Introduction to Statistical Mechanics

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Lecture – 26
Quantum Theory of EM Field- I

Okay, so what we did till now was that we discussed the creation and annihilation operator method, so we started off with the simplest system imaginable then that is the simple harmonic oscillator of a single mass tied to a single spring and you study that quantum mechanically and I explained to you how to study that using creation and annihilation operator.

So, the other thing you can do is you can study a chain of harmonic oscillators, mass spring that kind of a sequence, so you have a mass tied to a spring which is at the other end of the string, there is another mass which is followed by another spring and so on, so that type of system can also be studied using creation and annihilation operators and what we found for the simple harmonic oscillator for a single mass and a single spring was that the energy is quantized.

And the energy eigenvalues are basically, n plus 1/2 into h bar omega where n is an integer; 0, 1, 2, 3 and so on. So, similarly we could conclude for the case of a chain of mass spring, mass spring kind of sequence that the energy is actually now labelled by a; so it is kind of there are a continuous set of values that are possible, so it is not just the quanta have an additional label.

So, in the case of harmonic oscillators, so the moment you specify the number of quanta this, the state is uniquely fixed but here in addition to specifying the number of quanta, there is another label called Q which is basically the momentum associated because the translationally invariant system, so momentum is a good quantum number so, the momentum becomes a label along with, so the number of quanta has a label Q associated which is with it, which is the momentum.

So, you have to specify the number of quanta with momentum Q which and if you specify all those numbers, then you will specify the state of the system. So, in the same way we can study other systems which have similar properties and I mean I am going to actually come back to this fock space a little later.

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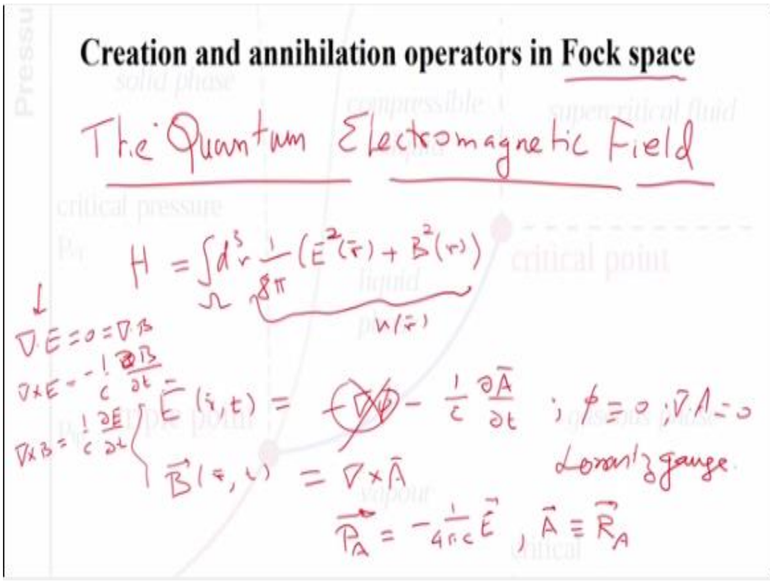

So, let me study instead the electromagnetic field, so first I am going to study the quantum electromagnetic field okay. So, the quantum theory of the electromagnetic field, so you know all of us are kind of taught from an early age that light is described you know, it is both the particle and a wave and there is the wave particle duality and then and all that sort of thing but nobody ever tells us why we should believe all these statements.

So, in particular the fact that light is a wave is believable because you know Maxwell introduces famous displacement current in his equations which allowed him to conclude that electromagnetic field in empty space you know always a wave equation and the solution of that propagating waves, they could be propagating waves with; so those are possible solutions.

And he found that the speed of those waves is a universal constant which is 1 over square root of mu 0 epsilon 0 where mu 0 is the permeability of free space and epsilon 0 is the permittivity of free space, so that really set the stage for the development of special relativity because that statement that the speed of an object or the speed of something is a universal constant is at odds with Galilean relativity.

So, Einstein used that idea to develop special relativity, so all this is this background story is well known to all of us and the fact that the light is an electromagnetic wave predates the idea that well of course, if you discount the old idea of Newton who kind of just by force of his authority declared that light is made of corpuscles, when in fact Young's experiment had shown otherwise.

So, if you discount that the legitimatize scientific ideas that were prevalent in the early late 19th century and early 20th century showed that basically you have to admit that light is an electromagnetic wave which propagates with the speed of light which is a universal constant, so given that background it is hard to understand why anyone would go back to old very old ideas of light being made of particles.

But then the advent of quantum theory showed that you need a dual description because even electrons for example, which we traditionally think of particles exhibit wave like properties and entities that exhibit wave like properties should logically also have particle like properties if maybe you have not looked hard enough. So, people did look hard enough and they found that there is a phenomenon known as photoelectric effect which naturally is explained by postulating you know electromagnetic radiation as being made of particles.

So and you know famously Einstein postulated that the energy contained in light itself comes in multiples of h Nu and which enables him to derive Planck's blackbody radiation so, Planck had erroneously assumed that the energy levels of the absorbing you know atoms of the walls of the container they are the ones that are quantized, so Einstein corrected him and said it is the light, the energy of light itself is quantized.

And of course, then that really forces us to think about maybe you know that set again, now that set odds with Maxwell's theory of electromagnetism which unequivocally talks about the electromagnetic field propagating in empty space being a wave. So, the question is where do these quanta come from, I mean what is the fundamental theory which explains where the quanta come from.

So, the answer to that is like everything else obtained by you know realizing that the that Maxwell's theory of the electromagnetic field is in essence a classical theory that means, it does not invoke quantum mechanical ideas that it is based on the idea that you know all observables commute and you can observe physical phenomena without disturbing that system on which the measurements are taking place.

So, all those are classical ideas and they need modification and now people realize that Maxwell’s theory of the electromagnetic field is classical and one has to you know develop a more sophisticated version of the electromagnetic theory which incorporates or takes into account quantum mechanics. So, the question is how would you go about doing that? So, the prescription is straightforward you know, as it is in most other systems that what you do is basically you identify if, so you start off with a classical system which is described in terms of certain variables that you identify as being equivalent to some kind of a position and some kind of a momentum,

So, you want them to be you know canonically conjugate in the sense of Hamiltonian mechanics. So, if you decide I mean, if you are successful in identifying variables in your systems that are canonically conjugate in the sense of Hamiltonian mechanics, then quantum mechanics affords a clear cut prescription or a path to developing the quantum version of such a theory.

And that involves merely promoting those variables that you have identified as position and momentum to operators and then imposing you know canonical commutation rules and then go ahead and describe the system quantum mechanically that way. So, let us take this point of view and see how far it can take us when we apply it to the electromagnetic field, so in order to do that I have to start off with some electromagnetic theory which I am going to assume you already know.

Because I cannot possibly teach you all of physics in this course because strictly speaking this is statistical mechanics and the topics that I am teaching you now are somewhat tangential to the subject of statistical mechanics but then you know, knowing where the quanta of electromagnetic field come from are important because when you have such a deeper understanding of where they come from, then will you be able to appreciate more you know the topic that is central to this course namely Planck's blackbody formula.

So that is the reason why I felt it necessary to introduce or include these topics which are kind of somewhat scarily called advanced but they are really all that advanced but they are quite fundamental even if they are advanced. So, I am going to start off by writing down the well-known expression for the Hamiltonian or the energy contained in the electromagnetic field in free space.

And I am going to use CGS units, so physicists prefer CGS units because in electromagnetism especially, we prefer CGS units because the electric and magnetic field are the same dimensions in CGS units, you do not have those pesky, Mu 0's and epsilon 0’s, which kind of very bothersome, so I am going to write down the Hamiltonian of the electromagnetic field of the energy.

So, we know that the energy density of the electromagnetic field in CGS units is writable in this fashion, this is the square of the electric field plus square because E and B have the same units, so I am going to simply add the squares and then I divide by 8pi which comes from the you know, by examining the continuity equation for the pointing vector and energy density of the electromagnetic field over a continuity equation.

And from there you can read off now that you know the pointing vector is a you know, C by 4pi into E cross h, you can read off what the energy density ought to be by staring at the continuity equation. So, this is basically the energy, so if the electromagnetic field occupies some kind of a volume omega, so you take the energy density which is u and then you integrate over that volume and you get your energy that is contained in the electromagnetic field.

And notice that we are talking about electromagnetic field in free space well, in general you can always write the electromagnetic field in terms of the appropriate potential, so you know that in electromagnetic theory, there are 2 types of potentials; one is called the scalar potential and the other is a vector potential. So, in terms of the electric field is writable in terms of both.

Whereas, the magnetic field is purely given in terms of the vector potential okay, it so happens that you know there is something called a choice of gauge that there is too much ambiguity you know well, the sacred quantities are E and B, so you can have different choices of Phi and A, which give you the same E and B. So, in particular you can get away, I am not going to explain all this because it is part of an electromagnetic theory course.

So, in particular you can get away by choosing Phi equal to 0 and this is called choosing the Lorentz gauge okay, so if you do not know what all this is, please look it up, go to your electromagnetic theory, Jackson is always a default choice you know JD Jackson classical electrodynamics or you could go for Landau Lifshitz also , you are bound to find a description of gauge transformation in those books.

But also Griffiths would have some descriptions electromagnetics by Griffiths would have something, so it is called Lorentz gauge; I am going to choose Phi equal to 0. So, the point is that so, this in turn you know because of the 4 Maxwell equations well remember that we are in empty space, so it is E and B have no divergence, curl of E is minus dB/dt, it is 1/c dB/dt and curl of B is 1/c dE/dt, okay.

So, this is these are the 4 Maxwell equations, so that Phi is anyway 0, it is not there, so the divergence of A can be said to be 0 because of this gauge choice okay. So, then you can you see the point I am trying to make is that this enables us to think of the one of the variables say, the electric field to be the momentum, so I am going to characterize the effective momentum.

So, I am going to think of these 2 as somewhat like conjugates, okay of course, what right I have to do that because you know again you will have to consult an appropriate book on classical field theory for that basically, it so happens that the Poisson bracket relations are recovered if I decide to think of; so remember that you know the mutual Poisson bracket between momentum and position is 1.

So, if I choose to think of the electric field as being proportional to the canonical momentum, then it so happens that then I can think of the vector potential as the position okay. So, this becomes my R, so I mean why this is I am going to kind of defer that to some other course.

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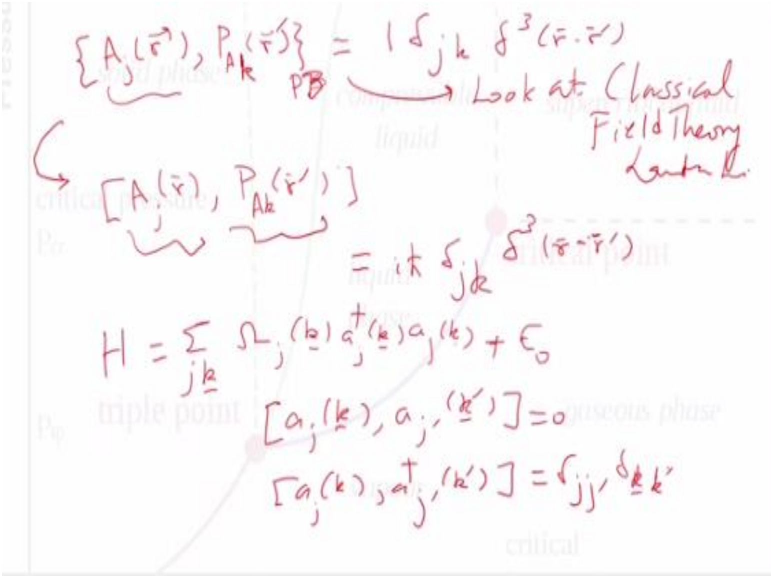

So, it so happens that you can verify this that the jth component of the vector potential, the Poisson bracket of that with the kth component of the electric field that is what P is, P is the canonical momentum which have identified with the electric field apart from this constant called minus 1 by 4pi c. So, the Poisson bracket; so this is the Poisson bracket, this is the classical Poisson bracket that is basically, going to be 1, with that delta j, k there.

So, this is the kth component of the canonical momentum, so this becomes a canonical position as it were, now this is so you know quantum mechanics allows us to have a corresponding quantum description of the system and the way you do that is you promote the Poisson bracket to the commutator and you end up writing this well okay, sorry, there is a Dirac delta function.

So, these are 2 different positions and so this allows me to; so do not ask me where this comes from, you will have to look up, look at maybe some book on classical field theory like Landau Lifshitz classical field theory, or you do it yourself whatever, so this is going to be ih bar delta jk times delta cube r minus r dash, so this becomes the commutator. Now, look we already have pinned down the canonical position and canonical momentum.

And at the back of our mind, we have this requirement that the electromagnetic field be describable in terms of quanta, so in terms of say creation and relation operators which describe creating and annihilating those quanta but we also know from our experience of studying the simple harmonic oscillator or a chain of harmonic oscillators that the position and moment are relatable to linear combinations of creation and annihilation operators.

So, given that background we should be able to write down this Hamiltonian in terms of, so remember that in the case of a chain of harmonic oscillators which have the mass followed by a spring followed by a mass again followed by spring, so that is a prototype of a field where you know you have a degree of freedom or different degrees of freedom at different locations, so that is the simplest example of a field.

So, when you quantize that system you have the simplest example of a quantum field, so here too we can suspect that and we also saw that for a system which has translational symmetry, the quantum field is more simply described in the Fourier space rather than in the real space and the Fourier space, the Hamiltonian becomes diagonal when expressed in terms of the creation and annihilation operators.

So, we are going to kind of borrow all that those ideas enmasse and apply it here and when you do that, we can; so this is an optimistic guess and it is going to be a valid guess as we will find out later. So, I am going to guess that apart from some zero point energies, my Hamiltonian is writable in this fashion and I am going to suspect that the creation and annihilation operators are going to obey this relation okay.

So, now I am going to do the following, I am going to rewrite this Hamiltonian, so the idea is that this Hamiltonian first has to be written in terms of canonical momentum and canonical position, so the canonical momentum is proportional to the electric field and the canonical position is proportional to the vector potential, the scalar potential has been selected to be 0 identically because of Lorentz gauge.

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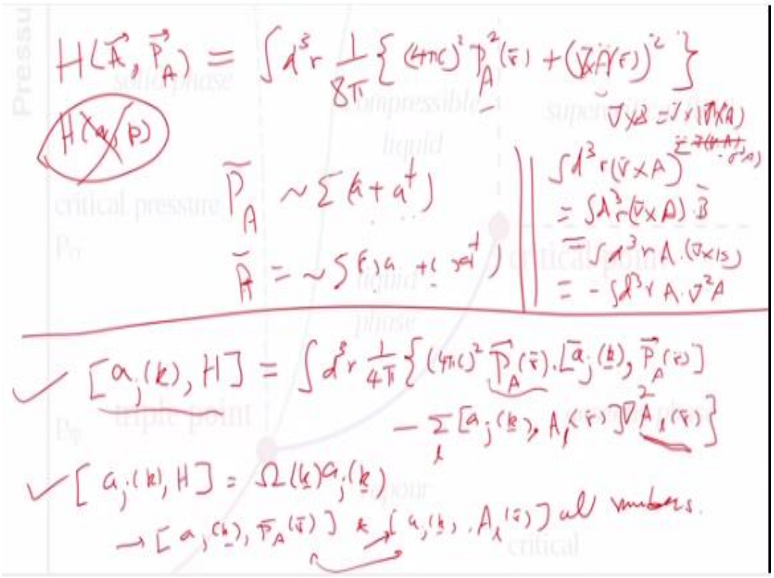

So, now the Hamiltonian is expressible in terms of the; so this is the generalized position and this is the generalized momentum, so this is you know the analogue of this, this is what we would be writing in the classical mechanic’s language, this is your phase space now remember that Q and P are operators and we are going to interpret it that way; PA square plus okay.

So that is I mean where does this come from; it just comes from you know taking the definition of PA and RA but I am not in using this, just take retain this as it is, this B squared is just curl of A squared, I have retained that as it is, so as just B squared, it is just that I have written E square in terms of PA, so this is position, there is a momentum okay. So, now the idea is that this PA is expressible as some kinds of linear combination of all these you know creation and annihilation operators.

And similarly, A; I mean the position is also expressible as some other you know something else, something along those lines, so the idea is to find all this. So, in order to do that I am going to do the following that I am going to do the usual thing, so I am going to start calculating a commutator such as this okay, I am going to start calculating commutators such as this.

So, I am going to calculate this, so this is going to be basically, this is going to be this, so also keep in mind that the integral of curl of A square can be written like this, you can firstly even write this as integral of curl of A times curl of A which is curl of A is B, so one of the curl of

A, I am going to write as B and what I am going to do is I am going to be able to write this as integral of A dot curl of B.

So, now curl of B is curl of curl of A but then keep in mind that the curl of curl of A; curl of B is curl of curl of A, so curl of curl of A is grad of A minus del squared A but then divergence of A is 0, so it is minus del squared A, so it is going to be like this, A dot del squared A, so I end up with this okay. So, I end up having to do all this same there okay, so that is my Hamiltonian.

See, this is on the one hand but on the other hand, keep in mind that this is what we have called H, so on the other hand a, j, k commutator H is also equal to omega times omega times ajk, put an index here but we will see later on that there is no I mean, does not depend on j, you might be wondering what this j is, so that is basically the polarization state. So, I am going to tell you right now what it is.

Look in the case of mass spring; mass spring that type of chain A, I mean the creation and annihilation operators had one label Q but now I have kind of expanded it out and said well there is a K which takes on the role of Q but there is an additional index J, I mean I am doing that to be on the safer side but later it will turn out that J is actually discrete and it has only 2 values and that is called polarization.

That is because you know the electromagnetic field in free space has 2 polarization states which are linearly independent, so those will also be the labels of the state of my electromagnetic field that is the reason but then the energy will not depend on the polarization state, okay, so that is why it does not carry that label. So, now keep in mind that these the commutators of A with P and a commutator of A with the vector potentials, they are all numbers, they are not operators.

Because see, we have kind of surmised that the position and momentum just like it is in the case of harmonic oscillator, we expect them to be linearly related to the creation and annihilation operators, so the commutator of the position and momentum with the creation and annihilation operators are numbers okay. So, since these communicators are just ordinary numbers, we expect the further commutator with this and other objects to be 0.

So, now what we are going to do is; see look what we have done here, we have taken the Hamiltonian we have found the commutator with respect to the annihilation operator wel, this is expressible as a linear combination of the canonical position and the canonical momentum but now, suppose I decide to take the commutator of this object with respect to either the position or the momentum, then we expect to get these numbers that I have already told you their numbers namely, these things.

Because associated with these numbers are these operators, so if I choose to find commutator with one or the other, then one or the other of these numbers are going to survive and the other drops out, so I am going to do that now.

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$$[[a_j(k), H], A_m(\vec{r}\,')]$$

$$= \int d^3r\ \frac{1}{4\pi}(4\pi c)^2\ \underbrace{[\vec{P}_{A_l}(\vec{r}), A_m(\vec{r}\,')]}_{-i\hbar\,\delta_{ml}\,\delta(\vec{r}-\vec{r}\,')} \cdot (a_j(k), \vec{P}_{A_l}(\vec{r}))$$

$$= -i\hbar \int d^3r\ \frac{1}{4\pi}(4\pi c)^2\ \delta^3(\vec{r}-\vec{r}\,')\ [a_j(k), P_{A_m}(\vec{r})]$$

$$= -i\hbar\ \frac{1}{4\pi}(4\pi c)^2\ [a_j(k), \underline{P_{A_m}(\vec{r}\,')}] \text{ —}$$

$$[\underbrace{[a_j(k), H]}_{\Omega(k)\,a_j(k)}, A_m(\vec{r}\,')] = \Omega(k)\,[\underline{a_j(k), A_m(\vec{r}\,')}]$$

And if I do that so, I am going to do this, so I am going to see if I can find this, so taking the commutator of this with respect to the canonical position will enable this to drop out because all the components or the position commute with all other components of the position, thus here you get you know ih bar because that is the canonical commutation rule, so I am going to get that.

So, I am going to be; so now, you see notice that this is nothing but minus ih bar delta, so I is going to be the appropriate component say, suppose this is k okay, let us call this l, this l, so this is going to be ml delta cubed r, r prime r, so that is what this is. So, I am going to be able to write this in this fashion okay, so then that l becomes the same as m and then this becomes m and so on.

So, this enables me to write this so, we call the Dirac delta function, this becomes minus ih bar 1 over 4pi times 4pi c whole squared times ajk commutator PAm r dash, so this is on the one hand but on the other hand, this so, remember that this is also equal to omega well sorry, omega k, ajk that is also equal to that. So, on the other hand this is equal to omega k ajk Am.

So, on the one hand it is this but on the other hand, it is this so, you see immediately I am able to relate the commutator of the annihilation operator with the canonical momentum to the commutator of the annihilation operator with the canonical position, so that is one relation that I have. So, in a similar vein we can do the other thing, so remember that what we have done here is we have taken this commutator of A with H.

And found the commutator of that object with the canonical position which is the basically the vector potential but we could also do the other thing which is take the commutator with the canonical momentum which is basically, the electric field so, let us do that. So, if you do that what you are going to get is this.

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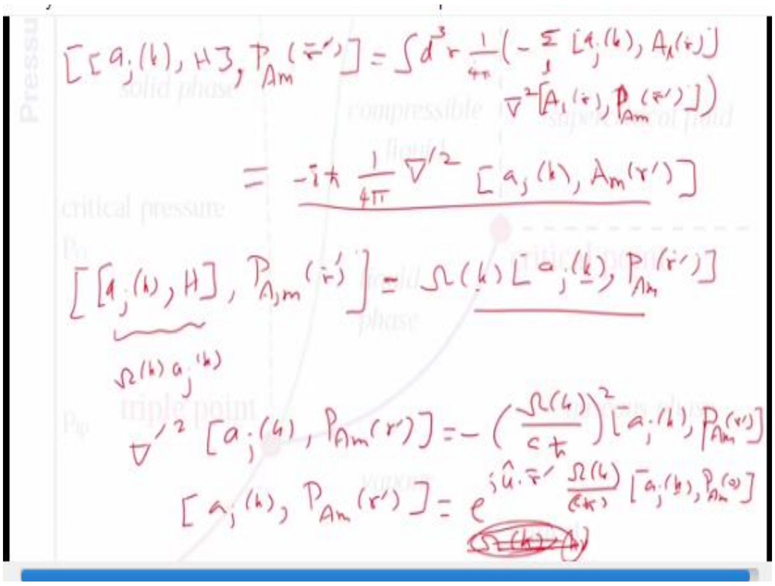

So, I am going to get this, so see remember first of all that s, A commutator H involves this and this but now I am going to take commutator P, so only this will survive sorry, I am missed del squared here, there is a del squared okay yeah, there is a del squared there, it did not matter for the earlier example because commutator of this with A will make this whole thing go away.

But now it matters now, there is a del squared I miss that okay, so that del squared comes because there is a del sitting there so because of that I am going to be able to write this in the following way; times the commutator okay, it is going to be this okay, so that is on the one hand but on the other hand because this is related to or it is equal to this, so on the other hand this is also equal to very simply just this.

So, by combining these 2, we can conclude that del squared of ajk commutator PAm r dash is given by minus omega k over ch bar squared, so what I am going to do is look, so this is equal to this but then notice that we have also succeeded in showing that this is equal to this divided by this constant, so I am going to take this relation and substitute here or the other way around.

So, basically I get this relation so, I get a relation purely for the del square of this commutator in terms of itself, so if I solve this, so this is easily solved, so I am going to be able to solve this using plane waves, some unit vector dot r dash times omega k over ch bar times this quantity at 0, okay. So, we will continue next time where we will show how all this leads to the; of an explicit formula for the energies that are possible for the quanta.

So, we expect this to be h Nu because that is what Einstein taught us but then we will have to actually derive this okay, so I am going to do that in the next class, so I hope you will join me for that okay. Thank you.