Introduction to Statistical Mechanics
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Lecture – 27
Quantum Theory of EM Field- II

Okay, so let us continue with our description of the quantum theory of the electromagnetic field, so if you remember that we had left off at a stage where we had to solve these 2 coupled equations, so we had postulated that the Hamiltonian of the electromagnetic field is writable in terms of creation and annihilation operators which we denoted by small a.

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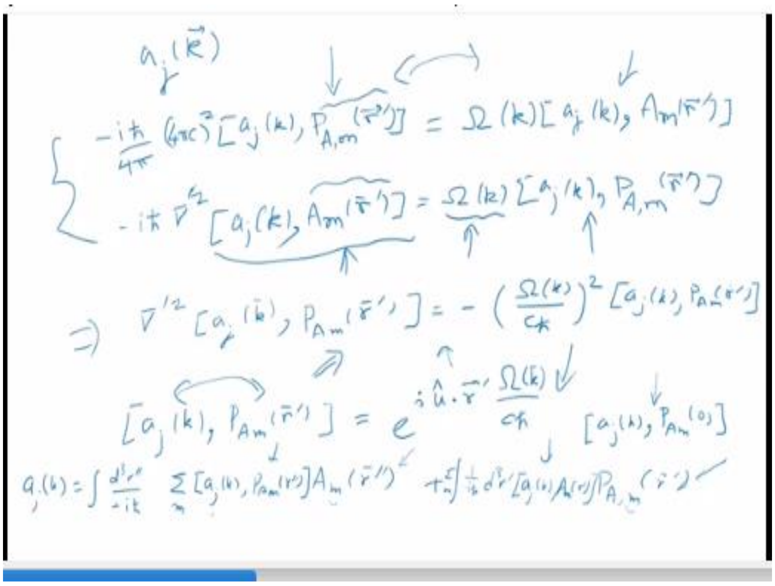

So, we denoted it by a subscript j,k, so this k refers to the fact that you are dealing with a continuum and j's will actually refer to the 2 polarization states which are mandatory for transverse electromagnetic field. So, now let me remind you what are the equations that we have to solve and they are the following; one is the equation which relates the commutator of the; the commutator of A with P.

So, if you remember that the P's and the position coordinate in for the electromagnetic field is just the vector potential and the conjugate momentum is basically, something like an electric field okay. So, what we are supposed to do; well, we are supposed to handle these kinds of equations and the energy Eigen value is going to be denoted by capital omega K times ajk commutator Am r dash, okay.

And then there is this equation which tells you, so this Poisson type of equation that we have to solve and so I am just reminding you where we left off last time because I had done all this so, I am just reminding you, so we are supposed to be solving this, this and okay, so like I have been repeating several times, it is important for you to try this out on your own, just do not take my word for anything that I said here, just work it out on your own.

So, now notice that this equation relates the commutator of A with the vector potential that is the annihilation operator of the photons as it were with the vector potential and that is what this commutator is and relates it to the commutator of the annihilation operator of the photons with the corresponding conjugate momentum. So, we need to know all these quantities explicitly this, this and also this.

Remember that we just still do not know what; we just know that it is an eigenvalue, so just like in all other problems in quantum mechanics, the eigenvalue is also to be found by solving the corresponding Schrodinger type of equations, so here two will be forced to, we would not know this a priori, we have to find out what it is. So, this is one equation, the other equation relates these 2 coefficients namely the commutator of A with a conjugate momentum to the commutator of A with the vector potential.

So, they are proportional and if you combine these 2 equations, you get a closed Poisson equation which you can easily then solve, so the closed Poisson equation is as follows well, in this case, well it is not really Poisson equation, it is more like Helmholtz equation okay, if there was some other variable here like a density, it would be Poisson equation ,actually it is Helmholtz equation.

Because this is the same as this one, well at this stage you can call it whatever you want but this is really Helmholtz equation okay, so it is going to look like this, so it is easy to now solve this. So, I have just combined these 2 equations and written this equation and then the easy solution to this is as follows; times the value of this at you know, r dash equals 0 if you wish, I mean this is the integration constant, this is the integration constant.

And this is clearly a simple solution of this and we are going to stick with this solution and this notice that u hat is an arbitrary unit vector which we will have to fix later, so how do you fix that and so see in order to fix this first we notice that this A itself is by definition a linear combination of the you know the generalized position which is this and the generalized momentum which is that.

So, just like is in the case of a simple harmonic oscillator you know, if you remember the annihilation operator was a linear combination of the position and moment; the canonical position and canonical momentum but in this case the so, the electromagnetic field the canonical position is really the entire field of vector potential and the canonical momentum is basically something like the electric field.

So, I am going to be able to write this in this fashion, so using that idea I can think of this as something into you know, it is some linear combination of A's and P's, so P subscript A implies that the momentum conjugate to A, this m is it carries over from this, the m is corresponding to this m. So, now how do you do this, I am going to convince you that this is correct by you know evaluating the commutators, okay.

I wrote too many integration signs here, d cubed r prime time's ajk commutator A m r dash, okay, so that is the m okay, I mean I am jumping and then I should have written okay, first let me write this in a; I do not want to swallow too many steps, so I am going to do it like this okay, this makes sense okay, this makes sense and this plus okay, this makes sense for what reason okay.

So, the reason why this makes sense is if you take ajk commutator suppose, you take Pm r dash, so what is going to happen is that A commutator P is going to be ih bar Dirac delta r, well firstly you should convert this to a dummy variable because see first of all A commutator P is only going to contribute because of this term, this term, first term, so it is not going to be contributing because of the second term.

Because the second term is already proportional to P and A commutator P is 0 and notice that A commutator vector potential is just a number and A commutator conjugate to vector potential is also just a number, this is very similar to the simple harmonic oscillator where the annihilation operator is a linear combination of X and P and the coefficients are of course, constants and these are those coefficients, okay, these are those coefficients.

And they are constant so, if I take the commutator of A and the conjugate to vector potential, this is going to drop out, this is going to survive and you are going to get an ih bar which is going to cancel out from here and you are going to get a Dirac delta of r dash, dash minus r dash, so in other words that is going to force r dash, dash to become equal to r dash and you get back and you basically you get an identity.

So, this is just you know it is just a way of representing the knowledge that A is linear in vector potential and its conjugate and the vector potential and its conjugate obeys canonical commutation rules okay. So, once you have understood that we can use this idea to pin down the quantities, we do not know about namely this, the nature of this unit vector u and the energy and so on.

So, I am going to substitute these kinds of expressions and so remember that once I know a commutator P from this relation, I also know a commutator vector potential so, I know everything as a function for r dash so long as I fix the other quantities there.

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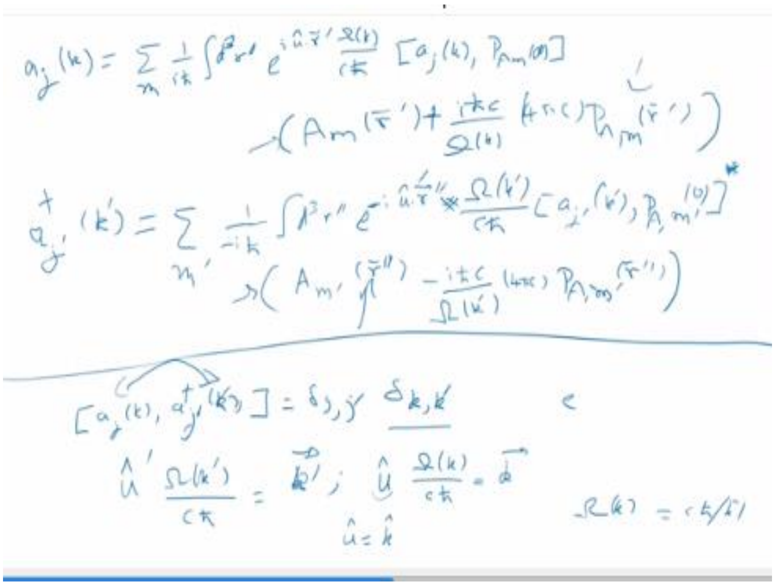

So, I am going to use these two and substitute in the expression for A and I get this result, okay, so I apologize for this I mean, I should really apologize because this subject is quite technical and the calculations are quite tedious and then notation becomes quite clumsy because of when you are dealing with fields, there are you know a large number of degrees of freedom and you have to invent subscript, superscripts and the arguments and so on.

So, very quickly the notation becomes unwieldy and it is not pretty to look at, no slide is pretty to look at, so that is the reason why I suggest that you should go ahead and work out all these ideas on your own, so that you become comfortable with what I am saying, so all right so I am going to just substitute the expressions and they are going to look like this; 4pi c times PAm r dash.

So, notice that this is the integration constant there okay, what I am going to do is that I am going to first use this to write down the corresponding conjugate with a different polarization index and a different wave vector and that is going to start to look like this, you can just take the hermitian conjugate of this expression change j to j dash and k to k dash and I get a new formula for the adjoint of A.

And that looks like this well; there is this term which says minus ih bar c over omega k dash times of 4pi c okay. So, now the idea is that I am going to enforce or impose canonical commutation rules, I am going to insist that this commutator is going to be j, j dash, delta of k, k dash. So, now you can see that look this is going to be the case only if u has the property that okay, so this is sorry, I mean this is times this, this is an exponent okay.

So, this becomes u dash, okay because everything becomes dash, so even the unit vectors because then finally, they are also getting summed over, so you will have to sum over the unit vectors as well, so now if you want this to be the case, then you see the only way this can be accomplished so, if you want to get a Kronecker delta, so basically, what you should get is the integral over, okay.

So, let me write it down and I will argue it out, so I am going to assert that the u dash unit vector times omega k dash divided by ch bar is equal to k dash vector okay and u vector times omega k over ch bar is k vectors, so notice that this is a unit vector so, if I decide to take the mod on both sides, the mod of unit vector is 1, so that immediately tells me that omega k is basically ch bar times mod k, okay so that is what it is.

So that implies that this unit vector is basically the k hat unit vector okay, so that immediately follows from this idea, okay. So, if I use this idea and I carry out these integrations, so I am going to skip some steps because it can become quite tedious, basically the idea that you have to evaluate these commutators and the way you do that is you take this expression and take the commutator with that.

So, a commutator P is some Dirac delta function times a Kronecker delta and similarly, this commutator with A is also like a Dirac delta times a Kronecker delta and so we will get a whole bunch of Dirac delta's and kronecker deltas.

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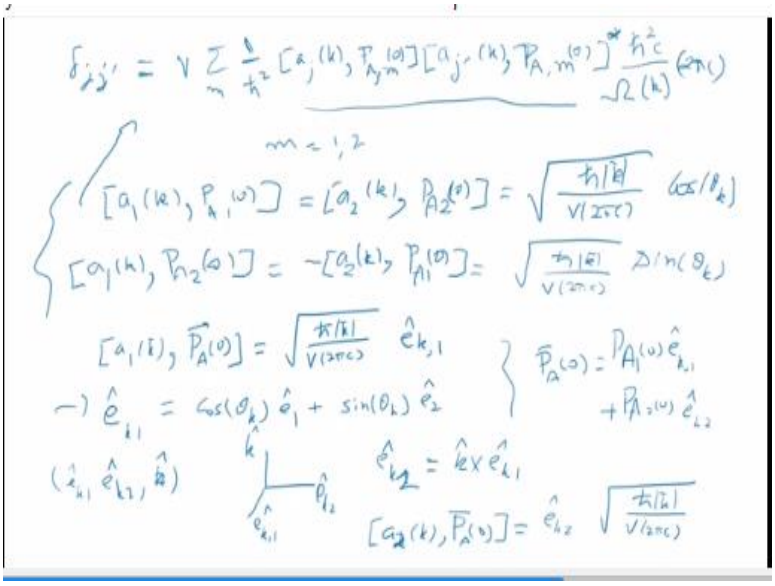

And so finally, you will end up with this constraint that the Kronecker delta that survives has to be equal to; so I am skipping a whole bunch of steps because like I told you it is, the subject is not very pretty and it is you know, it is easier to for me to show a typed document, it is much harder for me to write, so I urge you to fill in the gaps on your own, even if I flash you know pre-prepared transparencies you would still not be able to follow unless you work it out on your own.

So, basically the idea is that you will end up with a whole bunch of Dirac delta's and kronecker deltas and the constraint that survives is the following namely this, so the other point I want to make is that this course is not going to I mean this is strictly not part of the syllabus of this course in a sense that I am thinking of this as an advanced additional topic for you to learn.

But I am not going to test you in the examination based upon this understanding, so you do not have to necessarily prepare for this subject in order to pass the examination, so you can learn the subject at your leisure whenever you feel like doing so, so right now I am just you know giving you some pointers and telling you the direction in which you should move in order for you to understand the subject properly.

Because this is important which is why I am deciding to explain it but I won't be able to do full justice to the subject because you know it is not the kind of subject that you can just explain in a classroom very easily okay, so that is what it is. So, now you can go ahead and make the following so, this can be you know this constraint can be satisfied by the following assertion.

So, notice that this m index can be either 1 or 2, because there are 2 polarization states and so I am going to make the following assertions that a1 commutator PA1 is 0, is the same as a2 commutator PA20, which is; and the other result is a1 commutator PA20 is; so if I make these identifications you can see that this is obeyed, so I go ahead and substitute this here, I get what I am looking for.

So, this can be compactly written in the following, so these 2 can be written compactly in the following way, so I am going to write a1 k vector commutator P vector is square root h bar k over v times 2pi c times the polarization of the you know, this is a unit vector that would correspond to the first polarization direction. So, this would be writable so, basically you can always think of this as the; you know I can always write this in the following way.

So, basically it is just the linear combination of cos and sine, so it is cos theta k times e1 plus sine theta k, so you just verify this yourself, it is very easy, so where P1 is also by definition I mean, PA vector by definition is the PA1 e1 and okay like that so, now how do you find the ek2? So, ek2 is basically given as well, it is basically the ek1, ek2 and k for my right triad, so you can think of k vector like this and your ek1 is here and ek2 is there okay.

So, in other words you can think of ek2 as basically k cross ek1, so that is what it is, so once you know, as we know ek1, so you take the cross product of k and ek1, you get ek2, so that will immediately I will tell you that ak2 commutator P A 0 is going to be therefore, so you just work this out on your own, it is going to look like this, h bar k over v times 2pi c, okay, so that is what that is.

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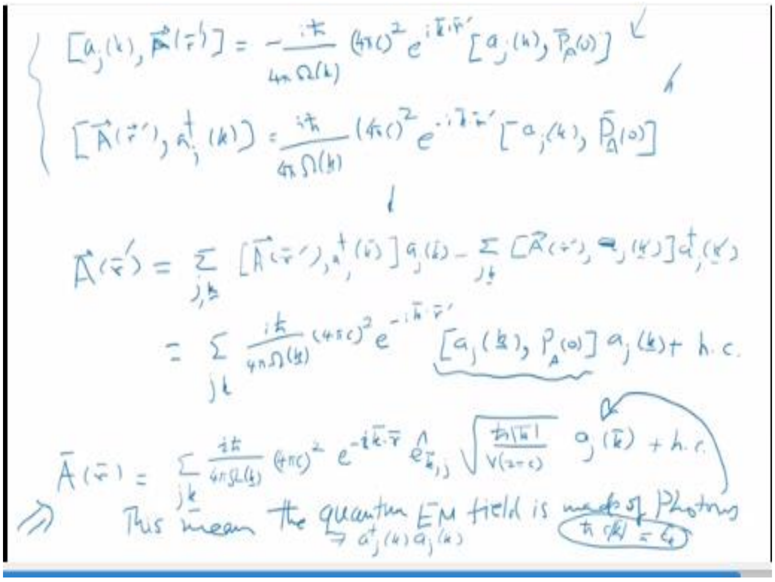

So, from here you can easily work out the commutator of the oscillators or the photon creation and annihilation operators for example, the photon annihilation operator commuting with the vector potential is going to now look like the 4pi c squared times e raised to ik dot r dash times ajk commutator PA0, okay so that is the that is one of the results, the other result is the conjugate to this you just take the dagger, you get this result.

So, notice the vector potential is real, so that dagger of that is itself, however the annihilation operators become their adjoint, when you take the conjugate and I end up with this result. So, somewhere a postulate that this is real and there is no loss of generality because then you know the phases can be absorbed by a redefinition of the phases of the creation and annihilation operators, okay.

So, this enables me to write for example, I can invert this I mean I can think of now the vector potential therefore as being related to the oscillators or the creation and annihilation operators of the photons okay, so this is actually the ultimate goal, so you must have forgotten by now what it is we are trying to do. So, what we are trying to do is basically describe or demonstrate or prove that the electromagnetic field is indeed made of photons.

So, the way you do it is to somehow finally convince yourselves that say the vector potential so, if you work in the radiation gauge and if it is free space, then the only thing you should worry about is the vector potential because the magnetic field is the curl of the vector potential and the electric field is the time derivative of the vector potential times minus 1 by c in CGS units.

So that is all you have to know the vector potential so, if you can convince yourself that the vector potential is somehow related to creation and annihilation operators of photons or basically some kind of oscillators; bosonic oscillators which you call photons, then you are more or less done because that is a confirmation that the quantum theory of the electromagnetic field is described in terms of quanta that obey a bosonic algebra or they have bosonic character, all right.

So, we are nearly there in other words, so by appropriately combining, so this implies basically that you can express the vector potential in terms of the creation and annihilation operators in the following way namely, see this is going to look like this okay. So, now I am going to substitute these formulas here and I get this result okay, so I am going to be able to write like this.

Sorry, this is A r dash, so this is going to look like this; plus the hermitian conjugate because A is real, so it is naturally the, whatever comes later is the hermitian conjugate of this. So, now remember that I have written down an explicit formula for ajk commutator P is 0 and that is this, so you know if it is j, it is basically is j, so this is what we have explicitly written down.

So, I am going to use that there and complete my calculation which basically, tells me that the vector potential is now expressible in terms of the creation and annihilation operator of photons okay. So, this is the complete story, so this tells me that this means the quantum EM field is made of photons and these are those photons, so these are the operators that create and annihilate photons and that is it that is pretty neat.

And what are the energies basically, the energies are you see the energy of each photon with momentum k and polarization j is basically h bar times ck; h bar k is the momentum and momentum times c which is a speed of light is the energy, so that is my energy of the photon, okay. So, the energy of the photon and the number of quanta would be basically it is an operator you see that this would be the number of quanta or just the number of photons with momentum k and polarization state j.

So, now this completes the description of the electromagnetic field in terms of photons, so strictly speaking this is not part of statistical mechanics per say but like I told you it is important to know the finer details or the deep reason for why the electromagnetic field is thought of as photons because remember that Planck's blackbody radiation was kind of accounted for by postulating that light is made of the energy content of light is in the form of quanta.

So, you see even though Einstein did not know all these quantum theory of the electromagnetic field which was done of course by Dirac much later, Einstein got it right I mean his intuition was fully correct and that the electromagnetic field should be thought of as carrying energy in packets; discrete packets and the deeper reason for why that is valid is because when you start with the electromagnetic field as a classical field.

And then you know try to apply the rules of quantum mechanics to a classical system the way you normally do in quantum mechanics, you naturally end up with that result which Einstein guessed namely that the electromagnetic field consists of photons. So, now this you may think of this description what I have given you till now which is attributable to Dirac as the rigorous explanation for why the electromagnetic field may be thought of as being made of photons, okay.

I hope you enjoyed this lecture even though it is not part of the syllabus of this course but it is kind of one of those appendices which you should know even though you do not need it to pass an examination. So, I will see you next time with another of these so called advance topics but I am sure you will enjoy it also because as you proceed in your education, you will find that most of these topics that I am discussing are actually quite important and you would not find them all that advanced any more okay, thank you let us meet again.