, you end up anti-symmetrizing it and that will be true of any pair okay. So this way of doing things, you know either creates a boson if s is +1 or creates a fermion if s is -1 okay. So that is how you create bosons and fermions at the point r.

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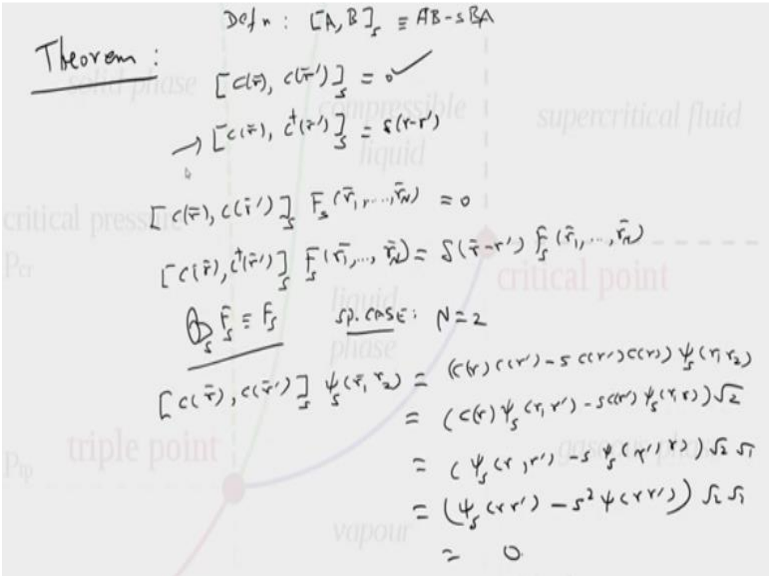

So now let me show you, I am going to prove this important theorem to you. So the theorem states, first I am going to define definition the s symmetric commutator, if you wish. So if s is +1 is the normal commutator, if s is -1 is the anti-commutator. So this is by definition. So if s is +1 it is the normal commutator, because it describes bosons and if s is -1, it is the anti-commutator. So it describes fermions.

So with that definition, I can make the following statement that C of r, C of r dash, the s commutator is 0, then C of r C dagger of r dash, the s commutator is delta of r – r dash okay. So how do you show this? So these are operators, so what this means is that this acts on some n particle wave function. So let me act it on an n particle wave function. So imagine that that wave function is denoted by capital F subscript s implying that it is already properly symmetrized okay.

So this is zero. Now if I take C of r C dagger of r dash and I act it on F of s, I get I mean the implication is that, this is an identity. So now of course b of s is redundant because it is already symmetrized. So that is the meaning of F of s. F of s is something that does not get affected by this symmetrization operator. So now consider a special case, n equals 2 that is our usual, you know, the simplest non-trivial case. So in that situation, so you can ask yourself how to do this.

So I am going to use my usual notation of psi of s now. So this is going to be C of r into C of r dash – s times C of r dash C of r acting on psi of s r1 r2. So if I annihilate a particle okay, I mean I am going to let you do this. So you just use these definitions that I have explained earlier, you know, how to annihilate particles and just go ahead and so for example it is you know, just you just annihilate the last one and let the first one be and you put square root of so that is easy.

So let us do that. So it is going to be psi of s r1 r dash – s psi of r dash psi of s r1 r okay square root of 2, because the square root of n there and that is square root of 2. Now we have no choice, but to just annihilate whatever is there. So it is going to be psi of s r. So it becomes r dash – s psi s r dash and r square root of 2 an square root of 1, because now in the second annihilation there is only 1 particle. So now given that r dash r is same as rr dash, but there is a factor of s okay.

So you can see that, so this is zero so this becomes psi of r r dash – s square psi of r r dash square root of 2 square root of 1 and then s squared is always 1, because s is +1 or -1. So this is 0 okay. So that verifies this, I mean verifies means in this special case n equal to 2, that is an example. So let us do the same thing for verifying the here. This is more difficult slightly because there is a C dagger s.

If you start with a 2-particle wave function and you involve daggers in your formula, so what is going to happen is you are going to introduce one more variable, because you are going to create a particle. So we have to put in some more work. It is not as easy as proving the first result. So it is worthwhile putting in that extra effort, because you know, you can be confident that what you are saying makes sense.

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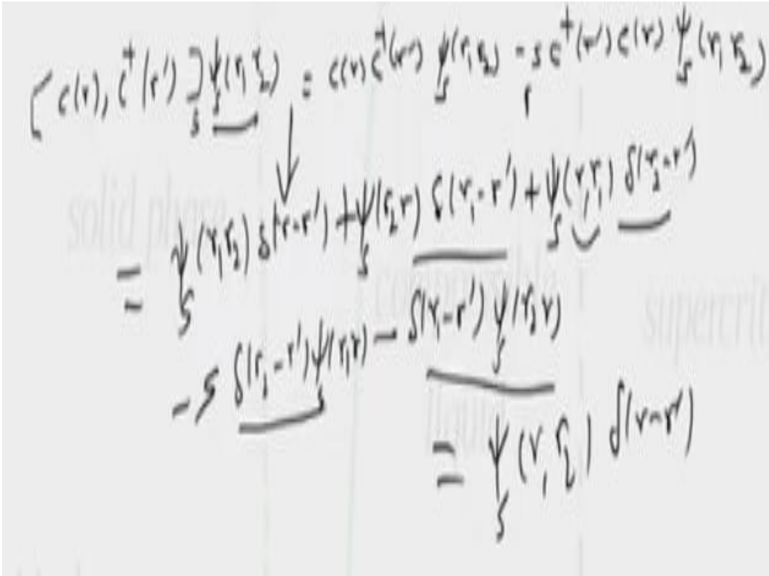

So let us put in that effort and see where we go from there. This is an isometric communicator and r2 okay. So that is by definition of this isometric commutator. So now I am going to start creating and annihilating things. So when I do that, I am going to allow you to fill in the rest of the gaps, if you find them to be incomplete. So I am going to write directly this result times psi r delta of r2 - r dash, then there is a minus s + now you are going to start doing this okay delta of r2 minus r dash psi s of r1 r - delta of r1 - r dash psi s r2 r.

And so now let us see what all cancels out. Look, there is an r comma r1 here, so if I interchange these 2. So remember that even if you freeze 2 of them, because they are already there, they were already anti-symmetric to begin with, they will continue to be anti-symmetric with respect to even the frozen variable, then if I flip you know that I am going to pick up an s there and that is going to cancel these 2 going to cancel out and there is an r2 r are already here, that anyway cancels out.

So what survives is this. So I this survives. So now that means that if you take the properly symmetrized psi of r1 r2 and you acted on the s symmetric commutator of C and C dagger, you get the same wave function back times the Dirac delta function. So that proves this theorem, which says that if s is +1, the Cs commute amongst themselves, except the C and C dagger commute to give you a Dirac delta function the different C's commute with each other.

However, if there are fermions they anti-commute if the they are the same Cs both are Cs of different rs as the arguments and they anti-commute if one of them is C, the other is C dagger with a Dirac delta between the 2 different rs. So that completes the description of the properties of quantum operators that create an annihilate quantum particles, be they bosons or fermions. So now what I am going to do in the next class is I am going to use this to define what is known as the Green’s function in many body physics.

So that is a very central concept in many body physics, which is defined using these creation and annihilation operators. So once I define the Green’s function, I will be able to tell you what is the physical meaning of the Green’s function, then I am going to also show or derive certain properties obeyed by the Green’s function specifically the boundary conditions in imaginary time and this so on and so forth.

So depending upon how it goes and time permitting, I might reach up to the stage where I derive what are Schwinger-Dyson equations okay. I am going to stop here now. Thank you.