Introduction to Statistical Mechanics
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Lecture - 10
Chandrasekhar Limit

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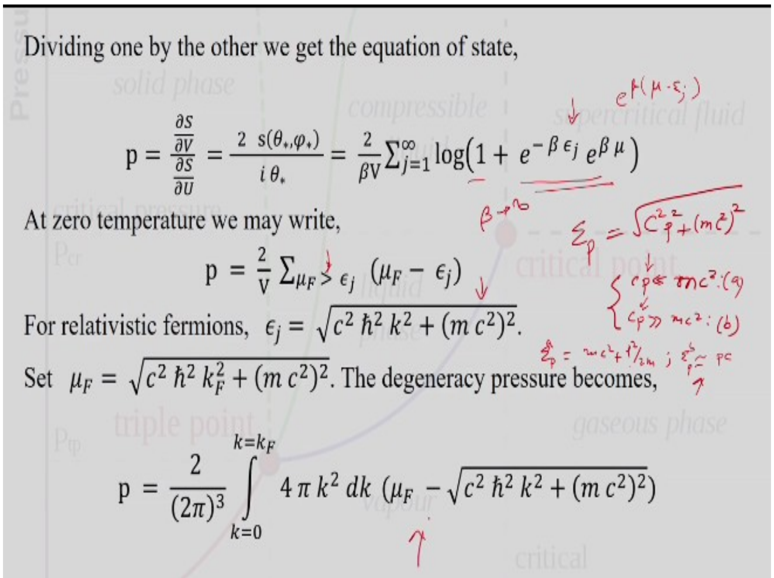

Okay, so, I had stopped somewhere here. So, I just want to refresh your memory. So, the point is that you know I was forced to re-derive the expression for the pressure of a degenerate Fermi gas because till now I had only studied or discussed non-relativistic quantum gases or classical gases regardless. So, in other words, the energy versus momentum dispersion was quadratic, so E = p^2/ 2m, but in the case of white dwarf, as I told you that we will have occasion to study both the limits.

So in other words, we will have occasion to study the conventional non-relativistic limit, which I am going to actually skip because that is also important for reasons that I'll mentioned later, but what is more interesting is the ultra-relativistic limit, where the energy is close to, so in other words, the energy versus momentum relation in general in relativity, as you know, is this. So, you have 2 different limits. So, when you have cp << mc^2, you get a certain limit, so you get this limit, and then you get cp >> mc^2.

So, these are the 2 limits. So, this is called a, this is called b. So, the energy in the a case is going to be mc^2 + p^2/ 2m. So, in other words, this is the non-relativistic limit, where the momentum is small compared to mc. So that is in a non-relativistic limited, the energy is mc ^{2} plus the non-relativistic correction, which is $p^2/2m$. However, in the ultra-relativistic limit where momentum is much greater than mc, the energy is going to be roughly cp.

So, actually, we should be studying both, but I am going to focus mainly on this. So, this is ultrarelativistic limit, but regardless you know, I have done it in general, you see I have taken this general result, and so if you recall that I had mentioned that this will be what we had derived earlier. So, this would be pressure and the point is that at 0 temperature, $\beta \rightarrow \infty$ and because $\beta \rightarrow \infty$, this is log of 1 + something. When $\beta \rightarrow \infty$ of this whole thing, this whole thing either goes to infinity or goes to 0.

If it goes to zero, this becomes log(1) and it will not contribute. So the only time it is going to contribute is when this thing goes to infinity rather than goes to 0. So, when does it go to infinity, it goes to infinity when $\mu - \varepsilon_j > 0$. So, in other words, that is this condition. So $\mu > \varepsilon_j$, and then only this one can be ignored in comparison with this, and when we take the log, the β cancels out and you get this expression.

So, now I am going to become ambitious and do the general case where I write ε_j in this fashion and then I can write down the formula for the degeneracy pressure.

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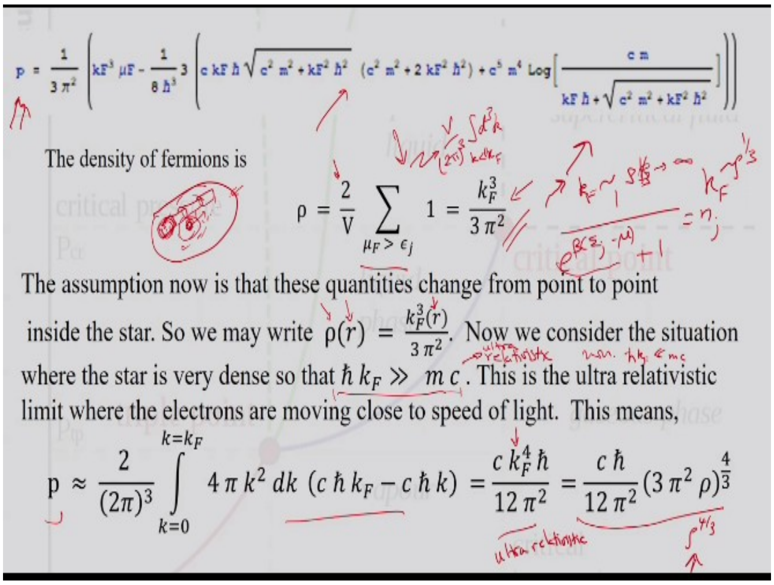

So it turns out that the general expression is of this nature, which I already mentioned, and you can go ahead and recast this. So notice that this is in terms of the Fermi momentum or the Fermi wave number k_F, the Fermi momentum will be $\hbar k_F$. So, however, it is more useful to rewrite everything in terms of the density of fermions and as you know, the density of fermions is just given by the Fermi distribution. In this case, the Fermi distribution is a step function. So this is important for you to appreciate why it is a step function.

So recall that this was my Fermi distribution and then recall that β is actually large, in case of 0 temperature, this β is large, so when β is large, you have this situation that if ε_j is greater than μ, this whole thing becomes very large, but it is in the denominator, so it is going to vanish, so the whole thing is going to vanish. So the only situation when it is not going to vanish is when β is large and $\mu > \varepsilon_j$, in which case this is exponentially suppressed and becomes close to 1.

So, $\mu > \varepsilon_j$ is the only situation where that n_j contributes and in those situations, n_j is actually close to 1 because β tends to infinity. So that is the situation that I have in mind. So now I am going to perform this summation as you very well know how to do this, because I told you how to do this already, it involves replacing the summation over ε_j by integration over the quantum numbers, in this case, the quantum numbers are the case.

So,I have in mind, you know particle in a box type of situation, where you have the $k = n \pi / L$, and then finally $L \rightarrow \infty$, so that the k's become continuous, and then you can integrate rather than sum over discrete case, and then when you integrate, you get this expression. So, I have proved this already. So please, if you forgotten, just go towards the beginning of this statistical mechanics lecture series, and then you will find it. So, I have already proved this to you.

So now of course, this integration is such that I am going to only integrate within this radius, which is k_F. So when I do that, I get this answer. So this is going to be the density of fermions. So, this is how density of fermions is related to the wave number. So this is very general. So, I want to impress upon you that this expression is very general, in the sense that it is valid for Fermi gas definitely, but it is valid for Fermi gas in three dimensions, but there is no restriction about what type of Fermi gas, if it is a non-relativistic or relativistic or something in between, it is always valid.

So, so long as it is a Fermi gas and this 2 is because of the spins, I have assumed that the fermions have spin half. So, up spin I have to count the states and for down spin also have to count the state. So I get a factor of 2 there. So, this is actually valid for spin half fermions, so that means that 2 spin projections up and down. So, this is valid for spin half fermions and in 3 dimensions okay and it does not matter whether it is relativistic, non-relativistic, or whatever.

So now, what Chandrasekhar assumed was that in the star, you see it is possible to kind of think of, see remember that the star is a, you know, it is a star, I mean it is a macroscopic object to put it mildly. So the point is that we are looking at the atomic description, that we are looking at the electronic degrees of freedom in a star. So obviously, if you are at some point r and you can of course, you know, demarcate a certain volume around that point and whatever volume you demarcate is going to be necessarily macroscopic, but however, it is going to be miniscule.

You can always have a situation where the volume that you mark out is incredibly tiny compared to the size of the star, but incredibly large compared to the electronic length scales which are involved. So in other words, like the thermal wavelength and that sort of thing. So in other words, you can suspect that there will be enormous number of electrons in this small volume. So as a result, what Chandrasekhar said was that it is possible to naïvely put an r dependence here so that you assume that this as a slowly varying function of r.

So, the density of electrons vary slowly as a function of r, so you can suspect that close to the center of the star, the density is huge and as you go farther and farther away, it falls, and then when you reach the boundary, you get a density which is 0 okay. So, like I was saying, so Chandrasekhar suggested that it is possible to and of course grain the region inside the star, so you kind of, divide it up into small pieces, then each piece is very small compared to the size of the star, but still has enormous number of electrons.

So as a result, you can kind of assume that the density of electrons is a slowly varying function of the distance from the center. So as a result, what happen is we can imagine that for a star like this, the density close to the center is going to be huge and it is going to fall off and become zero once you reach the boundary. So, this was, of course, a very general result and it would be nice if we could deal with this, but unfortunately, it is not possible to deal with this in the sense that, so let me tell you what I mean by deal with this.

So remember that this is the degeneracy pressure. So, this is the pressure that is being exerted by the degenerate Fermi gas as a result of Pauli exclusion principle. Now, this pressure is going to balance the inward pressure caused by the gravity that is trying to collapse the star. So as a result, when the two become equal, the star is going to reach an equilibrium. So that is what I mean by dealing with it. So that means, I want to equate this degeneracy pressure with the pressure of the incoming the falling star at a given point r.

So, then I will have to so that will give me implicitly, it will tell me what should be the distribution of density versus distance in the star and so on. So, it will tell me everything about the detailed distribution of matter inside the star. So that is very ambitious and it would be nice if we could do this in general, but it is not possible. So, what we have to do is we have to make further approξmations. So in fact, Chandrasekhar did both the calculations, namely, he studied the non-relativistic limit first, so this is relativistic limit.

So ultrarelativistic limit, see the non-relativistic limit would be the opposite, which is $\hbar$kF << mc. So in this case, E is close to $p^2/2m$ plus a constant and you can go ahead and do that and you get a result, I am going to tell you what results we will get for that, I will just, I will not derive it, I will just mention it, but I will derive, what I am going to derive is this ultrarelativistic limit, which leads to the famous Chandrasekhar's mass limit of the white dwarf. So let us get on with it.

So let us look at the Chandrasekhar's calculation where he did the second part where $\hbar$kF which is the Fermi momentum is much greater than mc. So in this limit, the fermions are moving close to speed of light. So, instead of dealing with this complicated equation, we can just go ahead and approξmate this by $\hbar$kc and then just do this integral. So, that is what I have done here. So, when I do this integration, I get this result.

So it is going to be k_F to the fourth power and recall that k_F is nothing but, so from here you can see that k_F is proportional to density of fermions raised to one-third. So, k_F is going to be density of fermions raised to one-third. So because pressure is k_F to the power 4, so for an ultrarelativistic gas, please recall that, remember that this is ultra-relativistic only. So for an ultrarelativistic Fermi guess, the pressure is proportional to the fourth power of the wave number and the wave number itself is always given in 3 dimensions by ρ, density raised to one-third.

So putting those together, you get expression, which is density raised to four-thirds. So, this is an example of what is known as a polytrope. So, the pressure that is exerted by a degenerate Fermi gas or in general, any type of gas, so if you express it in terms of the density, so that equation is referred to as a polytrope. So in this case, well, specifically if it is a power law, so that means if it is a power law, it is called a polytrope. So, in the extreme limits, in extreme non-relativistic limit and the extreme relativistic limit, in both cases, it is going to be a polytrope.

So, I would not do the non-relativistic calculation. So, you will have to believe the result that I tell you later, maybe if we have time, we will leave it for the exercises later on tutorials and so on okay. So, now remember that what I am supposed to do is that this is the pressure exerted by the gas, which is basically due to Pauli’s exclusion principle. So in other words, this is the pressure the electrons will exert if you try to compress those electrons. So, the origin of this pressure is basically Pauli’s exclusion principle and this is valid when the electrons are moving close to speed of light.

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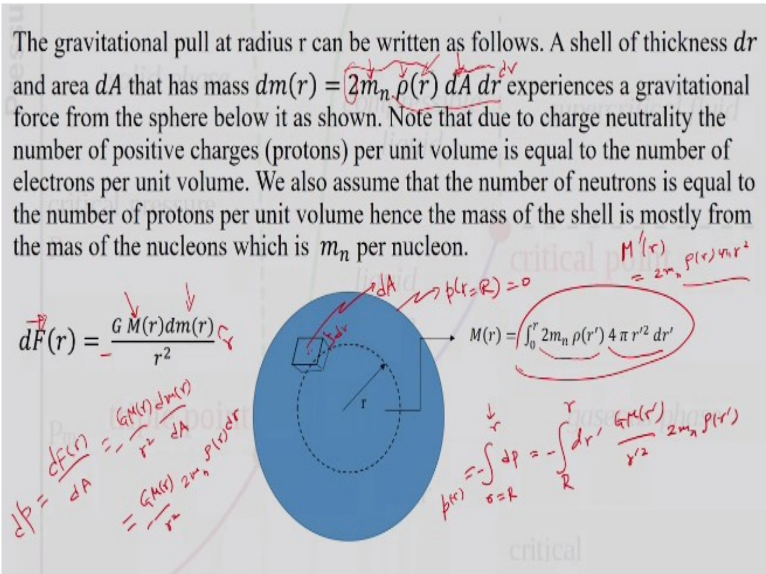

So, now this pressure has to balance the pressure, what is going to push it, so what is going to push it is basically the gravitational force. So what we have to do is we have to calculate the pressure caused by gravitational force and we have to equate that with the pressure that we just calculated, which is the outward pressure caused by the Pauli exclusion principle and degenerate Fermi gas where the electrons are trying to avoid being pushed together too closely. So how do we do that? It is very easy to do that.

So imagine, you have a small area like this. So this is my dA and this is my dr. So I have a volume dA into dr and this volume is experiencing a force. So, this volume contains this much mass. So okay I will have to say what is all this? So, this is clearly the volume dV okay. So, this is how much volume there is in this square, I mean this kind of cubicle or whatever box that I have written. So, now the point is that the rest of it is basically the mass density of the star.

So, the mass density of the star times the small volume is basically the small mass that is contained in this box. So what is the mass density of the star? See, recall that ρ is basically the electron density okay, so this is the number of electrons per unit volume. So remember that for every negative charge, there is a positive charge, because overall, the star is electrically neutral. So in other words, there are as many protons as there are electrons, but then, so let us assume that we have a situation where the number of neutrons is equal to the number of protons.

So notice that so what I am implying therefore is that see I have to calculate or I have to find out what is the mass, not the number of electrons, I want to find how heavy this box is. So how heavy it is, so let us see what is all in the box. So in the box, there are this many electrons, there are ρdV number of electrons, but then electrons are very light, right. So electrons are very light, so they do not contribute to the mass, but then along with the electrons for every electron, there is a proton also because otherwise the system will not be electrically neutral.

There is a neutral atom sitting there after all, but electrons have been dislodged from the atoms, they are freely moving about, yes, but still there are as many protons as there are electrons. So in fact, and protons being enormously heavy compared to the electrons, they are actually going to contribute to the mass, more or less fully to the mass. So, we are going to ignore the mass of the electrons, we are only going to consider the mass of the nucleons.

So, nucleons means protons and neutrons, but one should not forget the neutrons because they do not contribute to any electrical forces, but they contribute to gravitational force definitely because they are as massive as protons. So, let us assume a situation where you have an atom, which has as many neutrons as there are protons. So that is kind of typical for a stable atom. So typically, number of protons matches the number of neutrons and matches, of course, the number of electrons, that is always true because of charge neutrality. So, let us assume that this is the situation.

So in that case, the mass of the box is actually the number of nucleons which is like two, so one proton, one neutron, and I have assumed that the mass of the proton and mass of the neutron are roughly the same as two times the mass of the nucleon times the number of nucleon per unit volume, which is equal to the number of electrons per unit volume times the volume. So that is the mass that is contained in this box. So, I hope that is clear.

So, that is the dm the mass that is contained in this box, and this box, the mass that is contained in this box experiences a gravitational force and of course you know Newton showed this 300 or more years ago, that the force on this is actually due to a force contained inside this region. So, it does not matter what is outside. So, if you have a spherical symmetry, then the gravitational force exerted on this mass is all because of the mass inside, so this M(r) is the mass that is inside this dotted line, and what is that mass?

So, it is basically volume integrated from 0 to r times the density, so that is what it is. So, it is dmM/r^2 , but then this is the dm there. So now, this is the force, but you know that force per unit area is pressure. So what I have to do, so I have to divide by dA, so dF/dA is my pressure. So, I am going to divide by dA. So, I am going to calculate dF(r)/dA, which is my pressure. So, I am going to actually end up getting this answer, so I am going to get this okay, so let me go and get that.

So when I get that, so I will get a pressure to be this result. So let me do that $GM(r)/r^2$ and so this is still dp by the way because there is a dr there. So, it is the pressure due to that really small dr. So, the pressure is $2m_n$ ρ r dr. So, I want the pressure at some point r. So, what I have to do is I have to integrate from a point where the pressure I know to be 0, so which is pressure at r equal R is 0. So, I am going to integrate from a place where I know the pressure to be, so r equal R to r, so that is going to be integral dr to r GM(r’) over.

So I have just replaced r by r’ because it is a dummy variable, because I want to reserve r for the point of interest. So this is the pressure at r. So, this is what it is. So, that is what I have written there, so I have integrated from.

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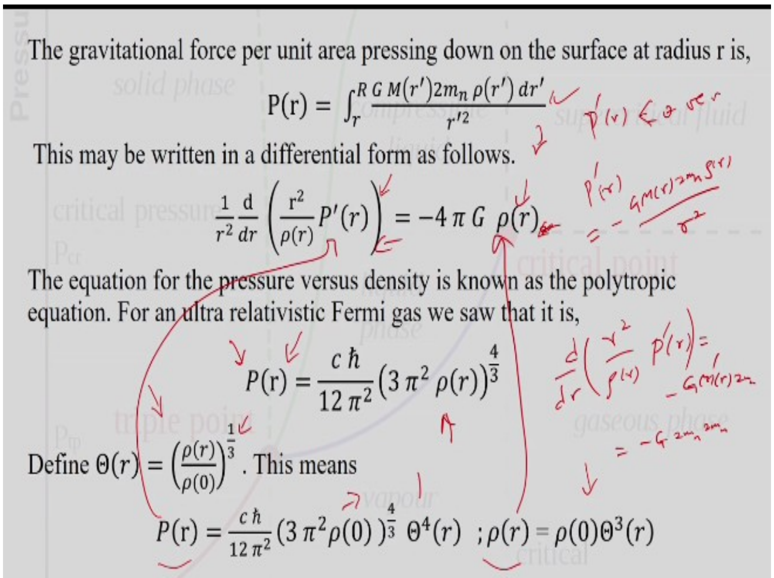

So the point is of let us fix the sign here. See, the point is that if I take the derivative with respect to r, what do I expect? So from here, you can see it is negative. So that means the pressure is actually decreasing as I go away from this, so I should have said a minus. So this force is attractive, so there should be a minus there, so that is the reason why I am not getting the right answer. So it is an attractive force, so pressure is like this, I mean the attractive means it is in minus r cap.

So this is if you put a vector there, it will be actually minus r cap, so r cap is out, minus r cap is in. So P dash r should be negative because pressure is decreasing as you go from the center through the surface. So now, I can rewrite this in this form. So I can first differentiate with respect to r, then I multiply by r squared divided by ρ and then again differentiate by r. So basically, look, it is a little bit of algebra. So what I can do here is that if I take P dash r, so P dash r is going to be minus GMr 2mn ρ r by r squared.

So now, I multiply by r squared and divide by ρ of r, I get this. So I get minus GMr into 2m. Now, keep in mind that M itself is nothing but this integration. So because of this, M’(r) is actually going to be $2m_n$ ρ of r into 4 π r^2 okay right. So if I just take the derivative of M of r, it is going to look like this. So, then I can take d / dr of this, so that is going to be M’(r). So, it is going to be minus G 2mn M dash r. So what is m M dash r? It is again to 2 mn. So it is going to be this to cut a long story short, okay.

So, I am going to skip the rest of the details, so you can figure it out yourself. So finally, I can rewrite this by appropriately differentiating however many times I want and I can always, so please convince yourself that this is what I got from my physics considerations by looking at this, the gravitational force exerted by this cube and all that. Now, this is from here to here is just some algebra. So you just make sure that you substitute this P here and show that it is an identity. So now, remember that for a polytrope, we just derived that the degeneracy pressure, so this is because of gravity.

So this is the pressure that gravity exerts and this is the pressure that is due to the Pauli's exclusion principle. So that is why I have called it by the same P, so even though this is due to different reasons. This pressure is due to gravity, this pressure is due to Pauli's exclusion principle, but I want the two to be equal because I want the equilibrium. So in equilibrium, these 2 pressures are the same. So now, I equate these two and so after this, it is a little bit technical, it is just a bunch of algebra.

So what you will have to bear with me because I find that the technical nuances are actually skipped in many of the books and because Chandrasekhar limit is such an exciting topic, you know, we all learn about it in our school days and we take great pride as being Indians to say that we know Chandrasekhar limit. So, we would be fooling ourselves if we did not make an effort to actually go through some of the steps that Chandrasekhar himself went through and recall that he was only 19 years old when he did these calculations.

It is amazing how he managed to do all this when he was only 19 and sailing a ship. Okay, so let me continue. So what I have to do is I am going to define dimensionless quantity, which depends on r. So ρ of zero is the density of electrons near the center of the star. So, I am going to define the ratio ρ of r by ρ zero, which is basically the ratio of which is the density of electrons at point r measured in units of the density at the origin okay, so and then there is a raised to one-third for reasons that are obvious because the polytrope always involves these type of ratios.

So, what I am going to do is I am going to use this correspondence and rewrite P in terms of this theta rather than in terms of the ρ. So, then I end up getting these 2 formula. So the P becomes related to the fourth power of theta and the ρ becomes related to the third power of theta, so that is the beauty of this theta definition because the pressure is some integer power of theta and the density is some other integer power of theta.

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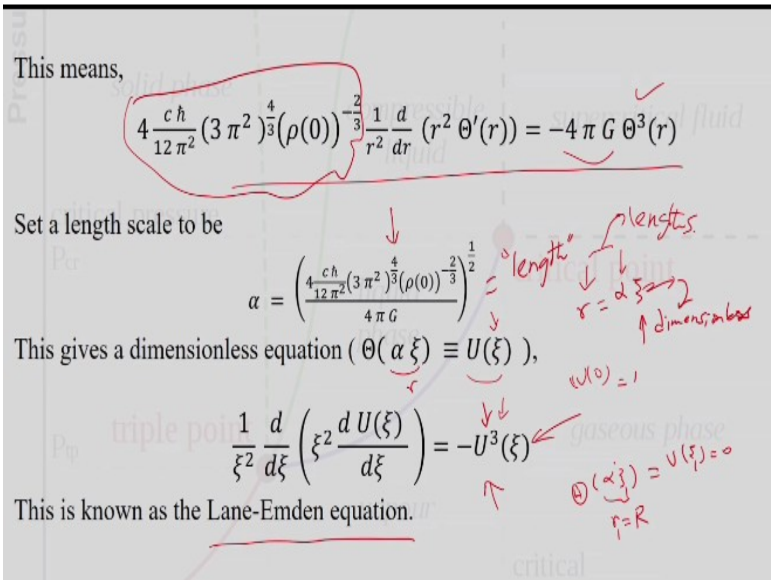

Okay, so now, I am going to go ahead and substitute this pressure here, rather here, I am going to substitute it there. So when I do that, I get this result okay. Well, I am going to do both, so I am going to substitute this pressure here and I am going to substitute this density there and then a whole bunch of thetas I am going to cancel out and finally I end up with this result. So see that this ugly looking constant has actually come out of the equation. So that is why I am going to take this down and call this whole thing as some alpha.

So this you can easily convince yourself this as the dimensions of length. So this is the length, so it has dimensions of length, some kind of a length. So, now I am going to define a new function. So I am going to define what is called alpha into ξ, so where this is now dimensionless. So this is distance and this is also distance, so both these are lengths. So both these are lengths and this is dimensionless. So, I am going to start calling, so this was my r, so theta of r, I am going to call it as U of ξ.

So if I choose to do that, then this equation can be written in this very beautiful form and this is called the Lane-Emden equation okay. So, this has to be solved. So this is a completely dimensionless equation, but then you see it is a second order nonlinear ordinary differential equation with non-constant coefficients, so that is a mouthful, and it is nonlinear. So if it was linear at least, if it is non-constant coefficients, it is already not easy to do as you very well know probably, those of you who have studied differential equations.

You know that if your non-constant coefficients, you have to use something called the Frobenius method to solve it, where if it is constant coefficients and it is second order, it is very easy. The solutions are always exponential or oscillatory or whatever, but if it is non-constant coefficients and linear, you can still if you get one solution, you can get the second by Wronskian method, but then none of those methods are going to work here because this is actually a nonlinear equation, so the superposition principle does not work.

So you know the second order equation, you expect 2 linearly independent solutions, but then if you know one, there is no way of figuring out the second one because you cannot use superposition principle, but fortunately, we do not need to go through all those, there is only going to be one physically meaningful solution.

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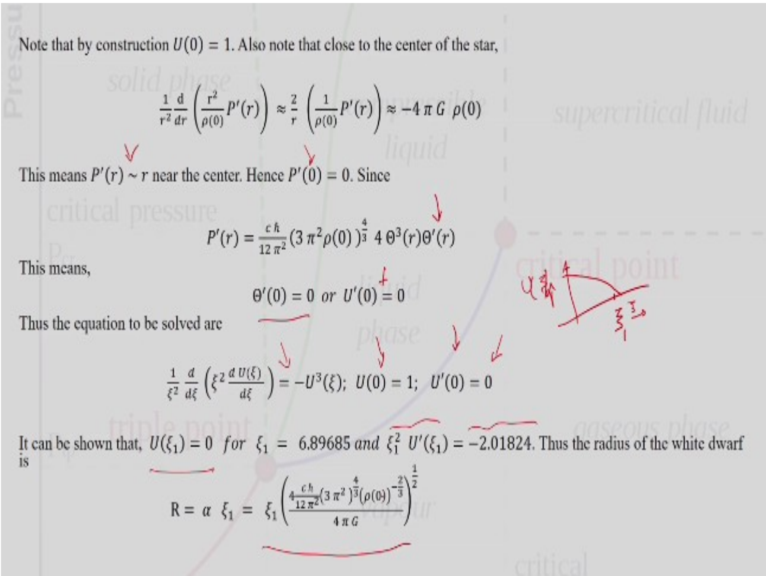

Alright, so this is the equation we have to solve. So now, what are the boundary conditions, of course, I mean I am not really looking for a general solution, I want a specific solution consistent with my physical boundary condition. So recall that theta of r it was nothing but this ratio. So if I put r equal to 0, I will get clearly theta of 0 equals 1, just by definition. So now that is one boundary. So this is a second order equation, and theta of 0 means U of 0 okay, so U of 0 and theta of 0 are the same things.

So U of 0 is 1, but then that is just one boundary condition, but then I need one more because this is a second order equation. So how do I deal with the second boundary condition? So in this equation, so if I go back here, so if I look at the small r limit, so if I look at the limit as r tends to 0, so when r tends to 0, you see the dominant term is going to be so when for r small, we can expect that this is going to be close to, P dash r is going to be close to P dash 0. So in this limit, p dash r is approξmately P dash 0 and ρ of r is approξmately a ρ of 0.

So the thing is that in this limit, this is going to be 1 by r, so this whole thing is going to become something like 2 because it is d by dr of r squared because all this will become constant and go out of the derivative. So it is d by dr of r squared, which is basically 2r, so 2r by r squared is 2 by r, 2 by r P dash 0 by ρ 0, that is what I get, but then that is equal to some ρ of 0. So in other words, it is actually what is telling me that the P dash 0 should tend to 0, that is P dash r should tend to 0 in the limit as r tends to 0 in such a way that P dash r is proportional to r.

So from here, you can conclude that P dash r has to be proportional to r as r tends to 0. So it is a little bit tricky, you should think about it more deeply. So as r tends to 0, you should convince yourself by staring at this equation that P dash r is proportional to r for when r is very small okay, just by staring at this you can convince yourself. Now so as a result when P dash r is close to r when r is very small, when r tends to 0, P dash 0 becomes 0 therefore, okay. So now, what is P dash r.

So P dash r remember that P dash r, P r itself was theta to the 4 and P dash r is nothing but theta dash r our times theta cubed and so on. So if P dash 0 is 0, so that means that because theta 0 is 1, we already know that, so that means theta dash of 0 should be 0 okay. So, theta dash of 0 is 0 same as saying U dash of 0 is 0. So, in other words, the Lane-Emden equation was this, but then I cannot solve this without supplying boundary conditions or initial conditions in this case.

So the initial conditions are going to be and because it is second order, I have to supply two initial conditions. So, the first initial condition has been U 0 is 1. The second initial condition is U dash 0 is 0. So it so happens that this equation looks formidable, but its solution is not that formidable and one can easily solve it, well it is not that easy, but it is not that hard either, and in fact, one can show that there is a ξ1, value ξ 1 for which U become 0. So, in other words, one can imagine that, remember that U dash 0 starts off being 1 and then finally it becomes 0.

So, there will be some ξ1. So, this ξ aξs and this is U aξs. So it starts off when ξ 0 is 1 then finally becomes. So the idea is that there is a ξ for which U become 0 and what is the physical meaning of that. So remember that U is nothing but theta, because U and theta are related. So theta of alpha ξ 1 equals U of ξ 1 equals 0. So that means, this is some r1, which I call basically r. So this is basically the radius of the star, why is this the radius of the star?

So the radius of the star is this ugly constant alpha times this fundamental number ξ 1, which makes U vanish, and so why is this the radius of the star because at that value if you put alpha ξ 1 here, so this theta is basically going to become 0, so that means at that value of r, when r equals alpha ξ 1, the pressure is 0, and the density is also 0 because this is 0. So the theta vanishes, so the pressure and density of the electrons is basically theta to some power.

So if theta becomes 0, the pressure becomes 0 and density becomes 0 and we can identify that that radius to be the radius of the star because we know that at the radius of the star, the pressure becomes 0 and there are no electrons left, that is the boundary of the star okay. So, we have found the boundary of the star and for later use, we will see that we will require this product. So ξ 1 square, U dash and ξ 1 can be evaluated and it is some number and remember that U dash is negative because the pressure and so on are basically decreasing functions of the r.

So our density and pressure are both decreasing as you go from the center to the surface of the star. So, this is the radius of the star okay. Now, so you can see that the radius of the star is basically, so it is kind of inversely related to that density okay. So larger the density, smaller the radius. If the density of the star at the center is very huge, the radius of the star is very small okay.

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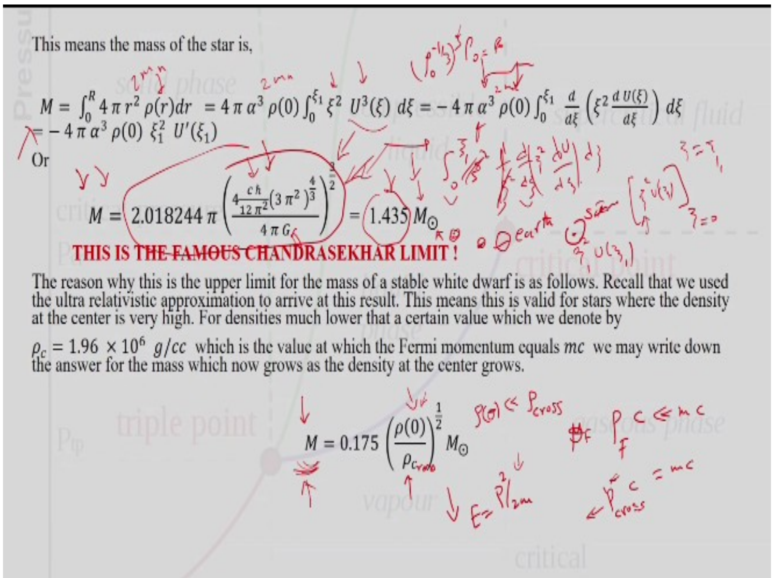

So let us see the mass of the star. So, this is not surprising that the radius of the star is inversely related to the density at the center, but what is really amazing is the mass, suppose you try to calculate the mass of the star, which is nothing but 4 pi r squared dr times the ρ, so it so happens that, so this of course I have to put a 2mn there, I forgot the 2mn, so 2 mn okay. So, the point is that if I calculate the mass of the star, okay, you can see that instead of ρ, I can start putting in my this equation.

So I can put my this result, so I will get this okay and keep in mind that U cubed, okay, so ρ is basically theta cubed, it is ρ zero into theta cubed, but then theta is basically directly related to U. So, this whole thing becomes just integral over U cubed ξ squared d ξ from 0 to ξ 1, which is the radius, I mean the radius in dimensionless units of the star. So remember that U cubed actually from this Lane-Emden equation can be written in terms of the derivative and the remarkable thing is this ξ squared is in the denominator in this ξ squared, they cancel, and then you end up with.

So if I do this ξ squared, so remember that if I do this kind of an integral, so it is going to be ξ squared, and what is U cubed? I forget the sign here. So it is 1 by ξ squared d by d ξ ξ squared okay dU by d ξ. So, this is what I have to do. So, now ξ squared ξ squared cancels and then I will end up having to do this. So, in other words, this is gone, this is gone, so I am trying to now integrate the derivative, so from 0 to ξ 1 and what is that answer?

So, the lower limit is just, so this is just going to be ξ squared U dash ξ evaluated from ξ equal to 0 to ξ equals ξ 1. So, now, you can see that the lower limit will not contribute because that makes ξ 0. So, the upper limit will contribute and upper limit is going to be U, U of ξ 1 squared, U of ξ 1, which is why I listed that earlier. So I told you I will require it and now is when I require it. So if I put this number there, you will see that finally, it will actually cancel out.

So the final answer does not involve ρ 0 at all because it has cancelled out okay. Why it has cancelled out because alpha cubed, remember that, there is a ρ 0 sitting here with an alpha, which is here. So you see what is alpha, so alpha is proportional to ρ of 0 raised to minus one-third okay, so that is what alpha is. Alpha is ρ 0 raised to minus one-third. Now, whereas here, what is this, this is alpha cubed. So alpha is ρ 0 raised to minus one-third. So alpha cubed is this cubed, so times ρ 0 okay, so that is what this is.

So it is alpha cubed, which is ρ 0 raised to minus one-third whole cube times ρ 0 and that is independent of ρ 0. So that is the amazing thing here. So now, if you calculate the total mass of the star, it is completely independent of the density at the center. So it only depends on a whole bunch of fundamental constants like Planck's constant and gravitational, not a whole bunch, just three of them; speed of light, Planck's constant and gravitational constant.

So if you work out the numbers, and this is the astronomical symbol, the M with a circle with a dot in the center is a universal ancient symbol for the sun okay, and this would be the earth. I mean, this would be earth and this is the sun. So, this is sun and this is the earth, so that is the ancient astronomical symbol for the sun. So now, you know what is the mass of the sun, you just work out, it becomes 1.44 times the mass of the sun and this is a kind of a universal constant independent of anything, I mean this whole thing is a universal constant, it is independent of anything else.

So, that was the remarkable result, see the reason why it is remarkable, contrast this with this result, which says that the radius of a white dwarf is actually inversely related to the density at the center raised to one-third. So in other words, as the density increases, the radius keeps shrinking, so that is very believable and nobody will question that. So it is this result that made a lot of people not believed this initially. So in fact, the famous story goes that the great astronomer Arthur Eddington who was very influential at that time, kind of ridiculed this idea.

So he called this you know, stellar buffoonery. So, he did not believe that the mass of a star, so Chandrasekhar himself remarked that, you know, so being able to write down the mass of a star in terms of laboratory constants that you find, you know, in the back cover of your high school textbook and that is the mass of a star is somewhat hard to believe, but it is nevertheless true that what this is saying is that in the ultrarelativistic limit, the mass of this star when it is stable.

So if that is a white dwarf that is stable because it is exerting the degeneracy pressure due to Pauli principle, which is balancing the gravity, which is trying to collapse it and it so happens that the mass of that star is actually unique, it is a fundamental constant. So, there is only one mass of that star. So, only such a star can survive. So that was the remarkable result of Chandrasekhar, which people did not believe initially, but finally, you know, when it was confirmed that there are no stars heavier than Chandrasekhar's limit.

After making detailed observations over a period of time, then people realized that there are no white dwarfs which are heavier than the Chandrasekhar limit, so they were forced to conclude that this is correct and as a result, Chandrasekhar won the Nobel Prize as you very well know. So now, let me conclude by pointing out why is this the limit? So, this calculation is just telling me that the white dwarf has a mass which is a fundamental constant, namely this, but then why is this the upper limit?

The reason is because you see we directly jumped into the ultrarelativistic calculation where the energy was cp. So what I should have strictly done is, I should have done both the calculations, I should have done the calculation where energy is P squared by 2m, and then compared it with the result when energy is cp. So you can imagine when that density is very small, so the smallness of the density is basically governed by how much this mu, that means if PFC is much less than mc okay, so that is when you should be using the non-relativistic approξmation.

So one can in fact define what is called the critical, it is not really critical but some kind of a crossover scale where it crosses over from the non-relativistic to the relativistic regime. So you can define what is called the crossover scale. So the crossover scale will be exactly when these two are equal. So, one can define this crossover density. So this will implies so this is a Fermi momentum for the crossover, this implies a certain density.

In fact, if you use your non-relativistic, if you use your E equals P squared by 2m calculation and recalculate the mass, which I am not going to do, but suppose you redo the whole thing, all the way up to this point, but using not equal to cP like I have done till now, but use E equal to P squared by 2m, I am going to get a mass, which is not a fundamental constant, but I am going to get a mass that depends upon the density at the center raised to one-half. So, it is going to be a density at the center raised to one-half.

So this is of course a fundamental constant because that is related to all these. So, there is going to be a situation if the mass of the star keeps increasing, this ρ is going to keep increasing, but notice that this result is valid only when the ρ is much less than this crossover okay. So if the density of the center is much less than the crossover, then only this is valid. So if you keep increasing the mass of the star, there is going to come a situation when it is going to cross over into the relativistic regime.

So, the non-relativistic approξmation will fail and then you will gradually start making it more relativistic. So, that is the reason why this is called the limit. So, what Chandrasekhar did, first he calculated this way. So, he did this calculation, where he showed that the mass of the star in the non-relativistic limit, so when the mass of the star is small, you can use non-relativistic limit and you show that the mass is proportional to the square root of the density at the center.

So, as you keep increasing the mass, the density at the center keeps increasing, but then once it goes much beyond the crossover limit, then you cannot use non-relativistic limit. So, you use ultrarelativistic limit and it will immediately tell you that you should stop when you reach the mass to be this number, which is the universal Chandrasekhar limit. So, you can keep increasing the mass of the star at least stable, stable, stable, stable, stable, and it will stop being stable once you cross this.

So, once you reach this, you are already in the ultrarelativistic limit. Once you reach Chandrasekhar limit here, you are already in the ultrarelativistic limit. So, what Chandrasekhar limit is telling you that you cannot cross this. So if you try to cross this, the star will not be stable. So that is the reason why it is the limit okay. So it took a long time for people to accept this because the Chandrasekhar limit is just a bunch of fundamental constants and it refers to the mass of a star.

So, I hope you enjoyed this presentation of Chandrasekhar limit. So it is a little difficult and you should show some respect because Chandrasekhar did this difficult calculation when he was only 19 years old as I keep pointing out. Okay, so let me close here and move on to some other topic next time. Thank you.