Introduction to Statistical Mechanics

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Lecture – 25
Introduction to Second Quantisation: Harmonic Oscillator

Okay, so I am going to start a new topic, so this is along the lines of the, so it belongs to the part which I have been calling some of the advanced topics, so in other words if you recall that I had classified some of the topics such as renormalization group, second quantization, Schwinger Dyson equations etc., as advanced topics and so I finished discussing renormalization group applied to the one- D using model.

So, now I am going to discuss the method of second quantization which is very powerful when dealing with interacting many particle systems, so I am going to start with a very simple example which you should all be familiar with but it is worthwhile doing it the way I am going to explain because I am going to use a similar method again and again okay. So, the model I am going to start with is the absolutely simple, one dimensional harmonic oscillator.

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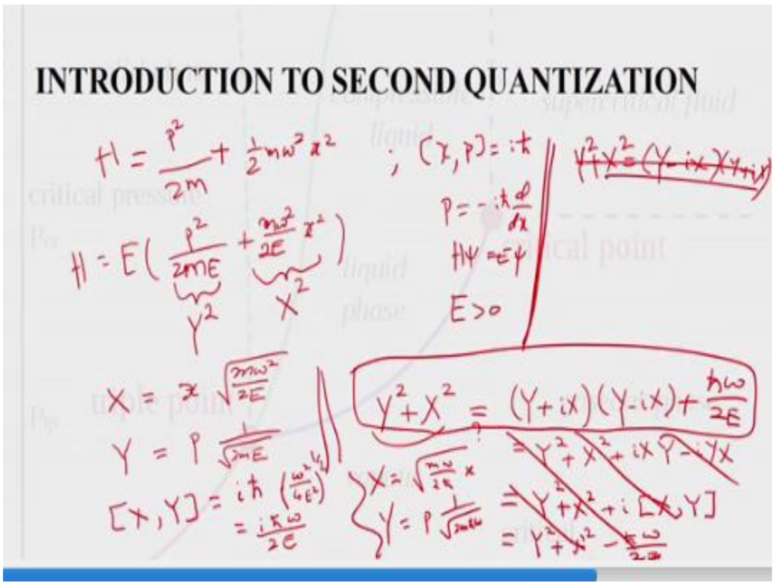

So, this is my Hamiltonian so, this is the simple p square by 2m plus 1/2 m omega square x squared, so this is my simple one D Hamiltonian and it is quantum mechanical, so that means the x and p are operators and they obey commutation rules such as this okay, so the question is; the goal is now to find the energy, meaning the eigenvalues of this Hamiltonian.

So, one way is to you know, write the Schrodinger ways to write p as minus ih bar d by dx and then just solve the further stationary states, so this is what one typically does but then you know there is a clever way of doing this, this is actually useful if you also want the Eigen functions along with the Eigen values but if you do not care about the Eigen functions, you only care about the eigenvalues, there is a quicker way of getting it.

And that is through what is called the second quantization method or so basically, it is involves using creation and annihilation operators, so this is; that is going to generalize to something in the context of field, so when I discuss so, this is just a single particle so, in the context of fields we are going to be using a similar method so, I am going to start off with a simpler one particle picture.

So, how do you deal with this; so notice that this has the form so, this resembles something like what I am going to do is the; I am going to first take the liberty of you know multiplying and dividing by a certain energy. So, I am going to multiply okay, so firstly let me do the following. So, I am going to pretend that there is some energy E okay, so which is positive.

And I am going to without loss of generality, multiply and divide by that so, I am going to fix this E later on okay, so I am going to tell you what this E is later on, for now it is just I am at liberty to do this, so long as E is not 0, I have also declared it to be positive so, so long as it is not 0 and it is positive I can always do this because E will cancel out. So, now if you stare at the thing in the bracket, you see this has the form, so it this resembles something like a capital Y squared and this resembles something like capital X squared, okay.

So, these are 2 new operators that I am going to now define, so I am going to define X as the operator X times as square root of m omega squared divided by 2E and the operator Y; I am going to define as p times 1 over square root of 2mE. So, now you see the question is; if I look I mean, if Y and X are numbers then, clearly I can always write like this, is not it?

So, I can always write Y minus ix, so if Y and X are numbers but if Y and X are not numbers, so in general what would this be? Suppose, if I expand this out what will it look like so, this is looks like Y squared right, Y into Y is Y square and if I multiply the ix with minus ix, it is going to look like x squared but then there is a cross term which is i into XY minus i into YX.

So, notice that if X and Y are numbers or they are just they are not operators, they are just commuting numbers, then X times Y is same as Y times X and this cancels out but then in this example X and Y are operators, so this is actually going to be Y squared plus X squared capital X squared that is, plus i times commutator of capital X capital Y. So, now what is the commutator of capital X, capital Y?

Just from this; from the definition this is going to look like so, small x commutator, small p is ih bar, so it is going to be ih bar times the rest of it okay, so the rest of it is basically raised to 1/2 times omega squared over 4E squared okay, so m cancels out and 2 becomes 2 times 2 is 4 and E times E is E square and so, this is going to be ih bar into omega over 2E, okay, so that is what that is going to be.

So, this is going to be Y squared plus X squared plus a constant, so that constant happens to be minus h bar omega over 2E, okay. So, now I can do the other way also, so in other words I can choose to do it this way, so notice that these 2; if X and Y were numbers it would not matter whether I write it like this or I write it like this, it is the same thing. So but then, so these 2 are not equal in other words, this is not equal to this.

So, what I have to do is I have to write this as this so, I could do this also; I get the different sign here, so I would not do this but I am just pointing out that you could do it this way or this way but I choose to do it this way. So, now Y squared plus X squared is not equal to Y plus iX into Y minus iX but rather it is Y plus iX into; so I have to get rid of this.

So, basically I have to write h bar omega plus 2E, so now it is correct okay, so now it is correct okay, so now it is this is correct okay. So, this is what I am going to use now, okay.

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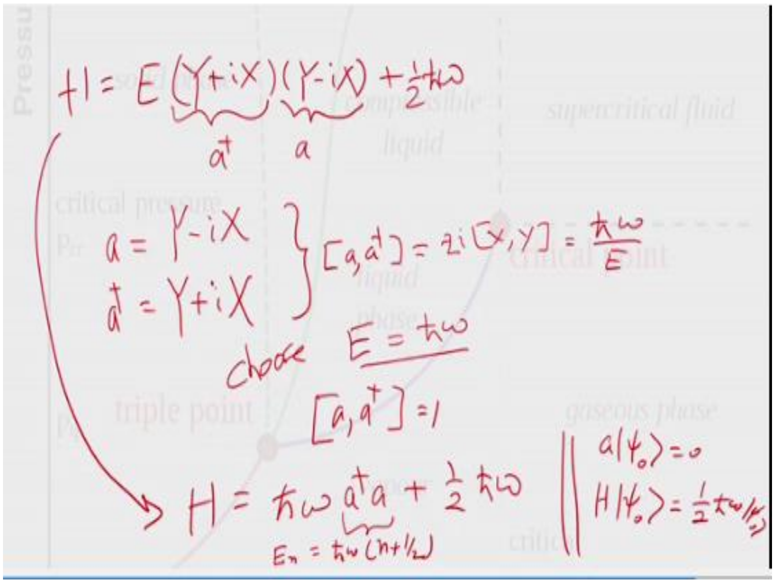

So, H is going to now look like E times; it is going to look like Y plus iX into Y minus iX times E times this, so it is going to be plus 1/2 h bar omega okay, so that is what that looks like. So, now what I am going to do is I am going to call this ‘a dagger’ and I am going to call this ‘a’ okay. So, a is basically this operator which is equal to Y minus iX and ‘a dagger’ is going to be Y plus iX okay.

So, dagger means basically, the complex conjugate of the transpose, so it is an operator, so it is the same dagger that you see in quantum mechanics, is the hermitian conjugate. So, now I can ask myself what is this commutator? Notice that this commutator will depend on E because that E was something that I just put in by hand, so I am at liberty to choose E in some convenient way, so to my convenience.

So, notice that this X commutator Y is ih bar omega over 2E, so a commutator, a dagger is basically going to be minus I X commutator Y, then it is going to be Y commutator iX, so there is going to be minus 2i there, so this is the answer okay, a communicator a dagger is minus 2i times X commutator Y. So, what is X commutator Y? It is ih bar omega over 2E, so it is the; so this 2 will cancel, i will cancel with minus i and you will get h bar omega over E.

So, what I am going to do is; I am going to choose E to be equal to h bar omega, so if I choose E to be h bar omega, what is going to happen is that this a, a dagger commutator will be 1 and that will fix the rest of it okay. So, then I will be able to write this Hamiltonian in this fashion, I will be able to write E as h bar omega because I have selected it that way, so and this is a dagger a plus 1/2 h bar omega okay.

So, now you can make out from here that this quantity is a self adjoint positive definite operator, so that means it is a hermitian operator that whose eigenvalues are positive or 0, so you can easily make out what the ground state of the system is because this is either positive or 0, clearly the smallest energy is for a state, so you can select a state where this is 0.

So, if you select a state where which is annihilated by a, so this operator if it annihilates the state, then that is going to be the ground state of the system because that is going to have the lowest energy, so that directly you can make out that has the lowest energy, okay. So, now the question is what about the other Eigen value? So, the ground state is 1/2 h bar omega, so how do you make out the other Eigen values?

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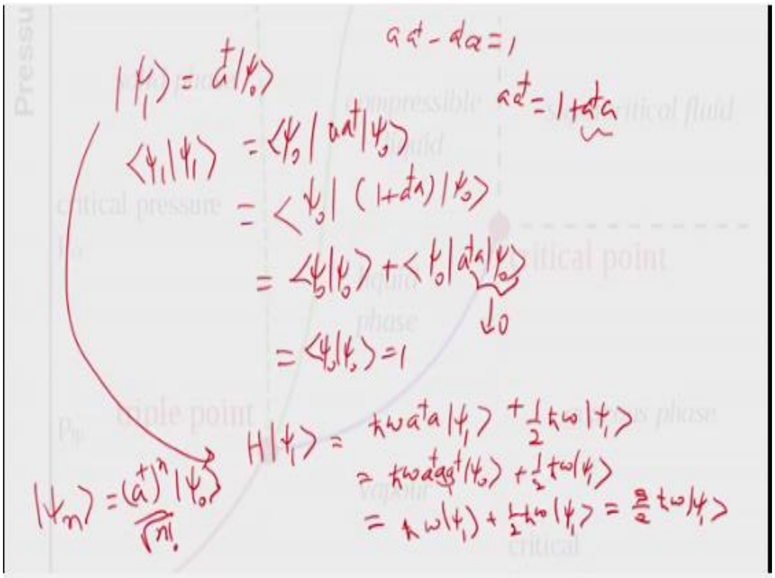

So, from the ground state you can construct the other states in the following way, so you can construct an excited state, so I am going to call psi 1, suppose I define psi 1 as a dagger psi 0, okay so, this is my definition. Now, you can see that this is if psi 0 is properly normalized, I have to make sure this psi 1 is normalized. So, if I take the adjoint and I take the norm like this, I get a times a dagger psi 0.

But then a times a dagger is; so remember a times a dagger minus a dagger times a is 1, so this is going to be psi dagger into 1 plus psi 0 okay, so this is going to be psi 0 psi 0 plus psi 0 a dagger a psi 0. Now, notice that this is 0, okay so, this is going to be psi 0 psi 0 which is 1, so this is properly normalized state. So, now I have to make out, I have to find out what is the energy of the state.

So, the energy of this state will be obtained, first I want to know whether this is the Eigen state or not so, if I take H psi 1, I will get h bar omega into a dagger a into psi 1 plus 1/2 h bar omega into psi 1. So, now this is h bar omega into a dagger a into a dagger psi 0 plus 1/2 omega h bar omega psi 1. So, now what is this again; I can rewrite this as; a, a dagger is 1 plus a dagger a.

So, now a dagger a acting on psi 0 is 0, so this is one contribute, so what will contribute will be 1, so it is h bar omega a dagger acting on, so this is effectively 1, so this a dagger behaves like 1 against acting on psi 0, so then a dagger acting on psi 0 is against psi 1 and this is 1/2 h bar omega psi 1 which is 3/2 h bar omega psi 1. So, like that you can generate all the states, so the nth state is going to be a dagger raised to n times psi 0.

But then you will have to normalize this, this will be up to a constant, so you can convince yourself that the correct normalization is this okay. So, notice that the; this is an abstract notation but you can also if you want the Schrodinger picture, you can express Y in terms of P and P as minus ih bar d/dx and you can go ahead and solve for this differential equations.

So, you see X is going to become m omega over 2h bar into small x and capital Y is going to become P into 1 over square root of 2m h bar omega okay. So, how would you write a? So, a is going to be this operator, so the operator a, that I am talking about so, let me write that operator a as; so a as Y minus iX.

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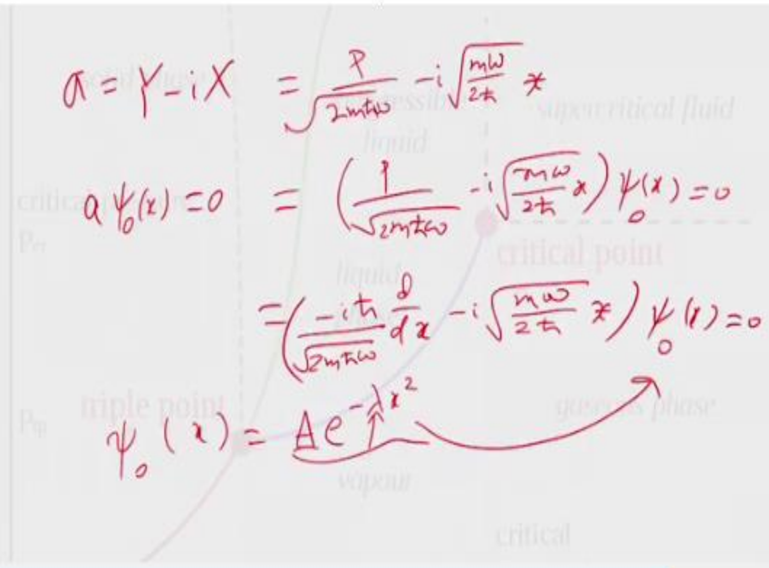

So, a is Y minus iX and so and what is Y; Y is P over square root of 2m h bar omega, so it is a P over square root of 2m h bar omega and minus i and small x was square root of m omega over 2h bar X, the square root of m omega over 2h bar times X. So, now the ground state is basically, the one which makes this 0, so now you can think of this as P times square root of 2m, m omega over 2h bar X acting on psi 0 to be 0.

Now, what is P? P is nothing but minus ih bar d by dx over; okay, so now you can go ahead and solve this and you will be able to write down what is psi 0, so psi 0 will be the solution to this equation which will be some constant into minus e raise to some other constant into x squared, okay, so you can figure out what that is by assuming it is this, at least you can find the lambda, this is the normalization which you cannot find.

But substitute that and you will be able to find lambda okay, so this is basically how you go about finding the answer to the ground state and the eigenvalues, so the Eigen value of this so that was the ground state Eigen function, so the eigenvalues of this nth state will be; it is going to be this is going to be an integer, so the eigenvalue will be h bar omega into n plus 1/2 because a dagger a is an integer, okay so, it is as simple as that.

So, this is how you do creation and annihilation operator approach to a harmonic oscillator okay. So, in the next topic I will try to generalize this and imagine a sequence of coupled one D harmonic oscillators, so that will be the next topic.

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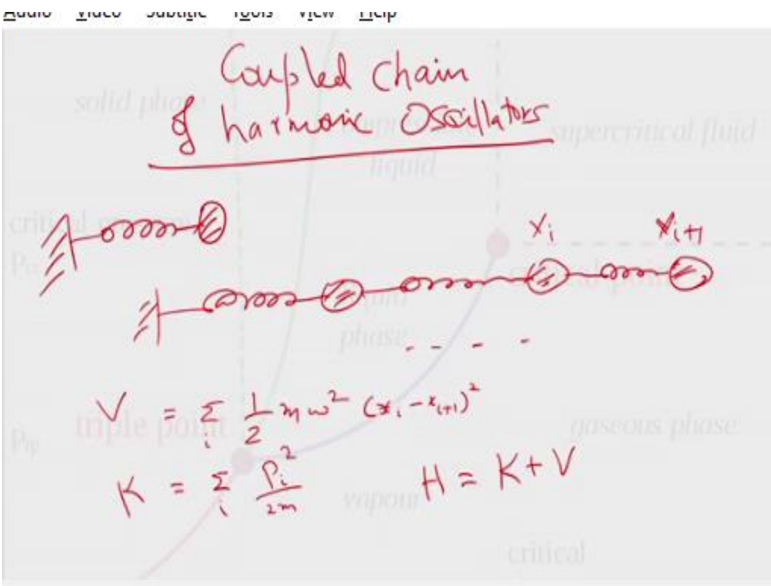

That is coupled chain of harmonic oscillators, so what I have in mind there is so, in the earlier example it was something like this, a mass tied to a spring okay, so that was a 1D harmonic oscillator but now what I have in mind is something like this, you have a mass tied to a spring which is again tied to another mass which is again tied to a spring, which is again tied to a mass and so on and so forth.

So, this is the situation that I have in mind and so I am going to imagine that this is the Xi position of the ith mass and this will be the position of Xi plus 1, so I am going to assume that Xi is the you know is in some sense the deviation from equilibrium okay, so deviation from the unstretched value of the spring. So, as a result you see the potential energy is going to be something like 1/2 m omega squared, so if all the springs are the same, so it is going to something look like this Xi, Xi plus 1 sorry, so the difference between these.

So, it is going to be the difference between these 2, so the distance square, so that is the that is how much potential energy is stored and you sum over all these, so this is the potential energy that is stored, the kinetic energy that is stored is clearly sigma i Pi squared over 2m. So, now the goal is to see if I can find the eigenvalues of the kinetic plus potential energy of this problem but using the creation and annihilation operator method, Thank you.