Introduction to Statistical Mechanics
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Lecture - 12
Thermodynamics of Black Holes

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The Poynting vector is $\boldsymbol{S} = \frac{c}{4\pi}\, \boldsymbol{E} \times \boldsymbol{H} = \frac{c}{4\pi}\, \boldsymbol{E} \times \left(\hat{k} \times \boldsymbol{E}\right) = \frac{c}{4\pi}\, \hat{k}\, \boldsymbol{E}^2$

The energy (minus the zero point energy) density is $u = \frac{1}{8\pi}\,(\boldsymbol{E}^2 + \boldsymbol{B}^2) = \frac{1}{4\pi}\, \boldsymbol{E}^2$ (in the time averaged sense).

Hence the Poynting vector for a given **k** is,

$$\boldsymbol{S}_k = c\,\hat{k}\; u = \frac{2}{V} \frac{c\,\hat{k}\,\hbar\,\omega_k}{e^{\frac{\hbar\omega_k}{T}} - 1}$$

The intensity due to all wavelengths moving in all directions is,

$$I = \sum_k \hat{k}\,.\boldsymbol{S}_k = \frac{2}{V} \sum_k \frac{c\,\hbar\,\omega_k}{e^{\frac{\hbar\omega_k}{T}} - 1} = \frac{2}{V} \sum_k \frac{c\,\hbar\,\omega_k}{e^{\frac{\hbar\omega_k}{T}} - 1} = \frac{2}{(2\pi)^3} \int_0^\infty \frac{c\,\hbar\,\omega_k}{e^{\frac{\hbar\omega_k}{T}} - 1}\, 4\,\pi\, k^2\, dk$$

But $\omega_k = 2\,\pi\,\nu = c\,k$. Hence,

$$I = \frac{8\pi}{c^2} \int_0^\infty \frac{h\,\nu}{e^{\frac{h\nu}{T}} - 1}\, \nu^2\, d\nu$$

The intensity in the frequency interval ν and $\nu + d\nu$ is,

$$I(\nu)d\nu = \frac{8\,\pi\,h\,\nu^3}{c^2} \frac{d\nu}{e^{\frac{h\,\nu}{T}} - 1}$$

THIS IS THE FAMOUS PLANCK'S BLACKBODY DISTRIBUTION!

Okay, so in the last class, we had discussed the thermodynamics of radiation. So we found several things where we were able to derive the black body spectrum, which was first guessed by Planck by looking at the experimental data. So, we derived this using a combination of statistical mechanics and Einstein's quantum hypothesis of radiation.

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But if you are interested in intensity hitting a flat surface we have to evaluate,

$$\frac{P}{A} = \sum_k \hat{z}\,.\boldsymbol{S}_k = \sum_{\hat{z}.\hat{k}>0} \frac{2}{V} \frac{c\hat{z}.\hat{k}\,\hbar\,\omega_k}{e^{\frac{\hbar\omega_k}{T}} - 1} = \frac{2}{(2\pi)^3} \int_0^\infty k^2\, dk \frac{c\,\hbar\,c\,k}{e^{\frac{\hbar c k}{T}} - 1} \int_{\hat{z}.\hat{k}>0} d\Omega \cos(\theta)$$

Or,

$$\frac{P}{A} = \int_0^\infty \frac{4\,\pi\,h\,\nu^3}{c^2} \frac{d\nu}{e^{\frac{h\,\nu}{T}} - 1} = \frac{2\pi h}{c^2} \left(\frac{T}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1}\, dx = \sigma\, T^4$$

where σ is called the Stefan Boltzmann constant.

$$\sigma = \frac{2\pi^5}{15\, c^2\, h^3}$$ (with temperature being measured in energy units)

So, the other thing we did was, we derived what is called a Stefan-Boltzmann law, which basically tells you the power that is radiated by black body as a function of temperature. So, the total power over all the frequencies that is radiated by the black body as a function of temperature and that is proportional to the fourth power of the absolute temperature of the black body.

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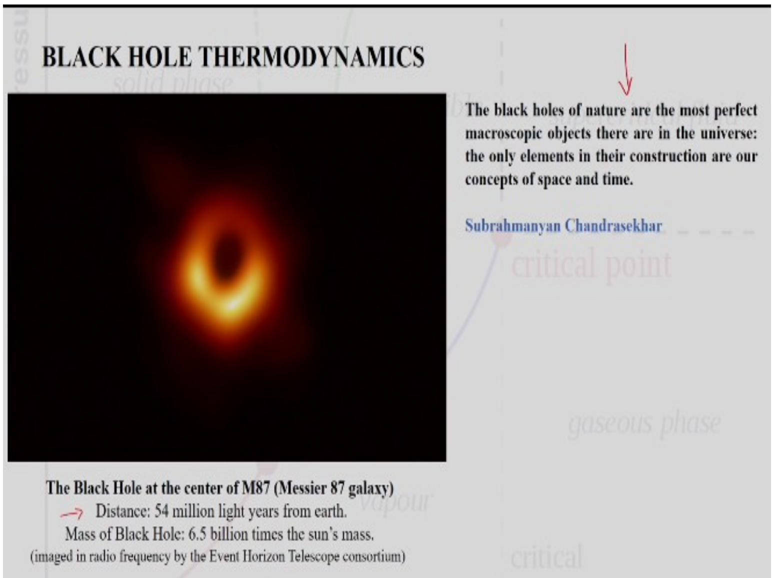

So now, we are going to use these ideas, that is the thermodynamics of radiation is going to be used in the next topic that I am going to discuss, which is a very interesting subject of the thermodynamics of black holes. So now, this is a very hot topic nowadays, because recently, the black hole, which is at the center of the galaxy M87, so it is the 87th galaxy in the Messier catalog, and it is at a distance of 54 million light years from earth and this is one of the most massive black holes ever seen in the universe.

In fact, it is 6.5 billion times the sun's mass and compare that with the black hole at the center of our own galaxy, which is the Milky Way, and that is only a few million times the sun's mass. So this is actually 1000 times heavier, so it is 6.5 billion times the sun's mass. So as you very well know, a black hole is basically the end result of the stars, just like people, stars are also born, they grow older, and then finally they die. So depending upon how heavy they were when they were living, so when they die, their final resting state depends upon how heavy they were before they died.

So if they were sufficiently light, they end up becoming what are called white dwarfs, but then if they were heavier than the Chandrasekhar limit to begin with, then they collapse further and they typically become what is called a neutron star, so where the matter is very dense, but it is mostly made of neutrons, you know. There is another limit that comes into play for neutron stars and if the star is even heavier than that limit, then gravity takes over and the star collapses and becomes a black hole.

So the matter just becomes a point, so that means it compresses itself to a point and the space time around it becomes extremely curved and generally, you will have to use general relativity to describe the space time around it. So in fact, as Chandrasekhar was one of the main persons who thought of this idea that, you know, stars may not be stable after all and he made this very beautiful remark that, you know, black holes of nature are the most perfect macroscopic objects there are in the universe.

The only elements in their construction are our concepts of space and time. So you see, it is such a beautiful idea that Chandrasekhar expressed and you can see why people are excited about this, and till now, it was not possible to study black holes properly, first of all because they are far away from us and secondly because by definition, they are black, so they do not emit anything, the only way you can study them is by looking at the effect of these black holes on nearby objects which are luminous like stars and other objects.

So people have been indirectly recognizing the existence of black hole, but it is only in the last couple of months that we have been able to actually see a black hole, actually image a black hole using radio astronomy and this was accomplished using what is called the Event Horizon telescope, which is an array of radio telescopes, located at various corners of the globe. So now, this is a beautiful picture of that Messier 87 galaxy’s black hole, which is at the center of the galaxy.

So you can see that there is this region where it is black here, the central hole there. So, you cannot really tell where the black hole starts because first of all, you know, black holes have something called an event horizon. So an event horizon is an imaginary surface, so if light exists below that surface, it cannot escape out of that surface. So, this is roughly of that order, the size of this is of that order of the event horizon.

So the bright gas here that you are seeing here, so the brightness here is actually the radiation that is emitted from the surroundings, especially the background that is from behind the black hole, that is in some sense lensed, so you see the intense gravity of the black hole causes space time to bend and it acts like a gravitational lens. So all the radiation gets collimated and concentrated in these lumps and then you are able to see the light that is coming out. So, it is all lensed light.

So first of all, it is not visible light, it is in the radio wavelength, so it has been converted to optical wavelengths for human beings to see, alright. So, this is all fine, but what I want to discuss in this class is something slightly different. I am not going to discuss radio astronomy, but I am going to discuss the thermodynamics of black holes because surprisingly, black holes not only have mass, they also have an entropy.

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According to Bekenstein and Hawking the entropy of a black hole is proportional to the area of its event horizon. This can be motivated by the following argument. Entropy is a measure of the number of microstates. We assert that these microstates are on the event horizon. We postulate that the smallest physical area that can store one bit of information is of the size of the square of Planck's length which is a quantity of dimension of length obtained by combining the fundamental constants.

$$\text{Planck length} \rightarrow l_P^2 = \frac{\hbar G}{c^3}$$

The number of states is 2^N is the number of microstates where N is the number of bits. The entropy is this number of bits. Detailed analysis by Hawking showed that the entropy is given by

$$S = \frac{A}{4 l_P^2}$$

where A is the area of the event horizon. Now the energy contained in a black hole is simply $U = M c^2$. Anything that has entropy and energy has a temperature.

So, in fact, this was first pointed out by Bekenstein and later on by Hawking that the entropy of a black hole is maybe thought of as being proportional to the area of the event horizon. So, you can motivate this idea. So you might be wondering how do you know that? So, how do you know that the entropy of a black hole is proportional to the area of the event horizon? So you can motivate this idea in the following way. So see, entropy after all is a measure of the number of microstates.

So, we may assert that these microstates are on the event horizon, because after all, see for a black hole, like Chandrasekhar said, the only thing going for a black hole is the nature of space and time itself. So the event horizon is the only structure that survives in the black hole. So the matter itself has shrunk to a point. So the only structure the extended object that survives is the event horizon. So it stands to reason then that these microstates live on the event horizon.

So what you can postulate is that the information, so that means that entropy is in some sense if you think of the information theoretic version of entropy is just a number of bits. Entropy is just a number of bits. So you just want to ask yourself, how many bits can I store on the event horizon? So the idea is that the area comes in discrete lumps because what happens in quantum gravity, that means if you look at, there is something called the Planck length, so which tells you the smallest length that is possible, you know, smallest length that has physical meaning.

So you can combine the fundamental constants of nature, which is Planck's constant, gravitational constant and the speed of light in this way and get a square of length. So l_P is called the Planck length. So this is called the Planck length, and the square of the Planck length is related to the fundamental constants. So now you can think of this as some kind of smallest area that is possible to exist in nature.

So if that is the case, then you can accommodate one bit on this area, so is the number of bits, so it is area divided by the area of this smallest piece of the event horizon, which is l_P^2 . So this factor of 4 cannot be derived by this argument. So this is $(1/4)A\,/l_P^2$. So other than that, you can kind of guess that this is what it should be. So now, we have vaguely convinced ourselves that black holes should have entropy, but it is more obvious, of course, that black holes have mass and therefore energy.

So because energy is mass according to special relativity, energy is Mc^2 . Now, anything that has entropy and energy, also has temperature. So that is what we have been telling ourselves till now in this course. So now that we have established that a black hole has entropy, by virtue of the fact that it has an event horizon, which has an area, and the entropy is proportional to that area and we also know that the black hole has mass and therefore it has energy, which is Mc^2 .

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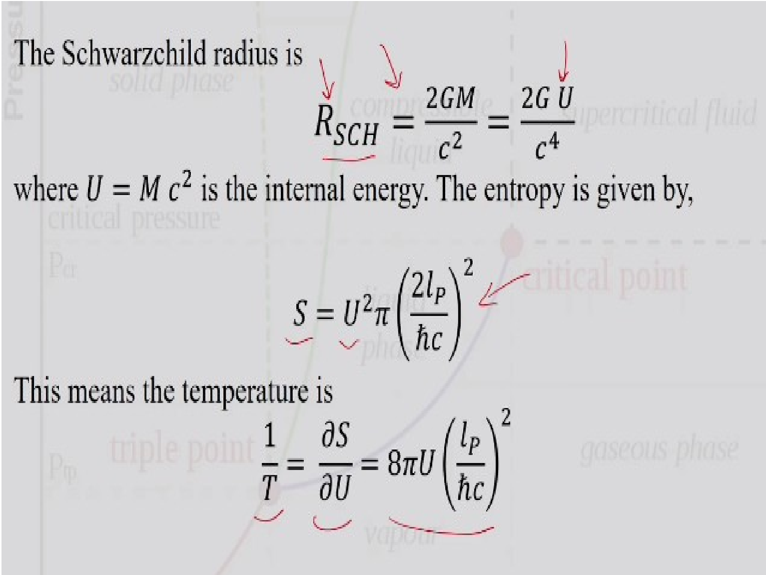

So now, we have a situation where the black hole has energy and it has entropy. So if it has energy and entropy both, then you know that such an object has temperature. So write down this, this constant is called Schwarzchild radius. So the reason why it is called that is because this is the radius of the event horizon basically. So below this, light cannot escape. So the entropy of the black hole is now expressible in terms of the energy because now you can rewrite.

So you can rewrite the Schwarzchild radius in terms energy and 4 πr^2 is the area of the event horizon, and you can therefore express entropy in terms of energy. Now once you know entropy in terms of energy, you can go ahead and differentiate this and you get an inverse temperature, so the absolute temperature, the thermodynamic temperature of a black hole can be derived as the slope of the entropy versus energy of the system and that comes out to be this expression. So, this is the temperature of a black hole, 1 by temperature of the black hole.

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Does entropy increase or decrease when two black holes collapse into one? What is the entropy change for the universe (in equivalent number of bits of information), when two solar mass black holes ($M_{\odot} \approx 2 \times 10^{30}\ kg$) coalesce?

When two black holes of mass M collapse into one, the entropy change is

$$\Delta S = ((2M)^2 - 2M^2)\,\pi\left(\frac{2\,c\,l_P}{\hbar}\right)^2 = 2M^2\,\pi\left(\frac{2\,c\,l_P}{\hbar}\right)^2 > 0$$

The number of microstates is $\Omega = 2^{n_B}$ where n_B is the number of bits. But the Number of microstates is also $\Omega = e^S$. Hence $n_B = \frac{S}{\log(2)}$. Hence the change in the number of bits is $\Delta n_B = \frac{\Delta S}{\log(2)} \sim 10^{77}$ bits are destroyed.

The thermal radiation near the event horizon produces particle hole pairs and one of them fall in the other escapes to infinity. The rate of this process is given by the rate at which energy is being emitted by the horizon (which is a black body type radiation). The total radiation emitted which is assumed to be converted to matter by pair production is,

$$c^2\frac{dM}{dt} = \frac{dE}{dt} = -\sigma\,T^4\,A; \quad A = \frac{16\,\pi\,G^2}{c^4}\,M^2\ ;T = \frac{\hbar\,c^3}{8\,\pi\,G\,M}$$

So, you can ask some interesting questions now. So suppose I have 2 black holes, say solar mass black holes. So the question is, of course, see you might be wondering how can it be a solar mass black hole because Chandrasekhar limit says that it has to be greater than 1.4 times the solar mass and even then you just do not get a white dwarf, you get something more dense than a white dwarf, which may not be even be a black hole. So of course, this is an order of magnitude calculation.

When I say solar mass black hole, I mean, maybe three times the solar mass, that is good enough to make it a black hole. So this is a rough back of the envelope order of magnitude calculation. So please do not take it too literally. So the question is suppose you have 2 solar mass black holes and they collide and merge into each other, so the question is what is the change in entropy? So, you expect entropy to be lost. In other words, you expect information to be lost, and so you can actually see that happening here because now you see the entropy is proportional to the square of the mass.

So, the final entropy is basically the 2M whole squared, which 2M is the mass of the combined black hole and M squared into 2 is the mass of the 2 separate black holes. Then if the difference is something positive, so these many bits have been destroyed. So if entropy is positive, that means there is an increase in disorder as it were and so the change in the number of bits, so you can express this in terms of the change in the number of bits because 2 raised to number of bits is e raised to entropy, which is the number of microstates.

So the number of microstates is 2 raised to number of bits, and the number of microstates is also e raised to S. So by combining these two, you can convince yourself that the change in the number of bits is entropy divided by log 2, which is of the order of 10 raised to 77 bits. So this is nice to know that if two black holes combine with each other, this many bits are destroyed. Now, what is more interesting is that, see by virtue of the fact that black holes have temperature, so we can imagine that there should be some kind of a black body radiation near the event horizon.

The reason is because what is going to happen is that see in elementary particle physics, there is something called pair production. So that means if energy of the vacuum is more than twice Mc squared, where say Mc squared is the rest energy of, say for example, the electron, so what is going to happen is that energy is going to disappear and reappear in the form of electron and a positron. So that is called a pair production.

So, what will happen is that you can have these kinds of virtual processes will become so you get a particle and an anti-particle and you may imagine roughly that one of them falls into the black hole and the other escapes to infinity. So that is the proposed mechanism for what is called Hawking’s radiation. So black holes lose mass through this process, so this is called Hawking radiation. So I would not go into too many details about how this exactly happens, but roughly this is what happens.

So now you know that the rate at which radiation is being emitted by a black body of temperature T is σT^4 . So this is power emitted per unit area. So if you multiply that by the area of the event horizon, then you get the rate at which energy is being lost by the black hole. Sorry, this is not capital T, it is small t, so this is time. So the rate at which energy is being lost is dE/dt , where $E = Mc^2$.

So the rate at which energy is being lost is, because it is losing energy there is a negative sign there, and it is $-\sigma T^4 A$ and we know that the area of the event horizon is nothing but it is proportional to a square of the mass of the black hole.

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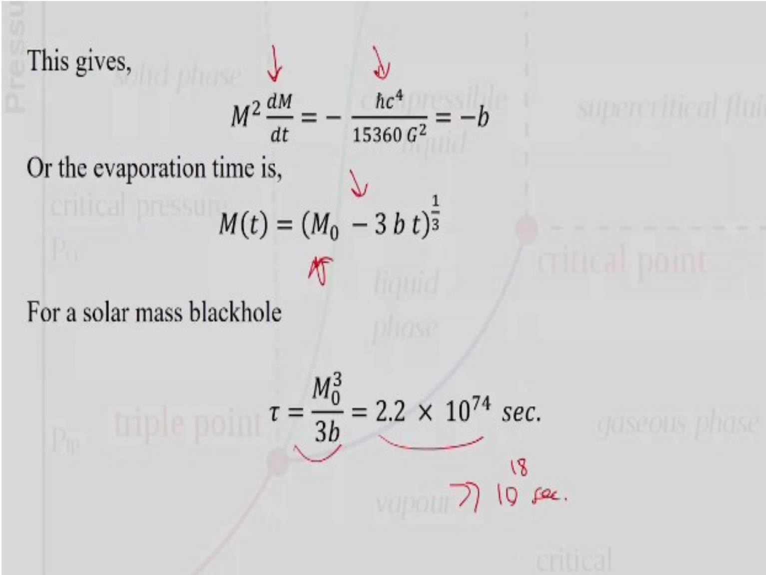

So because of this and we also know that the temperature of the black hole is inversely related to the mass, when you put them all together, you will be able to derive this equation how the mass kind of evaporates with time. So that means, the black hole actually evaporates slowly with time and the rate at which it evaporates is determined by this constant, and by solving this equation, you show that the time dependence of the mass of the black hole is given by this formula.

So if you start with a solar mass black hole, you can ask yourself, how long does it take before it all disappears? So the answer is basically this quantity and you can see that it is much larger than the age of the universe, which is of the order of 10^{18} seconds. So, black holes are going to live forever once they are created, so long as the astrophysical variety, which are mandated to be solar mass or above, but then it is possible to have a situation where you can create black holes, which are much tinier through processes, which are not related to astrophysical processes and those very tiny black holes actually evaporate very quickly.

So in fact, when the Large Hadron Collider was going to be built, there was a lot of excitement that these tiny black holes would be observed, but for some reason, they were not observed finally, but nevertheless, this is the story of the thermodynamics of black holes. So what we have learned in this lecture is that, firstly black holes are exciting because they have finally been imaged by a telescope on earth and it was a worldwide event, which was televised and streamed online and so that is the excitement about black holes.

Black holes have long known to have these basic features, namely, that it has an area which is basically the event horizon, it has a mass, and Hawking and Bekenstein said that entropy of the black hole is proportional to the area of the event horizon and anything which has mass, has same as energy. So anything which has energy and entropy has temperature, and anything that has temperature is going to behave like a black body and radiate. So the moment it starts radiating away its energy, its mass is going to slowly decay, and then finally, it is going to disappear.

So, what we have been able to calculate is, actually been able to calculate how long that takes and for astrophysical black holes, we are quite safe that it is going to last forever. So if once it is created, it is going to be around forever. So, that is the story of the thermodynamics of black holes, I hope you enjoyed this. So in the next class, we will discuss something more down to earth, which is basically the physics of the van der Waals fluid. The van der Waals fluid would be simple prototype of a non-ideal gas.

So I am going to revert back to the more classical topics of statistical mechanics and discuss van der Waals story. So, I hope you enjoyed this lecture. Hope to see you next time.