Introduction to Statistical Mechanics
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Lecture - 24
RG Method Ising Model

Okay so today I am going to see if I can discuss a new approach for understanding the properties of the 1-D Ising model.

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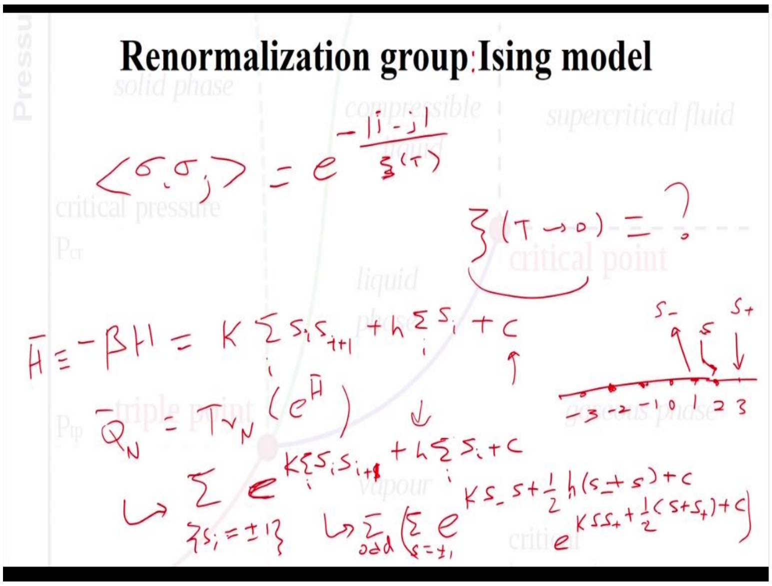

So, unlike the transfer matrix method which I had discussed earlier which is only applicable to largely to study 1-D Ising model. The method I am going to discuss now is generalizable to many systems, of course with some difficulty. So, this method is known as the renormalization group method and it actually is an old method basically invented to study second-order phase transitions.

So, the point is that in this method, the idea is that you study what is called the critical point. So, that means in the critical point what will happen is that the correlation lengths are infinite and the correlation functions are all power loss. So, the system kind of looks similar at all length scales. So, it stands to reason that in such situations that you can kind of adopt an approach which involves.

So, if you see if you recall that in the Ising model we typically had to calculate the partition function where you trace out all the spins. So, it stands to reason that in a situation where you are near the critical point where the system looks similar at all length scales, it stands to reason that you should be able to use a method which kind of exploits this property and so it enables you to map, you know the system at a certain length scale and then to a system at a different length scale.

So, if you want to take trace over say a bunch of variables. So, if you take, so you can kind of imagine all these bunch of variables, you separate them out into blocks so and you trace out each of those blocks basically you still be left with things to trace out, but then the point is that what the new effective partition function that you get will look similar to what you started off with.

So, this is basically the idea behind the renormalization group approach and it enables you to study this critical phenomena rather efficiently, but however the details can be complicated because it you know it does not necessarily, it is not very straightforward in general but in the case of the 1-D Ising model, it is particularly straightforward, but then that is to be expected because it is also exactly solvable using many other methods like transfer matrix methods.

So, this is not really, you know particularly useful application of this method, but then it is instructive as a tool for learning the technique itself okay. So, I am going to use this to solve the 1-D Ising model and explain to you basically solve means of interest or whatever what is basically the correlation functions and specifically I am interested in how the correlation the correlation length behaves with temperature.

So, that is typically what we are interested in so at low temperatures that is the, so you know that the critical point for 1-D Ising model is absolute zero. So, that is when I am interested in that situation. So, what is the correlation length as a function of temperature? So, remember how I define the correlation length as this. So, this is defined like this. So, this was the definition of the correlation length as a function of temperature.

So, my goal is to find out this thing. So, what is this? So, one way is to solve exactly using transfer matrix method and you will get this formula exactly but then the renormalization group method also enables you to write down a formula for this correlation length as a function of temperature at low temperatures in a nice way. I mean it is instructive to do this okay. So, let us see how we would do this.

So, the way we do this is first of all, see remember that in the, so let me start off with the model itself. So, imagine that the Hamiltonian that with a minus beta already build into it. I am going to write this as; I am going to write it like this, so using the usual periodic boundary condition okay. So, this is you know meant for I mean the original Hamiltonian there is no this independent or additive constant is not there.

But then in the renormalization group approach also known as the RG approach, this constant will have to be introduced because when you rescale the systems, these additional constants will appear on their own. So, we will have to deal with this okay. So, now let us try to write down the partition function which is basically, so I am going to define this H with a bar on top.

So, basically the partition function, the canonical partition function is basically that the trace over N of these things okay. So, that is what this is. So, that means it basically involves the sum over, so it involves taking you know a typical element such as this and you know well it actually involves you know summing over all the different spins. So, now the point is remember how we did this earlier, we use the transfer matrix method.

Now, instead of this, what I am going to do in this approach, in the RG approach is that I am going to, so imagine that this is the lattice of spins. So, you have say – 3, - 2, - 1, 0, then 1, then 2, 3 like that. So, what I am going to do is I am going to trace out say all the even spins. See in the Ising model, these 2 are coupled, - 1 is directly coupled to - 2 and - 1 is directly coupled to 0 but it is - 1 is not directly coupled to – 3.

So, suppose I trace out all the even things, so because you see the neighbors of an even spin are all the odd spins so and only those are involved in this Hamiltonian. So, I can safely and easily trace out all the even spins. So, it does not require any effort at all. So, in order to do that let me write this more instructively. So, imagine that say a given spin is say some s, so this is my s spin that I want to trace out.

So, I will call this, I will call this s + that is its right neighbor and this will be its left neighbor minus. So, in that sense, the quantity that I am trying to trace out is going to look like this s - s + 1/2 of h. So, it is going to look like this plus c times it is going to. So, that s spin you know couples to its left neighbor and also separately to its right neighbor, so that is what I am writing here. So, it looks like s + c okay.

So, I can kind of first say odd, odd will be outside. So, I can trace out all the even spins okay. So, I can decide to do that. So, tracing out the even spins is basically and then you just raise it to whichever power you want. So, that means however many even spins there are okay.

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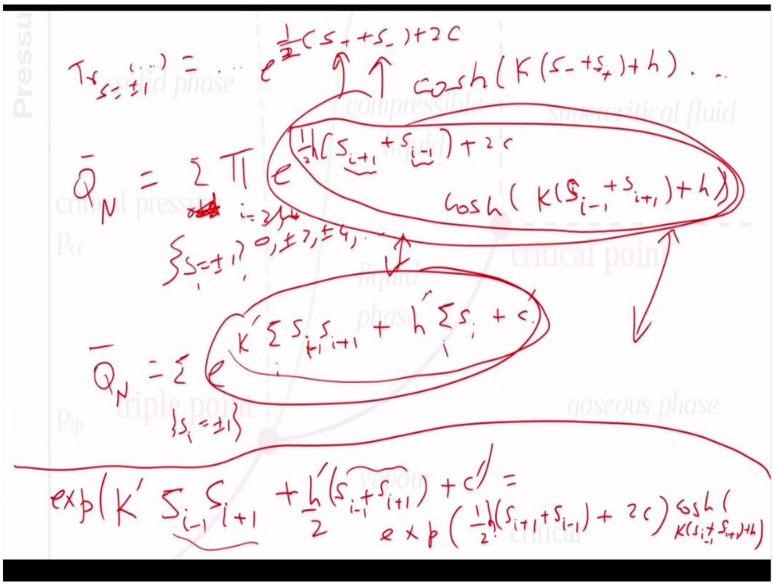

So, after tracing out over that particular spin, the result is going to be of this type, it is going to look like so it is. So, remember that this is perfectly there is linear in s, so it is very easy to trace it out. So, it is going to look like this, s + + s - + 2C then cosine hyperbolic times k into s - + s + + h so etcetera. So, that is what it looks like. So, the whole idea behind, so this is what it will look like.

And so then we will have to you know make sure that, so these are the odd guys see, so these are the odd spins, so because the even spins have been traced out. So, these will be the odd spins. So, the actual partition function will be actually over the trace of over all the odd spins of this type of thing okay, say i + 1 + i - 1 + 2C times cosine hyperbolic k into sorry s of i - 1 + s of i + 1 because I mean just imagine i is even in which case you have traced it out.

So, this is what remains. So, then I have to trace out over odds well okay in other words now that it is all odd, so this is say even 2 okay so this is 2, 4 okay sorry this is 0, + - 2, + - 4 like that so and then all the s's are + - 1. So, now you see this bears no resemblance to the actual partition function which is this I mean it appears that it bears no resemblance. Now, you will see that I can make it resemble it okay.

And okay so it appears that these two do not resemble each other at all, but now the idea is that I can define an effective, so I am going to postulate, I am going to demand that this B actually be the same as this. So, for some effective values of k and h and c. So, I am going to demand that there should be some effective values. So, there should be some effective value. So, notice that I cannot always satisfy this demand.

It so happens that in this 1-D Ising model because of its simplicity, I can kind of satisfy this demand. So, I am going to write this as i + i + 1, so it is, so that is my usual periodic boundary condition assumption. So, this is going to look like this. So, I am going to postulate that there should be some effective K's, effective h and effect to c such that it should look like this okay.

So, well exponential of this is going to be equal to the exponential of this times cosine hyperbolic k into si + si – 1 into si + 1 + h. So, this is what I am going to demand that this whole thing should look like this okay. So, I am going to demand that now. Well, this is cosine hyperbolic, so it is the sum of 2 exponentials. So, it is not just 1 exponential. So, as a result I am going to demand that these 2 should look the same or they should be actually equal to each other.

So, this sort of question is what does that mean practically? So, you see, you might be wondering how can this be achieved because after all there appears to be too many. So, the idea is that, you see now when you only have the odd sides remaining, so effectively now that you have traced out all the even sides, so basically only odd sides remain and they become effectively your nearest neighbor.

So, actually the effective nearest neighbor is now that the even sides are not there at all. So, actually your nearest neighbors will become the odd sides only. So, I should be doing this okay. So, I should actually be doing that. So, it is i – 1, so now it makes sense because otherwise it does not make sense. So, you might be wondering why I did this because after all this is not my original Q.

So, the whole idea is that you see now the even spins have all disappeared because I have traced them out. So, now the partition function involves only the spins on the odd sides, but then I wanted to look, I want the partition functions to look as if now it is original Ising model with the nearest neighbor interactions but now the nearest neighbor is between the nearest odd spins.

So, I am going to make it look like i – 1 and i + 1 because now after tracing out all the even spins, the nearest neighbors are actually only the odd spins. So, now it makes sense. So, I am going to force this to be the case, but now you might be wondering how is this possible because i - 1 has 2 independent possibilities + 1, - 1 and i + 1 has 2 independent possibilities + 1, - 1. So, put together there are 4 possibilities.

But then there are only 3 unknowns, which is k dash, h dash and c dash but I will get 4 equations, so it looks like it is over determined, but then you know it is not because they appear in this peculiar combination. So, it appears as either a sum or a product. So, it is really only 3 possibilities effectively. So, you can just think about it and effectively it is not 4 possibilities but only 3 possibilities.

So, anyway we can list them all and even if you list say 4 equations of which 2 will be identical, so you will only get 3 independent equations. So, basically how do you list all the equations?

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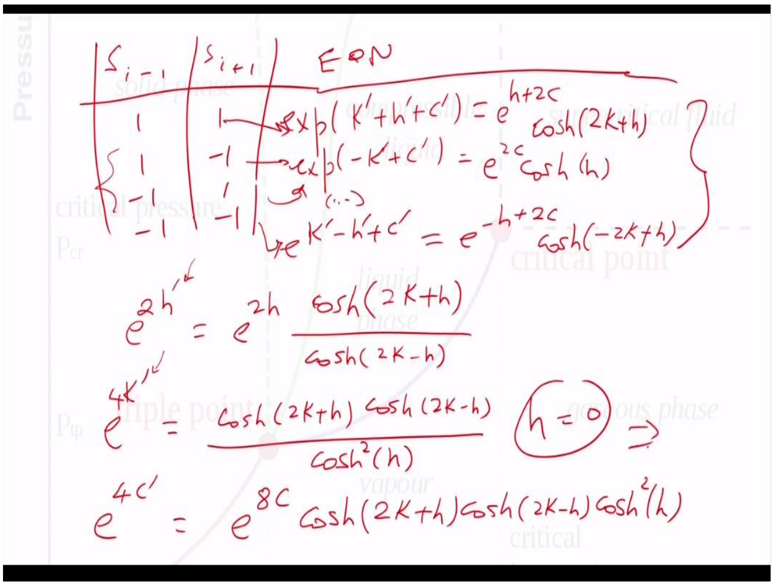

You either say Si – 1 and Si + 1, so you create a table like this and you call this 1 1 - 1 -1 1 - 1 – 1. So, if that is the case, you see if it is 1 1, so if it is 1 1 what does this look like? So, this looks like so that equation looks like this. So, exponential I missed an h there, so that was that h okay. So, it is going to look like this, so this is going to look like e raise to h + 2C times cosine hyperbolic 2k + h okay.

So, that is what it looks like because from here you can see that it is 2c + h times 2k + h, yeah so that corresponds to this. So, corresponding to this, so if it is 1 and – 1, it is exponential –k prime + c prime = e raise to 2c cosine hyperbolic h okay. So, because if they are opposite this cancels out and you know if it is this, you get back the same equation. So, that is the reason why you do not have 4 independent equations, you only have 3 but these 2 are the same equations.

So, the last one is going to give you something else and that is okay. So, now you can go ahead and solve these 4 equations and get the prime quantities explicitly in terms of the unprimed quantities and when you do that you get this result which is e raise to 2h dash = e raise to 2h times cosine hyperbolic 2k + h divided by cosine hyperbolic 2k – h. So, that is the equation for h dash in terms of h and the rest of it.

And then e raise to of k dash, so 4k dash will be in terms of the unprimed quantities which is cosine hyperbolic 2k + h times cosine hyperbolic 2k – h divided by cosine squared h. So, the last equation is 4c dash equals you get e raise to 8c times cosine hyperbolic 2k + h cosine hyperbolic 2k – h times cosine hyperbolic squared h okay. So, we are successful now in expressing the prime quantities in terms of the unprimed quantity.

So, just ask yourself what is the implication of this, see what this means is that look I have this Ising spin chain that it means a whole bunch of spins say on a circle. So, I am taking all the even spins and I am tracing it out. So, when I do that, I end up with a new Ising spin chain which has half the number of spins or now only at the odd sides okay. So, remarkably the spin chains that remains is also another Ising spin chain again with nearest neighbor interactions but now the nearest neighbor means because with whatever remains.

So, that means only the odd spins remain, so the spin number 1 now does not communicate with spin number 2 because that has been traced out. So, it now directly communicates with 3 with an effective coupling constant which is now called k dash instead of k. So, now that it so happens that this correspondence need not have been exact. That means you need not have been successful in concluding that the effective description when you trace out all the even spins would be identical apart from the coupling constants to the original description.

It need not have been that way, but it so happens that in the 1-D Ising model, it is that way but then for example if you apply to something more interesting like the 2-D Ising model, this will not be the case and you will get a whole bunch of other terms which you will have to learn how to deal with and that is something which is beyond the scope of these lectures. So, which I hope you will learn from some other more advanced course.

So, but then this kind of gives you a flavor for how this method works in this simple context okay. So, alright so now that we have been successful in you know tracing out all the even spins and concluding that the effective description is identical to the original description albeit with a new bunch of constants and new magnetic field and new coupling constant. So, now we have to decide how we are going to use this to draw our conclusions about the systems we are studying okay.

So, now you can, you see what we have done is now traced out all the even spins. Now, the effective description is only in terms of the odd spins. Now, you can there is nothing stopping you from repeating this procedure over and over again. So, the point is that you will, so the question that you have to ask yourself is that is there a situation am I going to reach a situation where the coupling constants you know will not change at all?

So, suppose you reach a situation where the coupling constant does not change, so if you rescale the system or trace out again, if the coupling constants do not change, so that means you have reached a critical point okay. So, the goal is to actually find that situation when you have reached that critical point okay. So, for example let us take the situation when there is no magnetic field okay. So, when there is no magnetic field, this equation can be written in this fashion.

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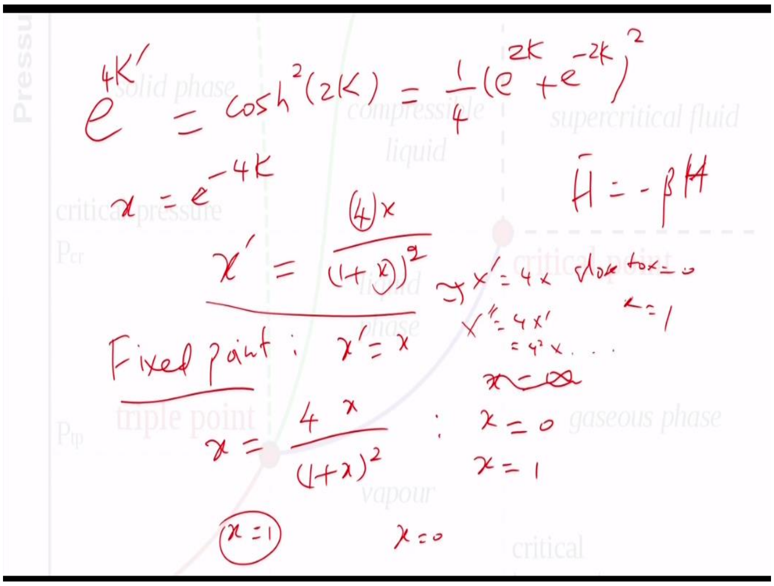

So, you can write this as when there is no magnetic field, the effective coupling between the spins after you trace out say the even spins. So, the effective coupling will become cosine hyperbolic squared times 2k. So, this is going to be basically 1/4th times e raise to 2k + e raise to – 2k times whole squared. So, now define x as e raise to - 4k okay. So, this will define this as e raise to - 4k.

In this case, the x value which is the exponential of the coupling between the spins is going to change upon rescaling in this fashion okay. So, now what I want is that, I want a fixed point. So, fixed point is a situation where after rescaling this does not change at all. So, it becomes the same. So, the fixed point equation is basically this okay. So, this has 2 solutions, x equals infinity okay or x equals okay sorry not x equals infinity, x equals 0 and x equals 1 okay.

So, these are the 2 solutions of this equation. So, these are the fixed points okay. So, x = 1 is called the trivial fixed point because that corresponds to the high temperature limit. So, x = 1 basically means remember that case has a temperature see H bar was - beta H. So, there was a temperature in the denominator implied in k, x equals 1 means k is 0. K is 0 basically means beta is 0, beta is 0 means high temperature.

So, this is the high temperature fixed point which is uninteresting. So, what is interesting is x = 0 fixed point. So, that is the low temperature fixed point. So, at the low temperature fixed point, so what we will have to do is we have to ask ourselves, so if I am near the low temperature fixed point okay and I expand it out, so how does this behave?

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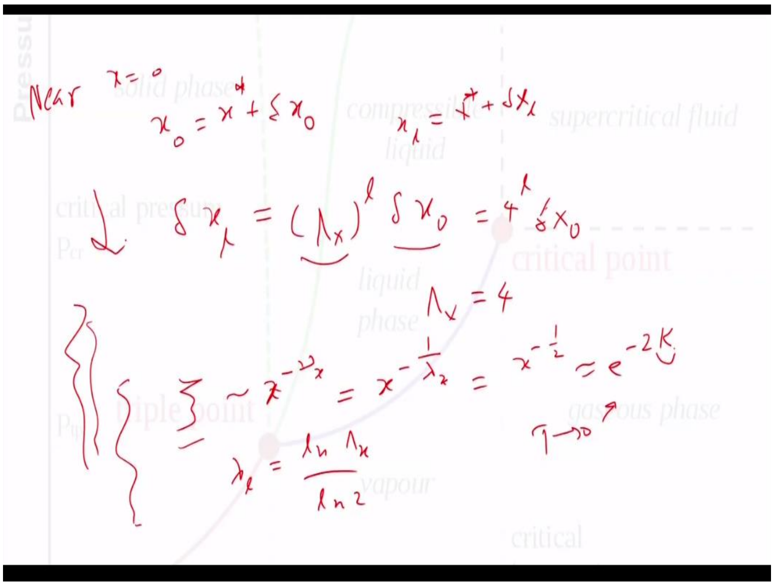

So, that means near x = 0 how does x behave? So, I am going to expand it out close to x equals 0 and you will see that it actually behaves in this fashion. So, I will write x0 is my fixed point x star, in this case it is 0, this is delta x0 okay. So, if I use the earlier equation, so I will be able to conclude that delta x will be equal to okay and so this is if you perform this once okay.

So, this is if you perform that rescaling once, it will look like this. So, now you can keep doing this rescaling say l times. So, if you do it l times, so you will get a fixed point which will be deviated from this like that okay. So, this will look like something raise to l because you will keep getting this factor here. So, when l is 0, you get back this and then you will start getting some factors in this fashion.

So, this lambda x will happen to be 4 in this case, so because of the nature of this equation. So, this is the 4 that is there, so that you will get that 4 there okay. So, you can work this out yourself. So, if you keep repeating this over and over again, so because see remember that x star is very small. So, x dash is basically 4x approximately close to close to x equals 0. So, that means that is non-trivial fixed point.

So, if you keep iterating this, this will get 4x dash which is 4 squared x and so on and so forth. So, you will get 4 raise to l x. So, this is basically 4 raise to l x delta original one okay, so that is what that is okay. So, the idea is that the correlation length okay, so the correlation length is basically going to be, so the correlation length is going to be, it is going to be given by this formula okay.

So, it is going to be this where x = -1 by lambda x and lambda x is going to be equal to ln lambda x by ln 2. So, I will derive these results, maybe I will you know walk you through this in some of the tutorials and so in any case this is an advanced topic which I did not really want to cover in great detail. I am just pointing it out that finally the basic idea of renormalization is that you will be able to express the effective, I mean the effective description after rescaling will look similar to what it was earlier.

And you use that idea to extract information about the temperature dependence of the correlation length and this becomes basically x raise to - 1/2 and remember what x was, x was e raise to - 4k, so this becomes e raise to - 2k. So, that is your correlation length in terms of certain temperature is implied in this. So, there was a temperature below it, so this as T tends to 0 that is your correlation length.

And you will see by taking the T tends to 0 limit of the solution that you obtain using transfer matrix, it is identical to this result okay. So, we are successful in reproducing, well maybe this part was not fully convincing. So, I will probably you know explain this to you and walk you through this in all the tutorials but bottom line is that this renormalization group method can, it is not convincing here because we have an exact solution in terms of transfer matrix with this which is much clearer.

But in situations where that method is not available, this you know this rescaling method can be quite powerful in extracting the correlation lengths as a function of temperature and near the critical point. So, I hope you understood a little bit at least. So, I will now stop here and move on to some other topics that I had promised that I will mention and such as second quantization and so on.

So, even that I will not just like here I did not do a thorough job of explaining this to you, so even there I am not going to be able to do a thorough job of explaining everything, so I will, so for the most part this course is going to end you know before this particular lecture. So, all examples that I worked out and so these are you know sort of optional advanced topics for you to think about and maybe I am just making you aware of the existence of these topics and also their importance, so that you will go ahead and learn them on your own okay.

So, if time permits later on, I will be able to maybe fill in some of the gaps such as this okay. So, hope you understood at least some of it and I will try to finish up some of the advanced topics that I had promised a little later so in the next few lectures okay. Thank you.